E. Etim: GENERALISED HERMITE POLYNOMIAL EXPANSION
IN THE THEORY OF INFRARED RADIATIVE CORRECTIONS.

E. Etim: GENERALISED HERMITE POLYNOMIAL EXPANSION IN THE THEORY OF INFRARED RADIATIVE CORRECTIONS.

ABSTRACT.

We derive the generalised Hermite polynomial expansion of the momentum distribution of infrared radiation in gauge theories and compare it with another expansion derived from a stochastic model in momentum space. The two agree in leading order.

The semi-classical method of administering radiative corrections in gauge theories is based on a stochastic model in Fock space[1]. The act of emission of photons or gluons is a Poisson process. The average number of emitted radiation

$$\langle n(p) \rangle = \int d^D k \, \delta(k^2) \left| j_\mu(p, k) \right|^2 \tag{1}$$

is given in terms of a classical current

$$j_\mu(p, k) = \frac{ig(k)}{(2\pi)^{(D-1)/2}} \sum_1 \epsilon_1 \frac{p_{1\mu}}{p_1 \cdot k} \tag{2}$$

where $p_{1\mu}$ is the momentum of the 1-th charged particle, k_μ that

of the emitted radiation and ϵ_1 is a signature factor equal to $+1$ for incoming particles and outgoing antiparticles and -1 otherwise. The coupling constant $g(k)$ is independent of momentum in QED but not in QCD. Space-time has dimension D.

The characteristic function of the Poisson process is

$$G(p, x) = \exp(-h(p, x))$$ (3)

where

$$h(p, x) = \int d^D k \, \delta(k^2) \left| j_\mu(p, k) \right|^2 (1 - e^{-ikx})$$ (4)

Only the energy spectrum of the emitted radiation is easy to compute in this approach[2]. Quite generally the Fourier transform of the characteristic function which yields the momentum distribution

$$W(p, q) = \frac{1}{(2\pi)^D} \int d^D x \, e^{iqx} G(p, x)$$ (5)

is extremely difficult to perform. This seems the end of the road unless one conjures up some approximation for $G(p, x)$[3]. Unfortunately since $G(p, x)$ is not experimentally accessible an approximation for it is not easy to motivate.

We have recently[4] argued this problem differently, namely that from the Poisson description the total momentum loss

$$q_\mu = \sum_k n_{\vec{k}} \, k_\mu$$ (6)

is itself a random variable, and, like the $n_{\vec{k}}$, has independent increments. The same is true infact of each momentum k_μ. Hence since the most general stochastic process with independent increments and continuous in probability is known[5] it follows that the emission of infrared radiation can also be modeled by a continuous Markov process (relativistic Orstein-Uhlenbeck process)[6] directly in momentum space. The solution found for $W(p, q)$ is a series

$$W(p, q) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \delta^D(q - \sum_{j=1}^{n} k_j) \prod_{j=1}^{n} \hat{W}(p, k_j) \, \delta(k_j^2) \, d^D k_j \qquad (7)$$

where

$$\hat{W}(p, k) = \prod_1 \left[W_0(p_1, k) \right]^{\epsilon_1} \qquad (8)$$

and

$$W_0(p_1, k) = A_1 (b_1 \cdot k) \exp(-b_1 \cdot k)$$

$$b_{1\mu} = p_{1\mu}/M^2 \qquad (9)$$

is the relativistic Maxwell-Boltzmann distribution[7]. M is a fixed mass scale. The relationship between this description and that in Fock space is through the drift and diffusion coefficients, $T_\mu(p, k)$ and λ respectively, of the Fokker-Planck equation [8] satisfied by $\hat{W}(p, k)$. We have

$$\lambda = \frac{Q^{\frac{D-4}{2}} |g(Q)|}{(2\pi)^{(D-1)/2}} \qquad (10)$$

$$T_\mu(p, k) = \lambda \frac{\partial}{\partial k_\mu} \ln \hat{W}(p, k) = \lambda \sum_1 \epsilon_1 \left(\frac{b_{1\mu}}{b_1 \cdot k} - b_{1\mu} \right) \qquad (10')$$

Q is the magnitude of the maximum of the momentum loss q_μ. For simplicity we have fixed $g(k)$ at this maximum. $T_\mu(p, k)$ is just the difference between the total current flowing in the reaction and the $j_\mu(p, k)$ in eq. (2).

While the difficulty in carrying out the summation in eq. (7) is of the same order as that of evaluating the Fourier transform in eq. (5) the approximations one may try for $W(p, q)$ are now not far-fetc hed. For instance one may cut-off the series in (7) after the first few terms or, even simpler replace $W_0(p, k)$ by the exponential $\exp(-b \cdot k)$. In view of these applications it is important to compare the series in eq. (7) with the generalised Hermite polynomial expansion obtainable from the characteristic function[9]. We find that up to second order,

i. e. with the neglect of third and higher order - quasi moments in the Hermite polynomial expansion, $W(p, q)$ is well approximated, not just by a Gaussian, but by

$$V(p, q) = A \exp(-F_\mu(b) q_\mu) \tag{11}$$

since terms quadratic in k have also been neglected in eqs. (2) and (10'). $F_\mu(b)$ is a linear combination of the vectors $b_{1\mu}$. This is in agreement with eq. (7). These exponentials are the leading terms in both expansions. A more detailed comparison is difficult.

To begin with consider the cumulant expansion of $h(p, x)$

$$-h(p, x) = \sum_{n=1}^{\infty} \frac{i^n}{n!} C_{\mu_1, \mu_2, \ldots, \mu_n}(p) x_{\mu_1} x_{\mu_2} \cdots x_{\mu_n} \tag{12}$$

$$C_{\mu_1 \mu_2 \cdots \mu_n}(p) = (-)^n \int d^D k \delta(k^2) \left| j_\mu(p, k) \right|^2 k_{\mu_1} k_{\mu_2} \cdots k_{\mu_n}$$

Repeated indices will always be understood to be summed over unless stated otherwise.

The relationship between the $C_{\mu_1 \cdots \mu_n}(p)$ and the moments

$$T_{\mu_1 \mu_2 \cdots \mu_n}(p) = \int d^D q\, W(p, q)\, q_{\mu_1} q_{\mu_2} \cdots q_{\mu_n} , \tag{13}$$

as well as much of what follows, is standard[9].

$$C_\mu = T_\mu$$

$$C_{\mu\nu} = T_{\mu\nu} - T_\mu T_\nu$$

$$C_{\mu\nu\varrho} = T_{\mu\nu\varrho} - (T_\mu T_{\nu\varrho} + T_\nu T_{\varrho\mu} + T_\varrho T_{\mu\nu}) + 2 T_\mu T_\nu T_\varrho$$

$$\begin{aligned}
C_{\mu\nu\varrho\sigma} = T_{\mu\nu\varrho\sigma} &- (T_{\mu\nu} T_{\varrho\sigma} + T_{\mu\varrho} T_{\nu\sigma} + T_{\mu\sigma} T_{\nu\varrho}) - \tag{14} \\
&- (T_\mu T_{\nu\varrho\sigma} + T_\nu T_{\varrho\sigma\mu} + T_\varrho T_{\sigma\mu\nu} + T_\sigma T_{\mu\nu\varrho}) + \\
&+ 2(T_\mu T_\nu T_{\varrho\sigma} + T_{\mu\varrho} T_{\sigma\nu} + T_\mu T_\sigma T_{\nu\varrho} + T_\nu T_\varrho T_{\mu\sigma} + \\
&+ T_\nu T_\sigma T_{\mu\varrho} + T_\varrho T_\sigma T_{\mu\nu}) - 6 T_\mu T_\nu T_\varrho T_\sigma
\end{aligned}$$

Now substitute eq. (12) into (3) and define quasi-moments $B_{\mu_1 \ldots}$
$\ldots \mu_n(p)$ by means of the expansion

$$G(p, x) = \exp\left(\sum_{n=1}^{\infty} \frac{i^n}{n!} C_{\mu_1 \ldots \mu_n}(p) \, x_{\mu_1} \ldots x_{\mu_n} \right) =$$

$$= \left(1 + \sum_{n=3}^{\infty} \frac{(-i)^n}{n!} B_{\mu_1 \ldots \mu_n}(p) \, x_{\mu_1} \ldots x_{\mu_n} \right) \cdot \tag{14}$$

$$\cdot \exp\left(i \, C_\mu(p) \, x_\mu - \frac{1}{2} C_{\mu\nu}(p) \, x_\mu x_\nu \right)$$

It is easy to see from the first three terms of the expansion of the right hand side of (14) that

$$B_{\mu_1 \ldots \mu_n} = (-)^n C_{\mu_1 \ldots \mu_n} \, , \qquad 3 \leq n \leq 5 \tag{15}$$

Substituting for $G(p, x)$ from eq. (14) in (5) we have

$$W(p, q) = \left(1 + \sum_{n=3}^{\infty} \frac{(-)^n}{n!} B_{\mu_1 \ldots \mu_n}(p) \frac{\partial}{\partial q_{\mu_1}} \cdots \frac{\partial}{\partial q_{\mu_n}} \right) W_2(p, q) \tag{16}$$

where $W_2(p, q)$ is the Gaussian

$$W_2(p, q) = \frac{1}{(2\pi)^D} \int d^D x \, e^{iqx} \exp\left(i \, C_\mu(p) \, x_\mu - \frac{1}{2} C_{\mu\nu}(p) \, x_\mu x_\nu \right) \tag{17}$$

For the convergence of the integral in eq. (17) the quadratic form

$$S(p, x) = C_{\mu\nu}(p) \, x_\mu x_\nu \tag{18}$$

must be positive semi-definite for all vectors x_μ. This may not be obvious in Minkowski space. Here then are a few properties of $C_{\mu\nu}(p)$.
From eqs. (12)-(14)

$$C_{\mu\nu}(p) = \int d^D q \, W(p, q) \, (q_\mu - T_\mu)(q_\nu - T_\nu) =$$

$$= \int d^D k \, \delta(k^2) \, |j_\varrho(p, k)|^2 \, k_\mu k_\nu \tag{19}$$

hence $C_{\mu\nu}(p)$ has vanishing trace

$$T_r(C_{\mu\nu}(p)) = 0 \tag{20}$$

and for $\mu = \nu$ we have (no sum over μ)

$$C_{\mu\mu}(p) > 0 ; \qquad \mu = 1, 2, \ldots, D \tag{21}$$

Lastly the matrix $C_{\mu\nu}(p)$ is non-singular

$$\det(C_{\mu\nu}(p)) \neq 0 \tag{22}$$

Now let R be the orthogonal matrix which reduces $C_{\mu\nu}$ to diagonal form

$$(RCR^{-1})_{\mu\nu} = K_{\mu\mu}\delta_{\mu\nu} \tag{23}$$

where $\delta_{\mu\nu}$ is the Kronecker delta and let y_μ be the transform of x_μ under R^{-1}. Then making use of eqs. (23) and (21) in (18) one finds

$$S(p, y) = K_{oo}y_o^2 + K_{jj}y_j^2 > 0 \tag{24}$$

The result of integrating eq. (17) is

$$W_2(p, q) = \left[(2\pi)^D \det(C_{\mu\nu}(p))\right]^{-1/2} \exp\left(-\frac{1}{2}\hat{C}_{\mu\nu}(p)u_\mu u_\nu\right) \tag{25}$$

$$u_\mu(q, p) = q_\mu + C_\mu(p) = q_\mu + T_\mu(p)$$

where $\hat{C}_{\mu\nu}(p)$ is the inverse of $C_{\mu\nu}(p)$.

Before substituting (25) into (16) consider the following generalisation of the Hermite polynomials[9]

$$H_{\mu_1 \cdots \mu_n}(u) = (-)^n \exp\left(\frac{1}{2}C_{\lambda\tau}u_\lambda u_\tau\right)\frac{\partial^n}{\partial q_{\mu_1} \cdots q_{\mu_n}} \cdot \tag{26}$$

$$\cdot \exp\left(-\frac{1}{2}C_{\varrho\sigma}u_\varrho u_\sigma\right)$$

In terms of the variable

$$z_\mu = \hat{C}_{\mu\nu} u_\nu \tag{27}$$

a few of them are

$$H_\mu(z) = z_\mu$$

$$H_{\mu\nu}(z) = z_\mu z_\nu - \hat{C}_{\mu\nu}$$

$$H_{\mu\nu\varrho}(z) = z_\mu z_\nu z_\varrho - (\hat{C}_{\mu\nu} z_\varrho + \hat{C}_{\mu\varrho} z_\nu + \hat{C}_{\nu\varrho} z_\mu) \tag{28}$$

$$H_{\mu\nu\varrho\sigma}(z) = z_\mu z_\nu z_\varrho z_\sigma - (\hat{C}_{\mu\nu} z_\varrho z_\sigma + \hat{C}_{\mu\varrho} z_\nu z_\sigma + \hat{C}_{\mu\sigma} z_\nu z_\varrho +$$

$$+ \hat{C}_{\nu\varrho} z_\mu z_\sigma + \hat{C}_{\nu\sigma} z_\mu z_\varrho + \hat{C}_{\varrho\sigma} z_\mu z_\nu) -$$

$$- (\hat{C}_{\mu\nu}\hat{C}_{\varrho\sigma} + \hat{C}_{\mu\varrho}\hat{C}_{\nu\sigma} + \hat{C}_{\mu\sigma}\hat{C}_{\nu\varrho})$$

Substituting (25) now into (16) and making use of (27) we have finally the expansion

$$W(p, q) = \left[1 + \sum_{n=3}^{\infty} \frac{1}{n!} B_{\mu_1 \cdots \mu_n}(p) H_{\mu_1 \cdots \mu_n}(u(p, q)) \right] W_2(p, q) \tag{29}$$

to be compared with eq. (7). Note that if $q^2 \neq 0$ the series in (7) also starts from $n = 2$. Neglecting the higher order terms ($n \geq 3$) eqs. (7) and (29) give

$$W_2(p, q) = \exp\left(- \sum_1 \epsilon_1 b_1 \cdot q\right) \frac{1}{2} \int d^D k\, \delta(k^2)\, \delta((q - k)^2) \cdot$$

$$\cdot \prod_1 \left[(b_1 \cdot k)(b_1 \cdot q) - (b_1 \cdot k)^2 \right]^{\epsilon_1} \tag{30}$$

Both sides of this equation agree in the dominant exponential factors apart from quadratic terms in q_μ. Moreover, exactly as in eq. (29), this exponential factor is common to all the terms of the series in eq. (7). Consequently the leading terms in eqs. (7) and (29) agree.

If to this order one approximates $W(p, q)$ itself by

$$W(p, q) = A(b \cdot q) \exp(-b \cdot q)$$

$$\tag{31}$$

$$b_\mu = \sum_1 \epsilon_1 b_1$$

then there are closed form expressions. For instance the q^2-distribution is given by

$$\frac{dW(p, q^2)}{dq^2} = A (2\pi q b)^{\nu} K_{\nu+1}(q b), \qquad \nu = \frac{1}{2}(D - 2) \qquad (32)$$

where on the right hand side q and b stand for the magnitudes of the vectors q and b respectively.

This is perhaps as far as one can go in the comparison. The distorted phase space integrals

$$\Omega_n(p, q) = \frac{1}{n!} \int \delta^D(q - \sum_{j=1}^{n} k_j) \prod_{j=1}^{n} \prod_{1} (b_1 \cdot k_j)^{\epsilon_1} \delta(k_j^2) d^D k_j \qquad (33)$$

are difficult to evaluate in the one case and the coefficients $B_{\mu_1 \ldots \mu_n}(p)$ are unknown in the other. It is nevertheless interesting that some progress can be made towards calculating with some rigour the momentum spectrum of infrared radiation. Having now understood the structure of this distribution some effort should be expended in calculating the correlation matrix $C_{\mu\nu}(p)$, at least for electron-positron annihilation. Finally the relationship between a discrete and a continuous Markov process with such physically relevant application is interesting in its own right.

REFERENCES

(1) - E. Etim, G. Pancheri and B. Touschek, Nuovo Cimento $\underline{51}$, 276 (1967) ; M. Greco, F. Palumbo, G. Pancheri-Srivastava and Y. Srivastava, Phys. Letters $\underline{77B}$, 282 (1978).

(2) - See first reference listed above.

(3) - G. Pancheri-Srivastava and Y. Srivastava, Frascati preprint LNF-78/46 (1978), to be published.

(4) - E. Etim, Frascati preprint LNF-79/9 (1979), to be published.

(5) - A. Blanc-Lapierre and R. Fortet, Theory of Random Functions (Gordon and Breach, 1965), vol. 1, Ch. 4, p. 153.

(6) - L. S. Orstein and G. E. Uhlenbeck, Phys. Rev. $\underline{36}$, 823 (1930).

(7) - F. Jüttner, Annalen der Physik $\underline{34}$, 856 (1911) ; J. L. Synge, The Relativistic Gas (North-Holland, 1957), p. 36.

(8) - See ref. (5), Ch. 7, p. 324.

(9) - R. L. Stratonovich, Topics in the Theory of Random Noise (Gordon and Breach, 1963), vol. 1, Ch. 3,