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M. Guidetti, G. Puddu and P. Quarati: NON-SPHERICAL CORRECTIONS TO THE MULTIPOLE POLARIZABILITY OF NUCLEI IN MUONIC ATOMS.

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NON-SPHERICAL CORRECTIONS TO THE MULTIPOLE POLARIZABILITY OF NUCLEI IN MUONIC ATOMS(*).

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SUMMARY.

We give the expressions of the different contributions to the muonic energy-level shifts $\Delta E_\mu(E1)$ in muonic atoms, due to the E1 multipole part of the muon-nucleus interaction. The shifts can be expressed in terms of spherical and non-spherical, one-body and two-body matrix elements. In the case of closed-shells nuclei, $\Delta E_\mu(E1)$ is proportional to the double-energy, weighted sum rule $\sigma_{-2}(E1)$ (spherical contribution). In the case of a nucleus composed of closed-shells plus one nucleon, the non-spherical contributions depend on the orientation of the muonic orbit respect to the orbit of the odd nucleon. We have not found cancellation between two-body and one-body non-spherical matrix elements. The most important correction to the term proportional to the σ_{-2} sum rule comes from the non-spherical, one-body matrix element.

1. - INTRODUCTION.

Negative muons can be used as probes for investigating the properties of atomic nuclei(1-4).

Nuclear quadrupole and hexadecupole moments can be determined with great precision, to less than one percent accuracy, from the hyperfine structure of intermediate orbits in muonic atoms(5-9).

Nuclear charge distributions, quadrupole moments, isotope shifts, isomer shifts and polarizability effects of heavy nuclei have been measured with a high degree of precision and analyzed very recently(10-13). The $2S_{1/2} - 2P_{3/2}$ splitting of the $(\mu^- {}^4He)^+$ muonic ion has been measured with a precision of 0.02 %(14).

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T. E. O. Ericson and J. Hüfner[15], J. Bernabéu and C. Jarlskog[16, 17] and E. M. Henley, F. R. Krejs and L. Wilets[18] have given extensive contributions to the study of the structure of light nuclei from muonic atoms using the closure approximation for muon and nuclear states.

More recently :

a) P. Vogel and V. R. Akylas[19] have generalized Chen's results ; by using a rotational model, they show that, with certain assumptions, the nuclear polarization correction may be expressed as a shift of all hyperfine components plus a renormalization of the even multipole hyperfine constant ;

b) J. Martorell and F. Scheck[20] have shown that the electric dipole polarizability gives rise to a cantre-of-gravity of the quadrupole multiplet and a shift of the quadrupole hyperfine constant.

The last term is shown to be zero in almost all the cases due to strong cancellation between some one-body and two-body matrix elements. However, looking at the Chen's results, the two-body terms, with a negative sign from exchange, cancel only some of the contributions of one-body terms. Therefore it seems to us that this point has to be clarified.

We wish to analyse if terms of non-spherical symmetry enter into the muonic energy levels shifts, in the electric dipole polarizability case and in the other electric multipoles cases. For this purpose, in this work, we are interested to nuclear polarizability corrections to the muon energy-level shifts. Usually one subdivides this correction into multipole contributions which are one-body and two-body matrix elements of the spherical and non-spherical parts of the muon-nucleus interaction.

First of all we give analytical expressions of the muon energy level shifts for all the multipoles. Expressions for the nuclear E1-polarizability with $1 \neq 0$ by means of sum rules techniques can be easily obtained. We limit ourselves to consider light and medium-light muonic atoms, so we develop a formalism valid in this case. Secondly, we verify Martorell-Scheck results analyzing, in detail, for some particular cases, the spherical and the non-spherical, one and two-body contributions.

We consider nuclei with closed shells and with closed shells plus one nucleon. As more extensively discussed in Section 2, we use a ls-coupling scheme for the muonic and the nuclear wave functions. It is straightforward to generalize our results to more-populated unfilled-shell nuclei. In Section 3 the energy-level shift are given. In Section 4 the properties of the angular factors are studied. In Section 5 we use harmonic oscillator sum rules to give some examples. At the end, in Section 6, the conclusions are reported.

2. - SCHEMES FOR THE MUON AND THE NUCLEUS WAVE FUNCTIONS.

In the case of intermediate hydrogen-like muon orbits, penetration effects are very small and one is allowed to treat separately muonic matrix elements from nuclear matrix elements. For sufficiently large muon orbits the interaction outside the nucleus can predominate and the interaction is expressed in terms of the conventional point-multipoles.

We assume for the nucleus a shell model with spherical symmetric potential and a harmonic oscillator basis. Therefore we can use harmonic oscillator sum rules, instead of the Migdal σ_{-2} sum rules, whose validity is not good for light and medium-light nuclei[21]. We describe the i-th nucleon state by means of the quantum numbers n_i, L_i, M_i, S_i, M_{S_i} (i. e. we adopt a LS-coupling).

In this scheme, if we sum to the nuclear hamiltonian

$$H_o = \sum_i \frac{1}{2} \hbar \omega_o (-\nabla_i^2 + r_i^2) \tag{1}$$

the one-body deformed hamiltonian

$$H_d = \sum_i \hbar \omega_o(0) \left[-a \frac{4}{3} \frac{\pi}{5} r_i^2 Y_{20} - 2 b \vec{L}_i \vec{S}_i - c L_i^2 \right] \tag{2}$$

(where a, b and c are parameters defined, for example, in ref. (22)), it is straigthforward to study deformed nuclei, because H_d can be treated as a perturbation.

This case will be studied in a second paper, together with the $l = 0$ polarizability, making use of perturbative sum rules techniques, as introduced by Delsanto and Quarati[23], and with extensive numerical calculations.

In the papers on muonic atoms mentioned up to now the j-j coupling is commonly used to describe the muon, and the h.f.s. is described by means of the quantum numbers F and M_F (total muon-nucleus spin and its third component).

Let us, instead, assume for the muon a ls-coupling. Therefore the decoupled muonic states are described by n_o, l_o, m_o, s_o, m_{s_o}. This assumption is equivalent to a "strong field approximation" for the muon or to a muonic Paschen-Back effect of h.f.s. where the external magnetic field is substituted with the muon-nucleus interaction. In the strong field approximation the spin-orbit interaction is weak and can be treated perturbatively. We assume that the spin-orbit interaction for the muon is weak enough so that the muonic states differ very little and as much as we like from the degenerate $|n_o, l_o, m_o, s_o, m_{s_o}\rangle$ states. In other words we take the spin-orbit interaction to be small enough to be negligeable to our order of approximation, but, at same time, large enough to remove degeneracy of the energy levels.

As it will be seen in the next Section, we have to use second order perturbation theory to calculate energy-level shifts.

It is easy to show that the muonic energy-level shifts $\Delta E_{n_o l_o m_o}$ are proportional to the double energy weighted photonuclear sum rule σ_{-2} (spherical contribution (s.)), when we disregard the total spin orientation of the muon or of the nucleus. Anyway the proportionality is always verified for closed-shells nuclei. Otherwise further terms contribute to $\Delta E_{n_o l_o m_o}$ expression (non-spherical contributions (n.s.)). If we limit ourselves, for simplicity, to the case of a nucleus with complete shells plus one odd nucleon we see that these terms depend on the orientation of the muon orbit respect to the odd nucleon orbit. More precisely, in this case, the level shifts can be identified by the quantum numbers n_o, l_o, m_o of the muon and L", M" of the odd nucleon.

In fact, if one disregards the orientation and sums and averages, there is no further contribution to the one proportional to σ_{-2}, except for a small contribution from two-body exchange matrix elements due to (closed shell) - (closed shell) terms[24], independent on the h.f.s. quantum numbers, which does not contribute to the muonic transitions between two levels.

The decoupled scheme assumed in this work for the muon-nucleus system describes a physical situation difficult to be obtained, at present, because the magnetic fields actually available have a magnitude lower than the one needed to decouple the system. However this scheme can be used in problems where the nuclei are polarized. In fact if we have, as an example, target nuclei polarized in the same direction of the spin of the muon beam, in practical calculations the muon-nucleus decoupled scheme can be used (Applications to nuclear physics problems will be developed elsewhere).

Concluding we have found :

a) in the non closed-shell nuclei the (n.s.) correction to the σ_{-2} term is quite important and not negligeable ;
b) the interaction hamiltonian between muon and nucleons gives rise to an effect which depends on the orientation of the muon orbit respect to the odd nucleon orbit. The sum over m_o or M is equivalent to sum and average over h.f.s. ;
c) our results are somehow in disagreement with Martorell-Scheck results in the sense that they found a complete cancellation, in a coupled scheme, between the one-body non-spherical contribution and the two-body non-spherical contribution.

3. - THE ENERGY-LEVEL SHIFTS.

The muon-nucleus interaction can be written as :

$$u = \sum_{z=1}^{Z} \frac{e^2}{|\bar{r}_z - \bar{r}_\mu|} = \sum_{l=0}^{\infty} u_l \qquad (3)$$

$$u_1 = - \sum_{z=1}^{Z} \sum_{m=-1}^{1} \frac{4\pi e^2}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\hat{r}_z) Y_{lm}^*(\hat{r}_\mu) \tag{4}$$

where $r_>$ ($r_<$) is the greater or lesser of $\bar{r}_z$ or $\bar{r}_\mu$, the proton and the muon coordinates respectively.

On calling u_o the spherically symmetric part of (3), the interaction hamiltonian is $(u - u_o)$. We assume that u_o contains all terms with $l = 0$, so that the interaction hamiltonian is :

$$U = - \sum_{z=1}^{Z} \sum_{l=1}^{\infty} \sum_{m=-1}^{1} \frac{4\pi e^2}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\hat{r}_z) Y_{lm}^*(\hat{r}_\mu) . \tag{5}$$

The muon-nucleus interaction does not involve the spin of the muon nor of the nucleons composing the nucleus.

We limit ourselves to the study of nuclear l-pole polarizability due to the terms u_l with $l \neq 0$.

Since the unperturbed hamiltonian H_o of the muonic atom does not couple muon and nuclear coordinates, the unperturbed solutions are direct product of the type $|\text{muon}\rangle / |\text{nucleus}\rangle$.

Let us limit ourselves to the more common cases having the diagonal first-order matrix elements of U zero. Therefore we are interested in second order correction to the muonic energy level γ :

$$\Delta E_\gamma = - \sum_\beta{}' \frac{|\langle \beta | U | \gamma \rangle|^2}{E_\beta^{(o)} - E_\gamma^{(o)}} \tag{6}$$

where $|\beta\rangle$, $|\gamma\rangle$ and $E_\beta^{(o)}$, $E_\gamma^{(o)}$ are eigenstates and eigenvalues of the muon-nucleus unperturbed system described by hamiltonian H_o. We assume that the atomic energy levels differences are much less than the nuclear ones.

Let us further assume for the ground state of the nucleus a Slater determinant with harmonic oscillators for single particle wave functions with quantum numbers n, L, M, M_S as explained in the Section 2. For details of the simple h.o. model wave function we use here see ref. (6) and (25).

By introducing (4) into (6) and extracting an average nuclear excitation energy $\hat{E}$, as usually is done in sum rules calculation, we have for the l-multipole interaction the following expression :

$$\Delta E_{n_o l_o m_o}(El) = - \frac{e^4}{\hat{E}} \left(\frac{4\pi}{2l+1}\right)^2 (n_o l_o) \left| r_\mu^{-(2l+2)} \right| n_o l_o) \{A + (B - C)\} \tag{7}$$

In eq. (7) the first matrix element contains the muon coordinate. If we make a non-relativistic reduction for the muon, separating spin from orbital angular momentum, the Coulomb interaction does not affect the muon spin. For explicit expression of the muon wave function we use here and its radial part see the paper of Cole of ref. (6), eq. (5. 3).

In eq. (7) A is the r^{2l} average value term :

$$A = \sum_{\substack{\text{complete shells} \\ \text{incomplete shells}}} \langle R_{nL} | r^{2l} | R_{nL} \rangle \langle LM | S_{[\hat{r}]} | LM \rangle (\text{matrix elem. spin part}) \tag{8}$$

R_{nL} are the h. o. radial functions and $|LM\rangle$ the angular part. The function $S_{[\hat{r}]}(El)$ is the following sum :

$$S_{[\hat{r}]}(El) = \sum_m Y_{lm}(\hat{r}) Y_{lm}^*(\hat{r}) F(l_o, m_o ; lm) \tag{9}$$

The function F is defined as follows :

$$F(l_o, m_o; l, m) = \sum_{\substack{k=0 \\ \text{even}}}^{\left[\min 2l,\, 2l_o\right]} \frac{(-1)^{m+m_o}}{4\pi} (2l_o + 1)(2l+1)(2k+1) \cdot \begin{pmatrix} l & l & k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_o & k & l_o \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l & k \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} l_o & k & l_o \\ -m_o & 0 & m_o \end{pmatrix} \tag{10}$$

More generally the factor F should be $F(l_o, m_o; l, l', m)$, defined as in eq. (12) but with $\sqrt{(2l+1)(2l'+1)}$ in place of $(2l+1)$, $\begin{pmatrix} l & l' & k \\ 0 & 0 & 0 \end{pmatrix}$ in place of $\begin{pmatrix} l & l & k \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} l & l' & k \\ m & -m & 0 \end{pmatrix}$ in place of $\begin{pmatrix} l & l & k \\ m & -m & 0 \end{pmatrix}$; we consider only the case $l = l'$ because interference El-El' effects on nuclear polarizability are neglected here.

It is useful to write F as the sum of the $(k = 0)$ spherically symmetric (s.) term, plus the remainder $[(k = 2 \text{ term}) + (k = 4 \text{ term}) + \cdots]$ which is non-spherically symmetric:

$$F(l_o, m_o; l, m) = \frac{1}{4\pi} + \sum_{\substack{k=2 \\ \text{even}}} \beta_k (-1)^m \begin{pmatrix} l & k & l \\ m & 0 & -m \end{pmatrix} \tag{11}$$

with

$$\beta_k(l_o, m_o) = \frac{(-1)^{m_o}}{4\pi} (2l_o+1)(2l+1)(2k+1) \begin{pmatrix} l & l & k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_o & k & l_o \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_o & k & l_o \\ -m_o & 0 & m_o \end{pmatrix} \tag{12}$$

We have then:

$$S_{[\hat{r}]}(El) = \frac{2l+1}{(4\pi)^2} + \sum_{\substack{k=2 \\ \text{even}}}^{\left[\min 2l,\, 2l_o\right]} \frac{\beta_k \sqrt{2k+1}}{\sqrt{4\pi}} \begin{pmatrix} l & k & l \\ 0 & 0 & 0 \end{pmatrix} Y_{ko}(\hat{r}) . \tag{13}$$

The first term $(k = 0)$ of (13) multiplied by the radial and spin part of (8) represents a spherically symmetric part of the shift, the second $(k \neq 0)$ the non-spherically symmetric part. We have then $k = 0$ matrix elements (s. matrix elements) and $k \neq 0$ (n. s. matrix elements).

The terms B and C can be evaluated by means of the rules on the two-body operators matrix elements. Only the contributions coming from nucleons outside the complete shells enter in the term B (in the case $l \neq 0$).

When $l \neq 0$, nuclei with closed shells have $B = 0$.

Let us write B, the direct term as:

$$B = \sum_{\text{complete shells}} \sum_{\text{incomplete shells}} \langle R_{nL} | r^l | R_{nL} \rangle \langle R_{n'L'} | r'^l | R_{n'L'} \rangle \cdot \\ \cdot \langle (LM)(L'M') | S_{[\hat{r}, \hat{r}']}(El) | (LM)(L'M') \rangle \text{ (spin part)} . \tag{14}$$

More attention must be paid to the term C as we shall see in what follows. In fact, for a complete-shells nucleus, the term C is different from zero and depends on the quantum numbers n and L of the shell. However if we subtract one energy level from another this term gives no contribution.

The exchange term C is defined as:

$$C = \sum_{\substack{\text{complete shells}\\ \text{incomplete shells}}} \sum \langle R_{nL} | r^{1} | R_{n'L'} \rangle \langle R_{n'L'} | r'^{1} | R_{nL} \rangle \cdot \tag{15}$$

$$\cdot \langle (LM)(L'M') | S_{[\hat{r},\hat{r}']} | (L'M')(LM) \rangle \text{ (spin part)} .$$

The two-body function $S_{[\hat{r},\hat{r}']}(El)$ which appears in B and C is:

$$S_{[\hat{r},\hat{r}']}(El) = \sum_{m} Y_{lm}(\hat{r}) Y^{*}_{lm}(\hat{r}') F(l_o, m_o; l, m) . \tag{16}$$

Also the two-body function (16) may be decomposed in a (s.) part plus a (n. s.) part. It turns out useful to write A, B and C as the sum of a (s.) part plus a (n. s.) part.

If we indicate with the index (1) the one-body operators and with (2) the two-body operators we have:

$$A + (B - C) = \left[A_{(1)}(s.) + (B_{(2)}(s.) - C_{(2)}(s.)) \right] + \left[A_{(1)}(n.s.) + (B_{(2)}(n.s.) - C_{(2)}(n.s.)) \right] \tag{17}$$

The first addendum of the sum contains only (s.) terms and is proportional to the δ_0 term of Martorell-Scheck and the second to the δA_2 term.

4. - PROPERTIES OF THE GEOMETRICAL ANGULAR FACTORS.

The average value of $S_r(El)$ in the state (LM) is:

$$\langle (LM) | S_{[\hat{r}]}(El) | (LM) \rangle = \frac{2l+1}{(4\pi)^2} + \sum_{\substack{k=2\\ \text{even}}} (-1)^M \beta_k \cdot \tag{18}$$

$$\cdot \frac{(2k+1)(2L+1)}{(4\pi)} \begin{pmatrix} 1 & k & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & k & L \\ -M & 0 & M \end{pmatrix} \begin{pmatrix} L & k & L \\ 0 & 0 & 0 \end{pmatrix} .$$

For instance in the case in which we only have the two terms k = 0 and k = 2 we have explicitely:

$$\langle (LM) | S_{[\hat{r}]}(El) | (LM) \rangle = \frac{2l+1}{(4\pi)^2} \left(1 + \frac{25 \cdot l(l+1) \left[3m_o^2 - l_o(l_o+1) \right] \left[3M^2 - L(L+1) \right]}{(2l+1)(2l-1)(2l+3)(2l_o-1)(2l_o+3)(2L-1)(2L+3)} \right) \tag{19}$$

By using the property that

$$\sum_{M} (-1)^M \begin{pmatrix} L & k & L \\ M & 0 & -M \end{pmatrix} = 0 \qquad \text{if } k \neq 0 \tag{20}$$

it is easy to show that the geometrical functions F and S have the following properties:

$$\frac{1}{2l_o + 1} \sum_{m_o} F(l_o, m_o; l, m) = \frac{1}{4\pi} \tag{21}$$

$$\frac{1}{2l_o + 1} \sum_{m_o} S_{[\hat{r}]}(El) = \frac{2l + 1}{(4\pi)^2} \tag{22}$$

(i.e. only the term with $k = 0$ gives a non zero contribution to the sum).

Let us now calculate the average values of $S_{[\hat{r}, \hat{r}']}(El)$.

The angular part of the direct term B gives :

$$\mathcal{D}_{[\hat{r}, \hat{r}']} = \langle (LM)(L'M') | S_{[\hat{r}, \hat{r}']}(El) | (LM)(L'M') \rangle = (-1)^{M+M'} \frac{2l+1}{4} (2L+1)(2L'+1) \cdot \begin{pmatrix} L & 1 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & 1 & L' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & 1 & L \\ M & 0 & -M \end{pmatrix} \begin{pmatrix} L' & 1 & L' \\ M' & 0 & -M' \end{pmatrix} F(l_o, m_o; l, 0) \tag{23}$$

To have a contribution different from zero this term l must be even, and one needs at least two nucleons outside closed shells. A sum over M and/or M' of $\mathcal{D}$ gives zero if $l \neq 0$:

$$\sum_{M \text{ and/or } M'} \mathcal{D}_{[\hat{r}, \hat{r}']} = 0 \tag{24}$$

The angular part of the exchange term C defined in eq. (5) is :

$$\mathcal{E}_{[\hat{r}, \hat{r}']} = \langle (LM)(L'M') | S_{[\hat{r}, \hat{r}']}(El) | (L'M')(LM) \rangle = \sum_m (-1)^{M+M'+m} \delta(m_s, m_{s'}) \cdot \frac{2l+1}{4\pi} (2L+1)(2L'+1) \begin{pmatrix} L' & 1 & L \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} L' & 1 & L \\ -M' & m & M \end{pmatrix}^2 \left[\frac{1}{4\pi} + \sum_{k=2} \beta_k (-1)^m \begin{pmatrix} 1 & k & 1 \\ m & 0 & -m \end{pmatrix} \right] \tag{25}$$

$L'+l+L$ must be even to have a result different from zero. In order to calculate C we have to sum $\mathcal{E}_{[\hat{r}, \hat{r}']}$ over complete and incomplete shells. Therefore we have to consider the following three contributions to the sum.

i) In case where the levels L and L' are complete we have to sum the angular matrix elements $\mathcal{E}_{[\hat{r}, \hat{r}']}$ over M and M'. This sum is different from zero, is spherically symmetric and is given by :

$$\sum_{M, M'} \mathcal{E}_{[\hat{r}, \hat{r}']}(El) = \frac{1}{(4\pi)^2} (2L+1)(2l+1)(2L'+1) \begin{pmatrix} L' & 1 & L \\ 0 & 0 & 0 \end{pmatrix}^2 \tag{26}$$

This result has to be used in case of closed-shells nuclei with, at least, two shells completed.

ii) If we have one nucleon only, described by the quantum numbers n'', L'', M'', M''_S, in an orbit outside the closed shell (n, L) we have :

$$\sum_{\substack{\text{over the complete} \\ \text{shell (nL)}}} \mathcal{E}_{[\hat{r}, \hat{r}'']} = \Gamma_o \sum_{m=-1}^{1} (-1)^m \begin{pmatrix} L'' & 1 & L \\ -M'' & m & M''-m \end{pmatrix}^2 + \sum_{\substack{k=2 \\ \text{even}}} \Gamma_k \sum_{m=-1}^{1} \cdot \begin{pmatrix} 1 & k & 1 \\ -m & 0 & m \end{pmatrix} \begin{pmatrix} L'' & 1 & L \\ -M'' & m & M''-m \end{pmatrix}^2 \tag{27}$$

where
$$\Gamma_o = \frac{1}{4\pi}\ \gamma(L, 1, L''),\qquad \Gamma_k = \beta_k \gamma(L, 1, L'') \qquad (k=2,4,\ldots)$$

and
$$\gamma(L, 1, L'') = \frac{1}{4\pi}(2L+1)(2l+1)(2L''+1)\begin{pmatrix} L'' & 1 & L \\ 0 & 0 & 0 \end{pmatrix}^2 \tag{28}$$

iii) Finally, in order that $\sum_{\text{incomplete shells}} \mathcal{E}_{[\hat{r},\hat{r}']}$ be different from zero we need at least two particles in the incomplete shell with different orbits. In this case we have to use eq. (25). C is generally the sum of the three terms considered.

Clearly, in the uncoupled basis we have adopted, to average matrix elements over nuclear and muon orientation is equivalent to average, in the coupled system, over hyperfine structures, described by the quantum numbers F and M_F.

Summarizing we have that :

a) the spherically symmetric, $k=0$ matrix elements are proportional to the σ_{-2} sum rule which will be introduced in the next section ;
b) the direct term contributes only for $l=0$ and l even, thus the direct term is zero in the electric dipole polarizability shift ;
c) the direct term gives also zero in the case of closed-shell nuclei or, in case of closed-shell nuclei plus one odd nucleon, if we disregard the orientation of the odd nucleon that it means to sum and average over M'' ; to average over possible orientations of the odd nucleon means to ignore the possibility of hyperfine structure ;
d) the contribution of the exchange term is non zero if $L'+l+L$ is even ; if it is so, in case of closed-shell nuclei, the exchange term contributes with its spherically symmetric, $k=0$ part ;
e) in nuclei having not complete shells the exchange term is composed of (s.) and (n. s.) parts ; to have a contribution from incomplete shells to B and C matrix elements one needs at least two nucleons outside complete shells with different orbitals ;
f) if we disregard the orientation of the muon and sum and average over m_o we have non zero contribution only from (s.) terms ;
g) the spherically symmetric part of the energy level shifts depends only on the quantum numbers l_o, m_o of the muon and it does not depend on the odd nucleon numbers.

The non-spherical part determines the hfs. The classification of the hfs levels is made here by means of the quantum numbers (l_o, m_o ; L'', M''), so that the effect of the relative orientation between muonic and odd nucleon orbits easily appears.

5. - THE USE OF SUM RULES AND EXAMPLES.

The spherical contribution to the shift, expressed in eq. (17) can be written by means of the harmonic oscillator sum rule $\sigma_{-2}^{h.o.}$. When the ground state wave function can be separated in an angular and a radial part we have for a nucleus of a given observed spin (otherwise we have to consider the factor $1/(2J+1)$, where J is the spin of the nucleus) that :

$$\sigma_{-2}^{h.o.}(El) = 2\pi^2 e^2 \frac{\hat{E}^{2l-3}_{MeV}(El)}{(197.26)^{2l-1}} \frac{(l+1)(2l+1)}{l\left[(2l+1)!!\right]^2} \cdot$$
$$\cdot \left\{ A(s.)\frac{(4\pi)^2}{2l+1} + \left[B(s.)(An.f.)_B^{-1} - C(s.)(An.f.)_C^{-1} \right] \right\} \frac{f^2}{MeV} \tag{29}$$

($e^2 = 1.44$ in f MeV, $\hat{E}(El)$ in MeV, and $(An.f.)_{B,C}$ are the expressions of the angular factors of equations (23) and (25) respectively).

In case of closed-shell nuclei, direct and exchange contributions are zero, the energy shift becomes :

$$\Delta E_{n_o l_o}(El) = -\frac{e^2}{2\pi^2}\frac{(2l-1)!!\ ^2 l}{(l+1)}\frac{(197.33)^{2l-1}}{\hat{E}^{2l-2}}\langle r_\mu^{-(2l+2)}\rangle_{n_o l_o}\ \sigma_{-2}^{h.o.}(El) \qquad (30)$$

and for the dipole electric case, for instance, of ${}^4He$:

$$\Delta E_{n_o l_o}(E1) = -\frac{e^2}{4\pi^2}\langle r_\mu^{-4}\rangle_{n_o l_o}\ 197.33\ \sigma_{-2}^{h.o.}(E1) = \frac{e^2}{2}\langle r^{-4}\rangle_{n_o l_o}\ 197.33\ \alpha^{(E1)} \qquad (31)$$

where $\alpha^{(E1)}$ is the electric dipole polarizability

$$\alpha^{(E1)} = \frac{1}{2\pi^2}\ \sigma_{-2}^{h.o.}(E1)\ .$$

Let us now examine the case of a nucleus with some complete shells, indicated by $(\mathcal{N}\ \mathcal{L})$, plus one odd nucleon (described by the quantum numbers L", M", ...) outside the last closed shell.

The general expression for the energy-level shifts $\Delta E_{n_o l_o m_o}(El;L",M")$ is given by the expression (7) where :

$$A = A(s.) + A(n.s.)$$

$$A(s.) = \frac{2l+1}{(4\pi)^2}\left\{\sum_{\substack{\text{complete}\\ \text{shells}(\mathcal{N}\mathcal{L})}} 2(2\mathcal{L}+1)\langle\mathcal{N}\mathcal{L}|r^{2l}|\mathcal{N}\mathcal{L}\rangle + \langle n"L"|r^{2l}|n"L"\rangle\right\} \qquad (32)$$

$$A(n.s.) = \frac{2l+1}{(4\pi)^2}\langle n"L"|r^{2l}|n"L"\rangle \sum_{\substack{k=2\\ \text{even}}} (-1)^{M"}\, 4\pi\,\beta_k\,\frac{1}{2l+1}\,(2k+1)(2L"+1)\ \cdot$$

$$\cdot\begin{pmatrix} l & k & l\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} L" & k & L"\\ -M" & 0 & M"\end{pmatrix}\begin{pmatrix} L" & k & L"\\ 0 & 0 & 0\end{pmatrix}$$

B : we do not report for this case the explicit expression of B because in the case of one nucleon only, outside the closed shells, the value of B is zero, also when l is even and different from zero.

$$C = \sum_{\substack{\text{complete}\\ \text{shells}(\mathcal{N}\mathcal{L})}} |\langle(\mathcal{N}\mathcal{L})|r^l|(n"L")\rangle|^2\left\{\frac{\gamma(\mathcal{L},l,L")}{4\pi}\sum_{m=-l}^{l}(-1)^m\begin{pmatrix} L" & l & \mathcal{L}\\ -M" & m & M"-m\end{pmatrix} +\right.$$

$$\left. + \sum_{\substack{k=2\\ \text{even}}}^{\begin{bmatrix}\min\\ 2l,\ 2l_o\end{bmatrix}} \beta_k(l_o,m_o)\,\gamma(\mathcal{L},l,L")\sum_{m=-l}^{l}\begin{pmatrix} l & k & l\\ -m & 0 & m\end{pmatrix}\begin{pmatrix} L" & l & \mathcal{L}\\ -M" & m & M"-m\end{pmatrix}^2\right\} + \text{constant} \qquad (33)$$

$$\text{constant} = \frac{1}{2}\sum_{\substack{(\mathcal{N}\mathcal{L})\neq(\mathcal{N}\mathcal{L})'\\ \text{all the couples}\\ \text{of the different}\\ \text{closed shells}}}\frac{(2l+1)}{(4\pi)^2}|\langle(\mathcal{N}\mathcal{L})|r^l|(\mathcal{N}\mathcal{L})'\rangle|^2\, 2(2\mathcal{L}+1)(2\mathcal{L}'+1)\begin{pmatrix}\mathcal{L}' & l & \mathcal{L}\\ 0 & 0 & 0\end{pmatrix}^2 \qquad (34)$$

where the first addendum represents the spherical part of the exchange contribution, the second the non-spherical part.

The constant defined in eq. (34) represents the (closed shell)-(closed shell) exchange contribution: it does not depend on n_o, l_o, m_o; therefore, when considering the difference between two muonic levels, this term does not contribute and, in this case, its value may not be taken into account.

If L and L' are the two complete shells forming the nucleus we note that: (L+1+ L") and (L'+1+L") must be event to have the functions $\gamma(\mathcal{L}, L, L")$ different from zero. The contribution from one or the other shell in eq. (33) is different from zero depending on 1 even or odd. If we choose, for example, dipole electric multipole 1 = 1 only the (last filled shell) - (odd nucleon) contribution to C is different from zero.

From the expression of the terms A, B, C we can see that (n. s.), one-body term does not eliminate the (n. s.), two-body contribution, as Martorell-Scheck found.

Let us note, now, that the factor $\dfrac{[3M"^2 - L"(L"+1)]}{[L"(2L"+1)(2L"-1)(L"+1)(2L"+3)]^{1/2}}$ can be written by means of the single-particle electric quadrupole moment $Q_2^{s.p.}$:

$$\frac{[3M"^2 - L"(L"+1)]}{[L"(2L"+1)(2L"-1)(L"+1)(2L"+3)]^{1/2}} = - \frac{Q_2^{s.p.} \, 4\pi[(2L"-1)(2L"+3)]^{1/2}}{e\langle r^2\rangle \sqrt{(2L"+1)^3}\,[L"(L"+1)]^{1/2}}$$

Let us now show some numerical examples. We whish to evaluate the different contribution to the shifts in the electric dipole case. As a first example we consider a nucleus with two filled shells and one odd nucleon having the following quantum numbers : L" = 2, M" = 2 ; the muon is in a state with $l_o = 2$, $m_o = 0$. The number of protons is Z = 9.

We have (we consider only the second order perturbation shift) :

$$\Delta E_{n_o,2,0}(1=1, L"=2, M"=2) = - \frac{e^4}{\hat{E}(E1)\ \mathrm{MeV}} \langle r_\mu^{-4}\rangle_{n_o,2} \frac{1}{9} \beta^{-1}\left[(64.53-4.32)+(1.08-2.16)-0.72\right] \qquad (35)$$

The terms including the factors 64.53 and (1.08) are the spherical contributions of A and C respectively, while (- 4.32) and (- 2.16) the (n. s.) contributions. The term which includes (- 0.72) is the (closed shell)-(closed shell) exchange spherical contribution, which can be considered a constant not depending on the quantum numbers of the muonic levels.

In the eq. (35) β is the harmonic oscillator parameter ($\beta^{-1} = \dfrac{\hbar}{M\omega_o}$ in fermi^2).

The sign of the correction to the "spherically sum rule term" depends on the value of m_o. In fact if we take $m_o = 2$ with M" = 2 the correction will be positive. Sign and magnitude of the one-body, (n. s.) correction depend on the orientation of the orbit of the odd nucleon respect to the muon orbit.

In conclusion in this example there is no cancellation among the (n. s.), one-body and the two-body terms. The most important correction to the term containing σ_{-2} is given by the one-body, (n. s.) term.

This case (Z = 9) may describe the energy-level shifts of the muonic atom $^{19}_{9}F$. Let us complete the numerical expression of eq. (35). As it is well known (except for $l_o = 0$):

$$\langle r_\mu^{-4}\rangle_{n_o,l_o} = a_\mu^{-4} \frac{Z^4 \frac{1}{2}(3n_o^2 - l_o(l_o+1))}{n_o^5(l_o+\frac{3}{2})(l_o+1)(l_o+\frac{1}{2})l_o(l_o-\frac{1}{2})} \qquad (36)$$

where $a_\mu = 260$ f. With $n_o = 3$, $l_o = 2$ we have :

$$\Delta E_{3,2,0}(E1, L''=2, M''=2) = -10.76 \frac{\beta^{-1}}{\hat{E}(E1)} \text{ MeV} \qquad (Z = 9) \qquad (37)$$

(in this case β^{-1} and $\hat{E}$ are h. o. parameters of the nucleus $^{19}_{9}F$).

Other examples for very light nuclei are

Z = 2 (closed shell nucleus) : (using the experimental value $\sigma_{-2}(E1) \approx 0.007 \frac{f^2}{MeV}$)

$$\Delta E_{2,1}(E1) = -0.79 \times 10^{-2} \text{ meV} \qquad (38)$$

Z = 3 :

$$\Delta E_{2,1,0}(E1, L''=1, M''=1) \quad 7.29 \frac{\beta^{-1}}{\hat{E}(E1)} \text{ meV} \qquad (39)$$

We calculate now, just as an example, and without discussing the numerical values obtained, the shift due to the one-body matrix element A of the closed-shell plus one nucleon nucleus $^{41}_{21}Sc$ (we neglect contributions from B and C) :

Z = 21:

$$\Delta E^{A}_{3,1,0}(E1, L''=3, M''=3) \quad (26.04 - 3.78) \frac{\beta^{-1}}{\hat{E}} = 22.26 \frac{\beta^{-1}}{\hat{E}} \text{ eV} \qquad (40)$$

where the first and the second addendum are the (s.), one-body and the (n. s.), one-body contribution respectively.

We calculate now the (s.), one-body contribution to the 3rd energy-level shift of the muonic atom $^{175}_{71}Lu$ (case considered by Martorell-Scheck) :

Z = 71 :

$$\Delta E^{A(s.)}_{3,2}(E1) = 1351.13 \frac{\beta^{-1}}{\hat{E}} \text{ eV} \qquad (41)$$

This last result is close to the δ_0 numerical value of Mertorell-Scheck (30 eV) only if there is another appropriate contribution, positive and different from zero, from B(s.) and C(s.) matrix elements (for the nucleus $^{175}_{71}Lu$ we have : $\beta^{-1} \approx 6.5 \text{ fm}^2$).

We note finally that the two parameters β^{-1} and $\hat{E}$ may be reduced to one parameter only, β, if we assume $\hat{E} = \hbar\omega_0 = 41.54\beta$ MeV. The average energy $\hat{E}$ should be written more exactly as $\hat{E} = 41.54\beta(1+\Delta)$, where Δ represents the correction introduced by the effect of residual interactions in the nuclear hamiltonian.

6. - CONCLUSIONS.

We have derived in this work the energy-level shifts of muon orbits in muonic atoms. We have utilized sum rules techniques as in the Martorell-Scheck work.

Our hypothesis are the following:

a) intermediate hydrogen-like orbits, and spin-orbit decoupled scheme for the muonic wave function;
b) LS-coupling for the nucleus : i. e. shell model with quantum numbers (n, L, M, S, m_S) plus harmonic oscillators basis.

We have obtained expressions of the shifts for all the multipoles of the muon-nucleus interaction.

If we sum and average over the orbital angular momentum of the muon or of the nucleus the shift is proportional to the sum rule σ_{-2}. Otherwise other terms contribute to the energy-level shifts.

We have examinated the case of closed shell nuclei plus one odd nucleon. Our result does not show, as in Martorell-Scheck, cancellation among different (n. s.), one-body and (n. s.) two-body terms. The most important correction to the (s.), one-body contribution is the (n. s.) one-body term.

Finally it seems worth to us :

a) to calculate the polarization shifts of deformed nuclei of light and medium-light muonic atoms by using the perturbative hamiltonian of eq. (2) ;

b) to verify, by means of the muonic energy-level shifts, the cluster structure of medium-light nuclei, by comparing cluster structure and shell model previsions on the energy-level shifts with the experimental data on the X-rays muonic atoms.

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