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E. Etim: A STOCHASTIC MODEL IN MOMENTUM SPACE FOR THE EMISSION OF INFRARED RADIATION.

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E. Etim: A STOCHASTIC MODEL IN MOMENTUM SPACE FOR THE EMISSION OF INFRARED RADIATION.

Abstract: The emission of infrared radiation in gauge theories can be modeled by a Poisson process in Fock space. We construct the corresponding Markov process in momentum space.

Infrared radiative corrections in gauge theories can be administered semi-classically using the method of coherent states[1]. This approach is conceptually very simple. It does not give up completely the perturbative counting of photons or gluons in Fock space. It does so with a difference: the number $n_{\vec{k}}$ of emitted radiation in the mode k_μ is considered as a random variable and the act of emission is modeled by a Poisson process.

The commulant generating function of the Poisson process

$$h(p, x) = \int \frac{d^3k}{2\omega} \left| j_\mu (p, k) \right|^2 (1 - e^{-ikx}) \tag{1}$$

is given in terms of a classical current

$$j_\mu (p, k) = \frac{i g(k)}{(2\pi)^{3/2}} \sum_l \varepsilon_l \frac{p_{l\mu}}{p_l k} \qquad (2)$$

where $p_{l\mu}$ is the 4 - momentum of the l - th charged particle, k_μ that of the emitted radiation, ε_l a signature factor equal to + 1 for an incoming (outgoing) particle (antiparticle) and - 1 for an outgoing (incoming) particle (antiparticle). The coupling constant g (k) is independent of k in QED but not in QCD.

The process in Fock space leads directly to the characteristic function

$$G(p, x) = \exp(-h(p, x)) \qquad (3)$$

This is essentially a space-time description . The Fourier transform of G(p, x) yields the momentum distribution

$$W(p, q) = \frac{1}{(2\pi)^4} \int d^4x\, e^{iqx}\, G(p, x) \qquad (4)$$

The transform in eq. (4) is difficult, if not impossible, to perform. At this point one naturally looks for a manageable approximation for G(p, x) and then compares the result for W(p, q) with perturbation theory. Since W(p, q) is the experimentally accessible function an approximation for it should be easier to motivate. It is interesting to start with the question: is there a momentum space stochastic model for the emission of soft radiation? The answer is yes. The purpose of this note is to construct such a process. We shall show that the momentum distribution is related to the relativistic Maxwell - Boltzmann distribution[2]. Mathematically the problem we have touched upon lightly here concerns the relationship between discrete and continuous relativistically invariant measures. Details will be considered elsewhere.

It follows from the Fock space description that the 4 - momentum

$$q_\mu = \sum_{\vec{k}} n_{\vec{k}} \; k_\mu \tag{5}$$

(in fact each momentum k_μ) due to infrared radiation is a random variable and like $n_{\vec{k}}$, has independent increments. The most general stochastic process with independent increments and continuous in probability is known[3]: it is the sum of a Wiener process (Brownian motion) and a Poisson process. However while $n_{\vec{k}}$ is discrete k_μ is continuous, hence: emission of infrared radiation may be modeled, by a Poisson process in Fock space as well as by a continuous Markov process in momentum space.

From eqs. (1) and (2) the average number of emitted radiation is given by

$$d < n(p,\omega) > \; = \; \beta(p) \; \frac{d\omega}{\omega} \tag{6}$$

where

$$\beta(p) = \frac{1}{2} \int d\,\Omega \;\; (\hat{k}) \; \omega^2 \left| j_\mu \;\; (p, k) \right|^2 \tag{7}$$

$\beta(p)$ is thus a measure of the average number of emitted radiation. It is larger the stronger the coupling $g(k)$. We shall consider the case of $\beta(p)$ large and then analytically continue to all $\beta(p)$.

In QCD $g(k)$ is not known as a function of k in the region of interest. For simplicity we shall fix it at the maximum allowed momentum Q_μ .

Let M be some mass scale and rewrite eq. (2) as

$$j_\mu (p, k) = \pm i \lambda \sum_l \varepsilon_l \frac{b_{l\mu}}{b_l k} \tag{8}$$

where

$$b_{l\mu} = p_{l\mu}/M^2$$

$$\lambda = \frac{|g(Q)|}{(2\pi)^{3/2}} \tag{9}$$

In the absence of radiation the current flow in the reaction is proportional to

$$j_\mu (p) = \pm i \lambda \sum_l \varepsilon_l b_{l\mu} \tag{10}$$

The difference between (8) and (10) is the net current

$$J_\mu (p, k) = \pm i \lambda \sum_l \varepsilon_l \left(\frac{b_{l\mu}}{b_l k} - b_{l\mu}\right) \tag{11}$$

due to the motion of all charges and corrected for the effect of their coupling to radiation. Our main observation is that these currents are gradients of scalar potentials. In particular

$$J_\mu (p, k) = \pm i \lambda \frac{\partial}{\partial k_\mu} \ln \hat{W}(b, k) \tag{12}$$

where

$$\hat{W}(b, k) = \prod_l \left[W_o (b_l, k)\right]^{\varepsilon_l} \tag{13}$$

and

$$W_o (b_l, k) = A_l (b_l k) \exp (- b_l k) \tag{14}$$

is the relativistic Maxwell - Boltzmann distribution[2]. It is the probability that a charged particle of momentum $M^2 b_{1\mu}$ emites radiation of momentum k_μ. Accordingly $\hat{W}(b, k)$ is the probability of the same k_μ being emitted in each of the statistically independent trials of probing several charged particles. It is easy to verify that $W_o(b, k)$ is an equilibrium solution of the Fokker-Planck equation[4].

$$\frac{W(\tau; b, k)}{\partial \tau} = \left(-\frac{\partial}{\partial k_\mu} T_\mu (b, k) + \lambda \frac{\partial^2}{\partial k_\mu^2}\right) W(\tau, b, k) \tag{15}$$

where τ is some invariant time variable and

$$T_\mu (b, k) = \lambda \frac{\partial}{\partial k_\mu} \ln W_o (b, k) = \lambda \left(\frac{b_\mu}{bk} - b_\mu\right) \tag{16}$$

Eq. (15) defines a continuous Markov process in momentum space with $T_\mu(b, k)$ and λ as the drift and diffusion coefficients respectively[3,4]. $\hat{W}(b, k)$ satisfies a similar equation with $\pm i J(p, k)$ as the drift current. Hence besides the Poisson process in Fock space there exists a stochastic description of the emission of infrared radiation directly in momentum space.

Its construct in terms of k_μ itself is given by the stochastic differential equation (equivalent in probability to eq. (15))[5]

$$\frac{dk_\mu}{d\tau} = \pm i J_\mu (p, k) + (2\lambda)^{1/2} \xi_\mu (p, \tau) \tag{17}$$

where the momenta p are parameters and $\xi_\mu (p, \tau)$ is a Gaussian random vector satisfying

$$\langle \xi_\mu(p, \tau) \rangle = 0$$
$$\langle \xi_\mu(p, \tau)\, \xi_\nu(p, \tau') \rangle = \delta_{\mu\nu}\, \delta(\tau - \tau') \qquad (18)$$

If eq. (10) is not corrected for the effect of the emission of radiation $W_o(b, k)$ becomes the simple exponential

$$V_o(b_1, k) = A_1 \exp(-b_1 k) \qquad (19)$$

Given $\hat{W}(b, k)$ the momentum distribution of soft radiation with total 4-momentum q_μ is

$$W(b, p) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \delta^4 \left(q - \sum_{j=1}^{n} k_j\right) \prod_{j=1}^{n} \hat{W}(b, k_j)\, d^4 k_j\, \delta(k_j^2) \qquad (20)$$

Eq. (20) is a formal solution for $W(b, q)$. The difficulty in carrying out the summation on its right hand side is of the same order as in evaluating the Fourier transform in eq. (4). In fact making use of (20) in (4) one finds

$$G(p, x) = \exp(\hat{Z}(b, x)) - 1 \qquad (21)$$

where

$$\hat{Z}(b, x) = \int d^4 k\, \delta(k^2)\, e^{-ikx}\, \hat{W}(b, k) \qquad (22)$$

Substitution of eq. (13) in (22) reveals immediately the extent and the nature of the problem involved. However it is now easier to see the range of approximations one may try for $W(b, q)$. For instance one may retain only the first few terms of the series in eq. (20).

As another approximation $V_o(b, k)$ may be substituted for $W_o(b, k)$. Perhaps the most interesting, at least from the point of view of strong interactions, is the following

$$W(b, q) \equiv W(b_1, b_2, \dots, q) \simeq W_o(\sum_l \varepsilon_l b_l, q) \tag{23}$$

suggested by the similarity of eq. (20) to the statistical bootstrap equation [6]. It assumes that the object with momentum $q_\mu = \sum_{\vec{k}} n_{\vec{k}} k_\mu$ behaves as an off-shell gluon and does not resolve the classical currents to which the k_μ couple into components.
Substituting (23) in (4) we have

$$h(p, x) = -\ln\left[\frac{16\pi A(b^2 + ib\, x)}{(b^2 + 2ib\, x - x^2)^3}\left(1 - \frac{1}{8} z^3 K_3(z)\right)\right] \tag{24}$$

$$z = (Q^2(b^2 + 2ib\, x - x^2))^{1/2}$$

$$b_\mu = \sum_l \varepsilon_l b_{l\mu} = \frac{1}{M^2}\sum_l \varepsilon_l p_{l\mu}$$

$K_3(z)$ is the modified Bessel function. It is interesting to compare the small x limit of both sides of (24). From eq. (1)

$$h(p, x_o, \vec{x} = 0) \xrightarrow[x_o \to 0]{} \beta(p, Q)\,(i p_o x_o) \tag{25}$$

The Q - dependence of $\beta(p, Q)$ comes from that in the coupling constant $g(Q)$. The integral over the magnitude of $\vec{k}$ in eq. (1) has been carried out up to the energy of the 4 - momentum $p_\mu = M^2 b_\mu$. Comparing (24) with (25) we get

$$\beta = 3 + 2\nu \tag{26}$$

where we have generalised to any number of space time dimensions $D = 2 + 2\nu$.

Using (26) to eliminate ν in favour of β we can finally rewrite eq. (24) in a form valid for all values of $\beta > 0$.

$$h(p,x) = -\ln\left[\frac{A\,2^{(1+\beta)/2}\,\Gamma\left(\frac{1+\beta}{2}\right)\pi^{(\beta-3)/2}\,(b^2+ib\,x)}{(b^2+2\,ib\,x-x^2)^{(1+\beta)/2}}\cdot\right.$$

$$\left.\cdot\left(1-\frac{z^{(1+\beta)/2}}{2^{(\beta-1)/2}\,\Gamma\left(\frac{1+\beta}{2}\right)}\,K_{\frac{1+\beta}{2}}(z)\right)\right] \qquad (27)$$

It can be shown, with some effort, that in QED a direct integration of eq. (1) leads to an expression which may be approximated by eq. (27).

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(3) - A. Blanc-Lapierre and R. Fortet; Theory of Random Functions Vol. 1; Gordon and Breach ch. 4 p. 153 (1965).

(4) - See Ref. (2) ch. 7, p. 234.

(5) - R. L. Stratonovich: Topics in the Theory of Random Noise, Vol. 1 Gordon and Breach p. 102 (1963). We are here making an additional assumption, involving a Wick rotation, since the equivalence between eqs. (15) and (17) holds, strictly speaking, in a Euclidean space.

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