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F. Palumbo: SPONTANEOUS BREAKING OF SUPERSYMMETRY AND SUPERCONDUCTIVITY.

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F. Palumbo: SPONTANEOUS BREAKING OF SUPERSYMMETRY AND SUPERCONDUCTIVITY(x).

Abstract.

It is conjectured that spontaneous breaking of supersymmetry implies superconductivity. This conjecture is investigated in the non relativistic domain, where it is shown that all the supersymmetric theories of fermions of a certain class have superconductive Hamiltonians.

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The theory of superconductivity has been formulated both as a field theory of fermions (electrons) and bosons (fonons), and as a field theory of selfinteracting fermions. In the second formulation a composite boson appears, formed by two electrons. The superconducting state results from a dynamical equilibrium where composite bosons are converted into electrons and vice-versa [(1)].

(x) - This paper is the development of the previous work LNF-78/17 (1978)

Supersymmetry[2] has been realized both in terms of boson and fermion fields changed (linearly) into each other by the spinorial generators, and in terms of a single fermion field[3] which is transformed nonlinearly in itself by the spinorial generator.

When supersymmetry is spontaneously broken, a Goldstone fermion appears belonging to a nonlinear realization, and some of the boson and fermion fields of the linear representation can be expressed as composite fields in terms of the Goldstone fermion field.

The above remarks suggest that spontaneous breaking of supersymmetry implies superconductivity, at least in the nonrelativistic domain[4]. This conjecture is also supported by the fact that the same U(1) symmetry which is broken in superconductivity must be broken in the presence of spontaneous supersymmetry breaking. This can be seen in the following way.

Since the low energy properties are determined by the selfinteracting Goldstone spinor, let us consider a supersymmetric Hamiltonian of fermions only. Let us denote by Q_α the spinorial generator (see eq. (6) below) and by $|\Phi_0\rangle$ the ground state. $|\Phi_0\rangle$ is not annihilated by Q_α and therefore the state $\bar{Q}Q|\Phi_0\rangle$ is also not vanishing and has the same energy as $|\Phi_0\rangle$, but a different number of particles, so that mass symmetry must be broken.

In the following we will show that nonrelativistic supersymmetric theories of fermions can actually be constructed. We will confine ourselves to Hamiltonians quartic in the fermion fields, giving up from the beginning Galilean invariance in order that they be exactly solvable. This is in the spirit of the "reduced" Hamiltonian of BCS. We will see that supersymmetric "reduced" Hamiltonians must necessarily be superconductive.

Let us consider the graded algebra of rotations and spce-time translations

$$[P_i , J_j] = i\,\varepsilon_{ij}{}^{K}\, P_K \quad ; \; [J_i , J_j] = i\,\varepsilon_{ij}{}^{K}\, J_K \qquad (1)$$

$$[Q_\alpha, P_i] = 0 \tag{2}$$

$$[Q_\alpha, J_i] = \frac{1}{2} \sigma_{i\alpha}{}^{\beta} Q_\beta \tag{3}$$

$$\{Q_\alpha, Q_\beta\} = 0 \tag{4}$$

$$\{Q_\alpha, Q_\beta^{+}\} = \pm \delta_{\alpha\beta} H \tag{5}$$

all the remaining commutators vanishing. In the above equations H is the Hamiltonian, P_k and J_k are the k-components of the total momentum resp., and σ_k are the Pauli matrices.

Since Q_α is a spinor it can only contain odd powers of fermion fields. In order that the Hamiltonian be quadratic and quartic in the fermion fields, we require that Q_α be linear and cubic. The most general expression for Q_α satisfying eq. (2) is therefore

$$Q_\alpha = \int dx \left[a_\alpha{}^{\beta} \psi_\beta^{+}(x) + b_\alpha{}^{\beta} \psi_\beta(x) \right] + \int dx_1 \int dx_2 \int dx_3$$
$$\Big\{ \psi_\beta^{+}(x_1) \psi_\gamma^{+}(x_2) \left[\psi_\delta(x_3) F_\alpha^{(1)\beta\gamma\delta}(x_i - x_j) + \psi_\delta^{+}(x_3) F_\alpha^{(2)\beta\gamma\delta}(x_i - x_j) \right]$$
$$+ \left[\psi_\delta^{+}(x_3) F_\alpha^{(3)\beta\gamma\delta}(x_i - x_j) + \psi_\delta(x_3) F_\alpha^{(4)\beta\gamma\delta}(x_i - x_j) \right] \psi_\gamma(x_2) \psi_\beta(x_1) \Big\} \tag{6}$$

where $\psi_\alpha(x)$ is the fermion field satisfying canonical anticommutation relations, $a_\alpha{}^{\beta}$ and $b_\alpha{}^{\beta}$ are constant matrices and $F_\alpha^{(k)\beta\gamma\delta}$ are arbitrary functions. We confine ourselves to the case that the $F_\alpha^{(k)\beta\gamma\delta}$ are square integrable even functions of a single argument

$$F_{\alpha}^{(k)\beta\gamma\delta}(x_i-x_j) = \frac{1}{2}\left[1-(-1)^{k}\right] t_{\alpha}^{(k)\beta\gamma\delta}(x_2-x_3) + $$
$$+ f_{\alpha}^{(k)\beta\gamma\delta}(x_1-x_2). \qquad (7)$$

It must be noted that the choice among the arguments x_1-x_2, x_2-x_3, x_3-x_1 is irrelevant to $F_{\alpha}^{(2)\beta\gamma\delta}$ and $F_{\alpha}^{(4)\beta\gamma\delta}$ because they multiply a totally antisymmetric operator, but there are two independent choices for $F_{\alpha}^{(1)\beta\gamma\delta}$ and $F_{\alpha}^{(3)\beta\gamma\delta}$ giving rise to the additional independent functions $t_{\alpha}^{(k)\beta\gamma\delta}$.

The most general form of the matrices $a_{\alpha}{}^{\beta}$ and $b_{\alpha}{}^{\beta}$ and of the functions $f_{\alpha}^{(k)\beta\gamma\delta}$ and $t_{\alpha}^{(k)\beta\gamma\delta}$ compatible with eq. (3) is

$$a_{\alpha}{}^{\beta} = a\,\delta_{\alpha}{}^{\beta} \quad ; \quad b_{\alpha}{}^{\beta} = b\,\varepsilon_{\alpha}{}^{\beta}$$

$$\begin{aligned}
f_{\alpha}^{(1)\beta\gamma\delta}(y) &= \varepsilon_{\alpha}{}^{\delta}\,\varepsilon^{\beta\gamma}\,f^{(1)}(y)\\
f_{\alpha}^{(2)\beta\gamma\delta}(y) &= \delta_{\alpha}{}^{\delta}\,\varepsilon^{\beta\gamma}\,f^{(2)}(y)\\
f_{\alpha}^{(3)\beta\gamma\delta}(y) &= \delta_{\alpha}{}^{\delta}\,\varepsilon^{\beta\gamma}\,f^{(3)}(y)\\
f_{\alpha}^{(4)\beta\gamma\delta}(y) &= \varepsilon_{\alpha}{}^{\delta}\,\varepsilon^{\beta\gamma}\,f^{(4)}(y)
\end{aligned} \qquad (8)$$

$$\left.\begin{aligned}
t_{\alpha}^{(1)\beta\gamma\delta}(y) &= \delta_{\alpha}{}^{\beta}\,\delta^{\gamma\delta}\,t^{(1)}(y)\\
t_{\alpha}^{(3)\beta\gamma\delta}(y) &= \varepsilon_{\alpha}{}^{\beta}\,\delta^{\gamma\delta}\,t^{(3)}(y)
\end{aligned}\right\} + \text{terms involving two Pauli matrices.}$$

where $\varepsilon^{12} = \varepsilon_1{}^2 = -\varepsilon^{21} = -\varepsilon_2{}^1 = 1$. We will omit the terms involving two Pauli matrices. They would modify the last term of eq. (10) below, without altering any of our conclusions. In order to find the con dions for Q_α to satify eqs. (4)-(5) we must distinguish the case i) in which eqs. (4)-(5) are to be satisfied for any value of the quantization box volume Ω, from the case ii) in which they are to be satisfied only for $\Omega \to \infty$. Case ii) is relevant to infinite systems and we will start from it. Precisely we require that

$$\lim_{\Omega\to\infty} \langle \Psi | \{ Q_\alpha, Q_\beta \} | \Psi' \rangle = 0$$
$$\lim_{\Omega\to\infty} \langle \Psi | \{ Q_\alpha, Q_\beta^+ \} - \delta_{\alpha\beta} H | \Psi' \rangle = 0, \tag{9}$$

Ψ and Ψ' being any couple of eigenstates of H. We identify H with the diagonal part of $\{Q_\alpha, Q_\beta^+\}$ getting

$$\begin{aligned}
\pm H = {} & \Omega(|a|^2 + |b|^2) + \int dx \int dy \{ t(y) \psi_\alpha^+(x) \psi_\alpha(x+y) + \\
& + [D(y) \psi_\alpha^+(x) \psi^{\alpha +}(x+y) + h.c.]\} + \\
& + \frac{1}{\Omega} \int dx \int dy \int dx' \int dy' \{ V(y,y') \psi_\alpha^+(x) \psi^{\alpha +}(x+y) \psi^\beta(x'+y') \psi_\beta(x') + \\
& + [U(y,y') \psi_\alpha^+(x) \psi^{\alpha +}(x+y) \psi_\beta^+(x') \psi^{\beta +}(x'+y') + h.c.] + \\
& + [W(y,y') \psi_\alpha^+(x) \psi^{\alpha +}(x+y) \psi_\beta^+(x') \psi_\beta(x'+y') + h.c.] + \\
& - Z(y,y') \psi_\alpha^+(x) \psi_\beta^+(x') \psi_\alpha(x+y') \psi_\beta(x'+y') \},
\end{aligned} \tag{10}$$

where $\psi^{\alpha}(x) = \varepsilon^{\alpha\beta}\psi_{\beta}(x)$ and

$$
\begin{aligned}
t(y) &= 2\Omega\left[a^* t^{(1)}(y) + b^* t^{(3)}(y) + c.c.\right] \\
D(y) &= \Omega\left[b^* f^{(1)}(y) + a^* f^{(2)}(y) + a\, f^{(3)*}(y) + b\, f^{(4)*}(y)\right] \\
V(y,y') &= \Omega^2\left[f^{(1)}(y)\, f^{(1)*}(y') + f^{(2)}(y)\, f^{(2)*}(y') + \right. \\
&\qquad \left. + f^{(3)*}(y)\, f^{(3)}(y') + f^{(4)*}(y)\, f^{(4)}(y')\right] \\
U(y,y') &= \Omega^2\left[f^{(1)}(y)\, f^{(4)*}(y') + f^{(3)*}(y)\, f^{(2)}(y')\right] \qquad (11) \\
W(y,y') &= 2\Omega^2\left[f^{(1)}(y)\, t^{(3)*}(y') + f^{(2)}(y)\, t^{(1)*}(y') + \right. \\
&\qquad \left. + f^{(3)*}(y)\, t^{(1)}(y') + f^{(4)*}(y)\, t^{(3)}(y')\right] \\
Z(y,y') &= 4\Omega^2\left[t^{(1)*}(y)\, t^{(1)}(y') + t^{(3)}(y)\, t^{(3)*}(y')\right].
\end{aligned}
$$

It follows from the above equations that we must take $t^{(i)} \sim f^{(i)} \sim \frac{1}{\Omega}$ and $a \sim b \sim 1$. The form of H in eq. (10) is such that we can apply Haag theorem, stating that the ground state is either a BCS state or the vacuum [5]. Now the terms violating eqs. (4) and (5) can be shown to be neagligeable (of the order $\frac{1}{\Omega}$ in the sense of eqs. (9)) only for states containing a number of fermions proportional to Ω. Therefore eqs. (4)-(5) are satisfied provided the constants and functions appearing in eqs. (8) are chosen in such a way that the ground state of H is a BCS state.

We will now give a specific example in which the Hamiltonian has the U(1) symmetry related to mass conservation. Let us take the minus sign in eq. (10) and put

$$\begin{aligned} f^{(3)}(y) &= f^{(4)}(y) = 0 \\ f^{(2)}(y) &= -i\, f^{(1)}(y) \\ t^{(3)}(y) &= -i\, t^{(1)}(y) \end{aligned} \tag{12}$$

with a, $f^{(1)}$ and $t^{(1)}$ real, so that

$$\begin{aligned} t(y) &= 8\Omega a\, t^{(1)}(y) & ;\quad D &= 0 \\ V(y,y') &= 2\Omega^2 f^{(1)}(y)\, f^{(1)}(y') & ;\quad U &= 0 \\ Z(y,y') &= 8\Omega^2 t^{(1)}(y)\; t^{(1)}(y') & ;\quad W &= 0. \end{aligned} \tag{13}$$

Let us further assume that in momentum space

$$t^{(1)}(p) = -\frac{1}{16\, a\, m\, \Omega}\; p^2\, \theta(P - p), \tag{14}$$

m being the fermion mass, P a cut-off parameter and θ (x) the step-function. The resulting Hamiltonian would be exactely the BCS Hamiltonian, if it were not for the last term in eq. (10). This term, however, can be made arbitrarily small by taking a large enough.

It remains to consider the case ii). This is much more complicated because there are all the additional terms of order $1/\Omega$ to be taken into account. We have not attempted to find the most general solution but we have further restricted the class of functions imposing $t^{(k)}(y) = 0$. We have found that necessary and sufficient conditions for eqs. (4) and (5) to be satisfied can, after a proper redefinition of the fields, be but in the form

$$
\begin{aligned}
& a = b = 0 \\
& f^{(4)}(y) = f^{(1)}(y) \\
& f^{(3)}(y) = f^{(2)}(y) \qquad\qquad (15) \\
& \int dy\, f^{(1)}(y) = 0 .
\end{aligned}
$$

The Hamiltonian is obtained up to a constant from eq. (10) introducing eqs. (15) in eqs. (11), with the exception of the function $t(y)$ which is now

$$t(y) = 4\Omega \left[|f^{(1)}(y)|^2 + |f^{(2)}(y)|^2 \right]. \qquad (16)$$

This term was absent in eqs. (11) because it is of order $1/\Omega$, and in fact we are in a strong coupling limit, with the additional severe constraint on "kinetic" and "potential" terms

$$V(y,y) = \frac{1}{2}\Omega\, t(y). \qquad (17)$$

The Hamiltonian is not U(1) invariant. What is interesting for the present purpose, however, is that it can be diagonalized exactely by a Bogoliubov-Valatin transformation.

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REFERENCES.

(1) - J. M. Blatt; Theory of superconductivity. Academic Press. New York and London 1964.

(2) - J. Wess and B. Zumino, Nucl. Phys. B70, 39 (1974). For a review see P. Fayet and S. Ferrara, Phys. Rep. 32C, 251 (1977).

(3) - D. V. Volkov and V. P. Akulov, Phys. Lett. 46B, 109, (1973).

(4) - The fact that the spectrum of a superconducting system has in general a nonvanishing energy gap is not a difficulty. In nonrelativistic systems, in fact, the spontaneous breakdown of a symmetry does not necessarily imply the existence of excitations of arbitrarily small energy (J. A. Swieca, Commun. Math. Phys. 4, 1 (1967).

(5) - R. Haag, Nuovo Cimento 25, 287 (1962). This is true unless we take all the $f^{(k)}(y) = 0$. In this case we have an essentially free Hamiltonian whose ground state can be neither a BCS state nor the vacuum.