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G. Parisi: AN INFINITE NUMBER OF ORDER PARAMETERS FOR SPIN GLASSES.

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ABSTRACT : We show that in the mean field approximation spin glasses must be described by an infinite number of order parameters in the framework of the replica theory.

From the theoretical point of view spin glasses are very important because they describe one of the most simple case of amorphous material.

The best framework to study spin glasses is the replica theory[1] : one introduces an order parameter $Q_{\alpha,\beta}$ which is the limit, for n going to zero, of an $n \times n$ matrix, zero on the diagonal. In the mean field approximation the statistical expectation value is different from zero only in the spin glass phase, at zero magnetic external field. In the Sherrington-Kirkpatrick (S-K) model the mean field approximation is exact in the thermodynamic limit[2].

An intriguing feature of this scheeme is the necessity of reaching the limit $n = 0$ as analytic continuation in n from positive integer n.

In the standard treatment of the S-K model one assumes :

$$Q_{\alpha,\beta} = q . \qquad (1)$$

q is the Edwards-Anderson[1] order parameter and it is understood that eq. (1) does not hold for $\alpha = \beta$: $Q_{\alpha,\alpha}$ is identically zero.

The Ansatz eq. (1) gives results in variance with the computer simulations[3] of the S-K model, moreover one obtains a negative entropy at zero temperature: $S(0) = -0.17$ (the entropy of the model must be non-negative by definition). It has been suggested that the wrong result is due to the fact that the true value of $Q_{\alpha,\beta}$ is not symmetric under permutations of the indeces[4]: the parametrization eq. (1) is not valid and the replica symmetry is broken. Various patners of symmetry breaking have been proposed[5,6]; in a previous paper[7] it has been noticed that the patner of symmetry breaking depends on a continous variable which must be treated as a variational parameter.

If the matrix $Q_{\alpha,\beta}$ is parametrized as a function of three variables, quite good results have been obtained for the S-K model[7]: the agreement with the computer simulations is excellent and the zero temperature entropy is quite small: $S(0) = -0.01$. Generalizing this approach, the matrix $Q_{\ ,}$ becomes a function of many parameters: the three variables case is only the first step toward this direction.

To clarify this issue it is convenient to study the S-K model, where the matrix $Q_{\alpha,\beta}$ is a function of 1, 3, 5, ... parameters, and to study if this sequence of approximations converges. In this note we have studied the cases with 1, 3, 5 and 7 parameters near the critical temperature T_c where notable simplifications are present. Indeed near T_c the order parameter $Q_{\alpha,\beta}$ is small and a Taylor expansion in Q is allowed; one finds that the Free Energy $F(\tau)$ is given by:

$$F(\tau) = \operatorname{Min} F(Q) ,$$
$$F(Q) = \lim_{n\to 0} \frac{1}{n}\left[\tau \operatorname{Tr}(Q^2) - \frac{1}{3}\operatorname{Tr}(Q^3) + \frac{1}{4}\sum_{\alpha,\beta} Q^4_{\alpha\beta}\right] , \qquad (2)$$

where τ is proportional to $T_c - T$. Other terms proportional to the fourth power of Q could be added without changing qualitatively the results: the term of fourth degree which we have retained, is the only one which is responsible for the breaking of the replica symmetry[4, 5, 6].

The matrix $Q_{\alpha,\beta}$ belongs to a zero dimensional space, so that it is not evident how to write down the generical matrix of this space. The only known procedure consists in doing simple ansatz for integer n which are analytically continued in n up to n = 0. One possibility is the following:

$$
\begin{aligned}
Q_{\alpha,\beta} &= q_0 && \text{If } I(\tfrac{\alpha}{m_1}) = I(\tfrac{\beta}{m_1}) , \\
Q_{\alpha,\beta} &= q_1 && \text{If } I(\tfrac{\alpha}{m_1}) \neq I(\tfrac{\beta}{m_1}) ; \; I(\tfrac{\alpha}{m_2}) = I(\tfrac{\beta}{m_2}) , \\
Q_{\alpha,\beta} &= q_2 && \text{If } I(\tfrac{\alpha}{m_1}) \neq I(\tfrac{\beta}{m_1}) ; \; I(\tfrac{\alpha}{m_2}) \neq I(\tfrac{\beta}{m_2}) ,
\end{aligned}
\tag{3}
$$

where m_1, m_2, m_2/m_1 and n/m_1 are all integers ; I(x) is an integer valued function: its value is the smallest integer greater or equal to x (e.g. I(0.5) = = 1).

The matrix $Q_{\alpha,\beta}$ depends on 5 parameters ; if $m_2 = n$, Q is independent from q_2 and we recover the case studied in ref. (7), if $m_1 = m_2 = n$ only q_o is relevant and the replica symmetry is unbroken. It is evident how to generalize eq. (3) by writing the matrix $Q_{\alpha,\beta}$ as a function of q_i, i = 0, N, and of m_i, i = 1, N (the total number of parameters being 2N + 1).

The free energy F(Q) can be obtained by substituting eq. (3) in eq. (2); after some algebra one gets in the limit $n \to 0$:

$$
\begin{aligned}
F(q_i, m_i) = & \sum_{i=0}^{N} (m_i - m_{i+1}) \left[-\tau q_i^2 - \frac{1}{4} q_i^4 + \frac{1}{3} (2m_i - m_{i+1}) q_i^3 \right] + \\
& + \sum_{i=0}^{N} \sum_{j=1+i}^{N} (m_i - m_{i+1}) (m_j - m_{j+1}) q_i q_j^2 ,
\end{aligned}
\tag{4}
$$

where $m_o = 1$, $m_{N+1} = 0$ and no restriction is put on the values of the m_i, which are now real numbers.

We must look for the minimum of $F(q_i, m_i)$ as function of τ; for small τ one finds:

$$
\begin{aligned}
q_o &= \tau + A_o\tau^2 + O(\tau^3) , & & \\
q_1 &= \tau A_1 + O(\tau^2) , & m_1 &= \tau B_1 + O(\tau^2) , \\
q_2 &= \tau A_2 + O(\tau^2) , & m_2 &= \tau B_2 + O(\tau^2) , \\
q_3 &= \tau A_3 + O(\tau^2) , & m_3 &= \tau B_2 + O(\tau^2) ,
\end{aligned}
\qquad (5)
$$

where:

$$
\begin{aligned}
N = 0 &: A_o = 1/2 , \\
N = 1 &: A_o = 25/18 , \quad A_1 = 1/3 , \quad B_1 = 2 , \\
N = 2 &: A_o \simeq 1.5 , \quad A_1 \simeq 0.6 , \quad B_1 \simeq 2.4 , \quad A_2 \simeq 0.2 , \quad B_2 \simeq 1.2 , \\
N = 3 &: A_o \simeq 1.5 , \quad A_1 \simeq 0.7 , \quad B_1 \simeq 2.7 , \quad A_2 \simeq 0.4 , \quad B_2 \simeq 1.8 , \\
& \quad A_3 \simeq 0.14 , \quad B_3 \simeq 0.9 .
\end{aligned}
\qquad (6)
$$

For $N = 0$ and 1 we have been able to find an analytic solution, for greater values of N, a numerical analysis has been done in order to avoid rather serious algebric difficulties.

Increasing N, the procedure does not look terrible convergent, however the parameters q_i and m_i do not have a direct physical interpretation. If we consider the free energy, which is a physically well defined quantity, we find:

$$
F(\tau) = \frac{1}{3}\tau^3 + F_4\tau^4 + F_5\tau^5 + O(\tau^6) \qquad (7)
$$

where F_4 is independent from $N\ (F_4 = 1/4)$ while F_5 depends on N:

$$
\begin{aligned}
F_5 &= 0.25 \quad N = 0 ; & F_5 &\simeq 0.4475 \quad N = 1 ; \\
F_5 &\simeq 0.4997 \quad N = 2 ; & F_5 &\simeq 0.4999 \quad N = 3 ;
\end{aligned}
\qquad (8)
$$

One can see that the convergence is very fast and that the bulk of the corrections are obtained for N as small as 1. The smallness of the corrections for going from $N = 1$ to higher values of N, explains why excellent results have been obtained in ref. (7) where only the case $N = 1$ has been considered.

The possibility of having an infinite number of order parameters is related to the negative value of the "replicon mass"[6] for finite N. We expect that only in the limit $N \to \infty$, the replicon susceptibility becomes infinite, a more careful study of this problem is needed.

One should also study the effects on an external magnetic field on the order parameters. We know that for large magnetic field H only q_o is different from zero; decreasing H, when H is equal to a critical value (H_1^c), also q_1 becomes different from zero. It is possible that there are an infinite number of critical magnetic fields H_l^c[8] ($H_l^c \to 0$ when $l \to \infty$) such that for $H_{N-1}^c < H < H_N^c$ only the first N q_i's are different from zero ($i = 0, N-1$). In the other words the degeneracy among the order parameters is removed by the magnetic field and an infinite number of phase transitions are found by decreasing the magnetic field toward zero. The study of the correctness of this conjecture goes beyond the aims of this note.

An infinite number of parameters is needed to describe spin glasses in the mean field approximation: the deep for the appearance of this phenomenum is rather misterious; it would be quite interesting to understand how to connect each of the q_i and m_i to physically well defined quantities, independently of the replica trick.

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