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B. Touschek: AN ANALYSIS OF STOCHASTIC COOLING.

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B. Touschek[+]: AN ANALYSIS OF STOCHASTIC COOLING.

A harmonic, but non linear, oscillator.

In most of the quasilinear oscillations, which one deales with in the theory of accelerators and storage rings the frequency depends in some way on the amplitude: $\omega = \omega(q^2)$, where q is the amplitude. In the simplest cases one has:

$$\omega = \omega_o(1 + \alpha q^2) \tag{1}$$

in which α is a constant of dimensions q^{-2}. If $\alpha > 0$ the oscillation will be called hyper-stable, if $\alpha < 0$ hypo-stable. The 1st case is often encountered in betatron oscillations, where the "non linearity" of the fields is due to higher magnetic poles (than quadripole); $\alpha < 0$ is encountered in synchrotron oscillations.

A consequence of (1), which is mostly without interest in the applications of "Routh's theory" in machine physics, is the appearance of higher harmonics in the Fourier expansion of the oscillations. The following is a model of an oscillator, where this complication does not occur.

I choose

$$\omega = \omega_o(1 + \eta H_o) \tag{2}$$

where η has dimensions erg^{-1} (and is > 0 for hyper, and < 0 for hypo-stability, and H_o is the Hamiltonian of a harmonic oscillator of

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+ Deceased.

frequency ω_o:

$$H_o = \frac{1}{2m}(p^2 + m^2\omega_o^2 q^2) \tag{3}$$

m is the mass of the oscillating body. There is the following correspondence between η and α:

$$\eta = \frac{\alpha}{m\omega_o^2} \tag{4}$$

which holds accurately if q^2 in (1) is replaced by $\langle q^2 \rangle$, i.e. its meanvalue.

The Hamiltonian H, which leads to the frequency dependence (3) will be a function of H_o and of H_o only, viz $H = H(H_o)$. The frequency ω is given by Hamiltonian mechanics as $\omega = 2\pi(dH/dJ)$ in which J is the action-integral (defined as $J = \oint p\,dq$).

$\therefore\ \omega = 2\pi \frac{dH}{dH_o}\frac{dH_o}{dJ} = \frac{dH}{dH_o}\omega_o$, since H_o describes an oscillator of frequency ω_o. From (3) it then follows that

$$H = H_o + \frac{1}{2}\eta H_o^2 \tag{5}$$

(by integration). The equations of motion for p and q are the equations of a Harmonic oscillator with ω_o replaced by ω. Indeed one has from the two equations $\dot{q} = \partial H/\partial p$, $\dot{p} = -\partial H/\partial q$ in which the 1st defines p as a function of the velocity $\dot{q}$ and the 2nd is simply Newton's 2nd law

$\dot{q} = \frac{dH}{dH_o}\frac{\partial H_o}{\partial p} = \frac{p}{m}(1 + \eta H_o)$ and $\dot{p} = -(1 + \eta H_o)m\omega_o^2 q$. Differentiating the 1st with respect to t and inserting into the 2nd one thus gets (using $\dot{H}_o = 0$)

$$\ddot{q} + \omega_o^2(1 + \eta H_o)^2 q = 0\ .$$

That this is a non linear equation is seen by remembering that H_o is given by equation (3).

The example shows that the appearance of higher Harmonics is a rather trivial and avoidable defect of non linear equations of motion. The oscillator described by (5) is still "harmonic".

The complex amplitudes.

The complex amplitudes are introduced by putting

$$J = a^* a\, 2\pi \tag{6}$$

$|a|$ is $1/\sqrt{2}$ x the radius of the circle which describes the trajectory in phase space. a has the dimensions of a $\sqrt{}$ of an "action" ($erg^{1/2}$ $sec^{1/2}$). We choose

$$a = \frac{1}{\sqrt{2m\omega_o}}(p - im\,\omega_o q) \tag{7}$$

and the corresponding relation with i replaced by -i for a^* the complex conj. Inverting (7) one gets

$$p = \sqrt{\frac{m\,\omega_o}{2}}\,(a + a^*)\ , \qquad q = \frac{+i}{\sqrt{2m\,\omega_o}}\,(a - a^*) \tag{8}$$

The momentum ∴ corresponds to the real part of a, the coordinate q to its imaginary part. The curve a = a(t) is a circle in the "Gaussian" plane. The representative point on this circle moves in a clockwise direction for a and anticlockcoise for a^*.

An important part in the theory of canonical transformation is played by the "Poisson Bracket"

$\{A, B\} = -\{B, A\} = \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial p}$, in which A and B are functions of the form f(p, q ; t). Obviously from the definition

$$\{p, q\} = 1\ . \tag{9}$$

The requirement that (9) holds also for "zero" momentum and space coordinates p' = p'(p, q), q' = q''p, q) is the basis for Liouville's theorem, since the det represented by { , } can be interpreted as the "Wronskian" or functional determinant which ensures that the element of phase-space dpdq is an invariant. With the help of the P-B Hamilton's equations follow automatically from

$$\dot{f} = \frac{\partial f}{\partial t} + \{H, f\} \tag{10}$$

valid for any function f(p, q ; t). The P.B. of the action amplitudes a and a^* can be derived from (7). One has, using (9)

$$\{a, a^*\} = i\ . \tag{9a}$$

The transformation which leads from p and q to a and a^* is not a canonical transformation. It resembles one. From (10) one can now derive the equations of motion for $H = \omega_o a^* a$. Indeed it follows from (9a) that

$$\dot{a} = \{H_o, a\} = -i\,\omega_o a\ . \tag{11}$$

Since $H_o = \omega_o a^* a$. This is the 1st order equation of motion for the amplitude; its complex conjugate replaces the 2nd of Hamilton's equations. With H given by (4) one gets $\{H, a\} = -i\omega a$, if equation (2) is used. ω itself is given by $\omega_o = \omega_o(1 + \eta\omega_o a^* a)$ and is ,'. - as wanted at the start-amplitude - dependent.

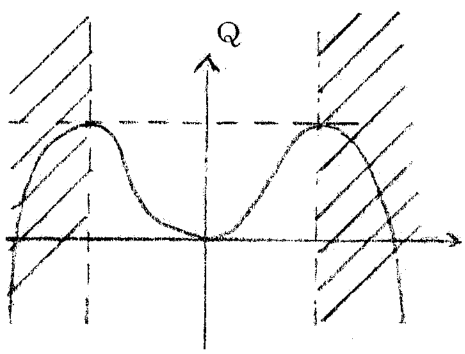
The "Ensemble" of N non linear oscillator.

Strictly speaking thermodynamics is - because of the absence of interaction between the oscillators - not applicable in the present case. It becomes applicable thro' the action of a P. U. electrode which connects with the collective degree of freedom, which in turn interacts with any one the single and in isolation independent oscillators. Indeed, the thermodynamical nomenclature is very suggestive when applied to the damping out of the oscillations of a beam of charged particles and is already implied in the name which Budker gave to the process: "cooling".

The key quantity to the thermodynamical situation with which we deal is the free energy $F = U - TS$ which is connected to the sum over states Z by $\beta F = -\log Z$. By Z we understand the s. o. states of an individual oscillator characterized by the Hamiltonian (4). H_o is connected to the action integral J by $H_o = (\omega_o/2\pi)J = \nu_o J$ and the free energy, in classical statistical mechanics is given by

$$Z = \int dJ\, e^{-\beta H(J)} \tag{12}$$

$\beta = 1/\theta$ and $\theta = kT$; k is Boltzmann's constant. The integration should be extended over J from 0 to ∞, but there one is beset by a difficulty: the motion of the oscillators is limited by the presence of the walls of the doughnut and for $\eta < 0$, which corresponds to the potential energy V shown in the drawing there may be, and there is with the integration extended to ∞, an "overflow" of the system into the unstable region which is shaded in the drawing (this happens at $H_o > 2/|\eta|$). In either case one has to make sure that this overflow is so slow that it does not effect equilibrium - it nearly always is.



After this precaution the integration in (12) can, as required by the theory, be extended from 0 to ∞. The sum over states in a classical theory is by definition given by an integral over phase-space as shown in (12). Putting $\beta\nu_o J = x$ one thus gets, using (4) for H

$Z = \frac{1}{\beta \nu_o} \int_0^\infty e^{-x} e^{-\frac{\eta}{2\beta} x^2} dx.$ If $|\eta|$ is sufficiently small ($\eta\theta \ll 1$) the convergence of the integral is due to the 1st factor e^{-x} of the integrand and the 2nd factor differs little from 1. One then gets :

$$Z \cong \frac{kT}{o} (1 - \eta kT) . \tag{13}$$

Note that in classical statistical thermodynamics Z has the dimension of an action. In quantum mechanics one has, in the case of 1 degree of freedom $Z_q = Z/h$ with h Planck's quantum of action.

Z_q defines the thermodynamical potential F, the so called free energy : $\beta F = - \log Z$. The free energy is the potential which allow one to determine the thermodynamical behaviour of a small part of a big system in terms of the temperature of the big system. This is exactly the situation which one is facing when considering stochastic damping. The small system is the collective mode on which P.U. and feed back operate. The "big system" must be big enough to allow the definition of a temperature.

The mean energy $\langle E \rangle$ of the small system in a heat bath T is given by

$$\langle \varepsilon \rangle = T^2 \frac{\partial}{\partial T} \frac{F}{T} \tag{14}$$

with $\log F = kT (\log T - \eta kT)$ one gets

$$\langle \varepsilon \rangle = kT (1 - \eta kT) = \theta (1 - \eta \theta) \tag{15}$$

for the mean energy/degree of freedom, i.e. - e.g. for the mean energy of the collective mode in equilibrium with the rest of the beam.

For S.D. $\langle \varepsilon^2 \rangle$, the average of the square of the energy is of some interest. Using $\langle \varepsilon^2 \rangle = Z''/Z$ (just as for ε one has $\langle \varepsilon \rangle = Z'/Z$ and neglecting terms $O(q^2)$ one finds - $Z'' = d^2Z/d\beta^2$

$$\langle \varepsilon^2 \rangle = 2 (\theta^2 - 2\eta\theta^3) \cong 2\theta^2 (1 - \eta\theta)^2 \tag{15a}$$

and therefore for the fluctuation

$$\langle \delta\varepsilon^2 \rangle = \langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2 = \theta^2 (1 - \eta\theta)^2 . \tag{15b}$$

With this we have exhausted the assumptions for the application of linear theory to S.D.

Transport theory.

In this section transport theory is applied to the interplay between the N oscillators and the collective mode. As a first step feed back and an "ohmic" P.U. are not considered, so that the whole system, whose energy is E, is closed: $\dot{E} = 0$. E has N degrees of freedom (2N in the counting of the law of equipartion).

The energy of the whole can be divided in its parts

$$E = R + \varepsilon \tag{16}$$

in which ε is the energy of the collective mode and R the energy of the rest with its N - 1 degrees of freedom. The transport (of energy) equations are in this case:

$$\dot{R} + \frac{\lambda}{N-1} R - \lambda\varepsilon = 0 , \qquad \dot{\varepsilon} - \frac{\lambda}{N-1} R + \lambda\varepsilon = 0 \tag{17}$$

in which λ is a constant of dimensions sec^{-1}, the decay constant of Landau-damping.

Adding the two equations (17) one finds $(R+\varepsilon)^{\bullet} = \dot{E}$, the total energy of the system is a constant. In the stationary case one has $\dot{R} = \dot{\varepsilon} = 0$ and $\therefore$ $R_{st} = (N-1)\varepsilon_{st}$ from either equation (17). This gives $E = N\varepsilon_{st}$, which just expresses the fact that the whole system has N, and ε one degree of freedom.

The problem is that of determining λ for which some preliminaries have yet to be prepared. We define the amplitude A(t) of the collective mode as

$$A(t) = \frac{1}{\sqrt{N}} \sum_{s=1}^{N} a_s e^{-i\omega_s t} \tag{18}$$

in which, as we have seen ω_s is itself a function of $|a_s|^2$. Indeed, one may write $\omega_s = \omega_o + \delta\omega_s$ corresponding to equation (2): $\delta\omega_s = \eta\omega_o^2 |a_s|^2 = \eta\omega_o^2 \frac{J}{2\pi} = \eta\omega_o H_o$ with $H_o = \nu_o J$.

The timedependence of $|A(t)|^2$, which will replace the action integral for the collective mode (tho' this integral does not exist, the motion of A not being periodical). One has

$$|A(t)|^2 = \frac{1}{N} \sum_{r,s} a_r^* a_s e^{i(\omega_r - \omega_s)t} = \frac{1}{N} \sum_{r,s} a_r^* a_s e^{i(\delta\omega_r - \delta\omega_s)t} \tag{19}$$

It is seen that $|A(t)|^2$ consists of two parts. One of them which corresponds to the diagonal in the double sum (19) and one which is formed from the non-diagonal elements. The 1st is constant and ∴ equal to its average value the 2nd is time-dependent and responsible for the fluctuations of $|A^2(t)|$. For the 1st one has :

$$(|A(t)|^2)_1 = \langle |A(t)|^2 \rangle = \frac{1}{N} \sum_{s=1}^{N} |a_s|^2 = \langle |a_s|^2 \rangle$$

where the 2nd parenthesis $\langle \ldots \rangle$ indicates the ensemble average (the 1st is the time average). In this way one finds, using eqs. (4) and (14)

$$\langle |A(t)|^2 \rangle = \frac{\theta}{\omega_o} \left(1 - \frac{3}{2}\eta\theta\right) . \tag{20}$$

The same results is obtained for $\langle H_o \rangle$: indeed, it follows from the previous considerations that $\langle H_o \rangle = \omega_o \langle |A(t)|^2 \rangle$. The difference $\langle H \rangle - \langle H_o \rangle = +(\eta/2)\theta^2$ is explained by equation (4). The variable part of $|A|^2$, which shall be called V(t) (it induces the alternating voltage on the P. U.) is given by

$$V(t) = \frac{1}{N} \sum_{r \neq s} a_r^{*} a_s \, e^{i(\delta\omega_r - \delta\omega_s)t} \tag{19a}$$

(comp. (19)). One has

$$\langle V(t) \rangle = 0 . \tag{19c}$$

in the time average. One sees that:

$$\dot{V}(t) = \frac{i}{N} \sum_{r \neq s} (\delta\omega_r - \delta\omega_s)\, a_r^{*} a_s \, e^{i(\delta\omega_r - \delta\omega_s)t} .$$

Also the ave of $\dot{V}(t) = 0$. To obtain something that is not we form $\langle |V(t)|^2 \rangle$ as well as $\langle |\dot{V}(t)|^2$. In the 1st case this gives:

$$\langle |V(t)|^2 \rangle = \frac{1}{N^2} \sum_{rs,\, r's'} \langle a_{s'}^{*} a_{r'} a_r^{*} a_s \rangle \, e^{i(\delta\omega_r - \delta\omega_{r'} - \delta\omega_s + \delta\omega_{s'})t} .$$

The constant part i. e. the of $V(t)^2$ is given by the terms in which $0 = \delta\omega_r - \delta\omega_{r'} - \delta\omega_s + \delta\omega_{s'}$. We assume that there exist no rational selection between the $\delta\omega_r$ and we have already implicitely assumed in writing (2) that all $\delta\omega_r > 0$ for $\eta > 0$ and < 0 for η smaller than zero. The vanishing of the exponential therefore leads to a factor $\delta_{ss'}\delta_{rr'}$ in the sum used for forming $\langle |V(t)|^2 \rangle$ and ∴ $\langle |V(t)|^2 \rangle =$ $\frac{1}{N} \sum_{r,s} |a_s|^2 |a_r|^2$. Again this sum can be divided in a diagonal and a non diagonal part. The diagonal part is $D = \frac{1}{N^2} \sum_r |a_r|^4$ $= \frac{1}{N} \langle |a|^4 \rangle$. For the N. D. part : $N = \frac{1}{N^2} \sum_{r \neq s} \langle |a_r|^2 \rangle \langle |a_s|^2 \rangle =$

$= \frac{N-1}{N} \langle |a_r|^2 \rangle^2$. Summing this expression to D and neglecting the terms due to the postulated magnetic non-linearity one gets

$$\langle |V(t)|^2 \rangle = \langle |A(t)|^4 \rangle = \frac{(N+1)}{N} \frac{\theta^2}{\omega_o^2} \tag{21}$$

Notice that from (20) and (21) it follows that

$$\langle (\delta |A(t)|^2)^2 \rangle = \langle |A(t)|^4 \rangle - \langle |A(t)|^2 \rangle^2 = \frac{\theta^2}{\omega_o^2} \frac{1}{N} \tag{21a}$$

The same argument can be applied to $\langle |\dot{V}(t)|^2 \rangle$; only the diagonal elements in the quadrupole sum will be zero in this case. It follows that

$$\langle |\dot{V}(t)|^2 \rangle = \frac{1}{N^2} \sum_{r \neq s} \langle (\delta\omega_r - \delta\omega_s)^2 |a_r|^2 |a_s|^2 \rangle =$$

$$= \frac{1}{N^2} \sum_{r \neq s} \langle (\delta\omega_r^2 + \delta\omega_s^2 - 2\,\delta\omega_r\,\delta\omega_s) |a_r|^2 |a_s|^2 \rangle =$$

$$= \frac{N-1}{N^2} 2(\langle \delta\omega_r^2 |a|^2 \rangle \langle |a|^2 \rangle - 1 \langle \delta\omega |a|^2 \rangle^2) =$$

$$= \frac{2}{\omega_o} \frac{N-1}{N} (\langle \delta\omega^2 H_o \rangle \langle H_o \rangle - \langle \delta\omega H_o \rangle^2) .$$

Now, with (2) one has

$$\delta\omega = \eta\omega_o H_o \quad \text{and}$$

$$\langle |\dot{V}(t)|^2 \rangle = 2 \frac{N-1}{N} \eta^2 (\langle H_o^3 \rangle \langle H_o \rangle - \langle H_o^2 \rangle^2) .$$

Using $\langle H_o^n \rangle = n! \langle H_o \rangle^n$ - (Euler's integral it follows that

$\langle |\dot{V}(t)|^2 \rangle = \frac{N-1}{N} \eta^2 \times 2(6-4) \langle H_o \rangle^4$ and ∴ finally and with $\langle H_o \rangle = \theta$:

$$\langle |\dot{V}(t)|^2 \rangle \cong 4\eta^2 \theta^4 . \tag{22}$$

The Landau damping constant λ (sec^{-1}) is now given by

$$\langle |\dot{V}(t)|^2\rangle = \lambda^2 \langle |V(t)|^2\rangle ; \qquad \lambda^2 = \langle |\dot{V}(t)|^2\rangle / \langle |V(t)|^2\rangle =$$

$$= 4\eta^2 \theta^2 \omega_o^2 \quad \text{and}$$

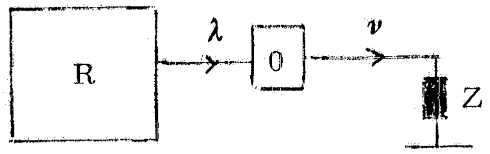
$$\lambda = 2|\eta|\theta\, \omega_o \tag{23}$$

The importance of this result is the fact that λ decreases with decreasing temperature. It follows that stochastic cooling does not follow an exponential law ($\theta = \theta_o e^{-\lambda t}$) but rather $\theta = \theta_o(1 + 2|\eta|\theta_o \omega_o t)^{-1}$.
The profound difference between the Newtonian exponential and the inverse power law is shown in the Table which gives θ/θ_o as a function of time.

$x = 2\|\eta\|\theta_o\omega_o t$ (or λt)	e^{-x}	$(1+x)^{-1}$
0	1.0	1.0
0.5	0.6065	0.6667
1.0	0.3679	0.5
1.5	0.2231	0.4
2.0	0.1353	0.3333
2.5	0.0821	0.2857
3.0	0.0498	0.25

After three time unit (x = 3) the inv. power has only cooled the system to a temperature which is more than 5 times larger than would follow from an exponential law.

At this point we go back to the transport problem, assuming the following scheme.



ν stands for P.U and feed back. It is a damping constant like $\lambda(\text{sec}^{-1})$.
Z = 0 and ohmic or perfect feed back corresponds to $\nu \to +\infty$.

The conclusion, i.e. the validity of an inverse power-law for stochastic cooling is reached by writing down the transport equations (17) for the case $\nu \neq 0$. They are

$$\dot{R} + \frac{\lambda}{N-1} R - \lambda \varepsilon = 0 , \qquad \dot{\varepsilon} - \frac{\lambda}{N-1} R + (\lambda + \nu) \varepsilon = 0 . \qquad (17a)$$

This set of linear equations is solved by finding the roots ϱ_1 and ϱ_2 of the secular equation

$$0 = \det \begin{pmatrix} \frac{\lambda}{N-1} - \varrho & -\lambda \\ -\frac{\lambda}{N-1} & \lambda + \nu - \varrho \end{pmatrix} = \varrho^2 - \left(\frac{\lambda N}{N-1} + \nu \right) \varrho + \frac{\lambda \nu}{N-1}$$

i.e.

$$\varrho_{1,2} = \frac{1}{2} \left(\nu + \frac{\lambda N}{N-1} \right) \pm D \qquad (24)$$

where the discriminat D is :

$$D = \frac{1}{2} \sqrt{\left(\frac{\lambda N}{N-1} + \nu \right)^2 - \frac{4 \lambda \nu}{N-1}} . \qquad (24a)$$

We are particularly interested in the case $\nu >> \lambda$ which corresponds to a perfect feed back.

In this case one has finally

$$\varrho_1 = \nu , \qquad \varrho_2 = \frac{\lambda}{N-1} \qquad (25)$$

We recognize in $\varrho_1 = \nu$ the past transient, with which the collective mode adjust itself to compromise between the requirements of the outside intervention and the "inside" supply of energy furnished by R. $\varrho_2 = \lambda / N - 1 \sim \lambda / N$ is the cooling mode. The conclusion of an inverse power law anticipated in the previous paragraph is ∴ confirmed.