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E. Etim: FIELD THEORY APPROACH TO THE STATISTICAL BOOTSTRAP

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FIELD THEORY APPROACH TO THE STATISTICAL BOOTSTRAP

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1. INTRODUCTION

There is currently a new trend in particle physics as is evidenced by the rapidly growing interest in the dynamics of extended structures and the mass spectrum they sustain.

It is assumed that these structures can be described by quantum field theory in which an infinite summation over some suitable collective modes takes into account the essential non-perturbative features of the many-body problem. The new approach therefore goes beyond canonical perturbation theory.

The Statistical Bootstrap Model was devised to describe precisely the kind of extended objects (fireballs = hadrons) for which the above field theory description is assumed to apply. From this point of view it is natural to enquire if a consistent field theory

[+] On leave of absence from Laboratori Nazionali Frascati, Italy; Present Address: Z.i.F., University of Bielefeld, Germany underlies the bootstrap idea. This is indeed the case. The bootstrap constraint can infact be respected by a non-local field theory. We therefore propose to generalize the statistical bootstrap model to a bootstrap field theory of which we shall construct here the simplest non-trivial element - the two-point function.

2. HAGEDORN-FRAUTSCHI AND YELLIN REPRESENTATIONS

It was first hinted by Yellin that there is a formal analogy between the statistical bootstrap model and a non-polynomial Lagrangian field theory. Unfortunately for the simplest and most studied non-polynomial Lagrangian

$$\mathcal{L}_{int}(x) = \Lambda e^{i\lambda\varphi_0(x)} \tag{1}$$

where Λ is the major and λ the minor coupling constant and $\varphi_0(x)$ a free scalar field, the spectral function of the "superpropagator"

$$\Delta(p^2) = e^{\lambda^2\Delta_F(p^2)} \tag{2}$$

grows less than exponentially for large $m = \sqrt{p^2}$

$$\mathrm{Im}\left[\Delta(p^2)\right]_{REG} \xrightarrow[p^2\to\infty]{} \exp(a\,m^{2/3}) \tag{3}$$

There is no doubt that one can find a non-polynomial Lagrangian with a spectral function which grows exponentially as required or at worst one that over-shoots it. However quite apart from the inherent complexity of working with non-polynomial Lagrangians, the main difficulty is in the approach itself - the specification of an interaction Lagrangian and consequently an equation of motion. In the bootstrap approach, the bootstrap constraint is the "equation of motion".

Its most familiar formulation is in the feed-back

A HADRON

is a composite of an undetermined
number of all kinds of HADRONS, each
of which in turn

from which it is clear that our business is with the way hadrons are assumed to be constituted. An infinite hierarchy belies this structure. In order to appreciate this point of view it is instructive to examine briefly how one would construct a hadron otherwise.

It is enough for our purposes to compare the phase space density of the composite system with the density of mass states of the constituents.

The density of states of N independent particles in a box of volume V_0 + and total energy E is given by

$$\sigma_N(E, V_0) = \left[\frac{V_0}{(2\pi)^3}\right]^N \int \delta\Big(E - \sum_{i=1}^{N} E_i\Big)\, \delta^{(3)}\Big(\sum_{i=1}^{N} \vec{p}_i\Big) \prod_{i=1}^{N} d^3\vec{p}_i \qquad (4)$$

If the N particles are identical a factor of 1/N! is also required in the RHS.

For simplicity we ignore quantum numbers, charge, spin, isospin, etc.

+ is necessary for "confinement"

(a) For a meson considered as a quark-antiquark pair, $q\,\bar{q}$, N = 2 and

$$\sigma_2(E, V_0) \xrightarrow[E\to\infty]{} E^2 \tag{5a}$$

while the mass spectrum is just a delta function.

$$\rho(m) = \delta(m - m_q) \tag{5b}$$

(b) For a baryon as a quark triplet N = 3 and

$$\sigma_3(E, V_0) \xrightarrow[E]{} E^5 \tag{6a}$$

The mass spectrum on the other hand is a sum of delta functions

$$\rho(m) = \sum_{i=1}^{3} \delta(m - m_i) \tag{6b}$$

(c) For the slightly more interesting case of a meson formed of a gas of an elementary input boson b, $N = \infty$, ie meson $\equiv$ bb, bbb, bbbb,.. Nb, ($N \to \infty$) one finds

$$\sigma_\infty(E, V_0) = \sum_{N=2}^{\infty} \frac{1}{N!}\, \sigma_N(E, V_c)$$

$$\xrightarrow[E\to\infty]{} \exp(aE^{3/4}) \tag{7a}$$

$$\rho(m) = \delta(m - m_b) \tag{7b}$$

From these simple examples it is evident that infinite compositeness is necessary if the density of all states of motion $\sigma(E, V_0$) must grow fast enough to approach an exponential. This condition is however far from sufficient to ensure the bootstrap equality

$$\sigma(E, V_0) = \rho(E = m) \tag{8}$$

as the last example clearly shows. The special strong interaction dynamics responsible for (8) does not therefore rely only on the wealth of elementary constituents available. Secondly eq. (8) cannot be valid for arbitrary volume V_0 . In fact if the volume V_0 is very small the contribution of the N-particle cluster to the LHS of (8) is suppressed by the factor V_0^N as against the linear V_0 dependence of the states contributing to $\rho(m)$ (cf. eq.(4)). If on the other hand V_0 is very large the N-particle clusters dominate overwhelmingly in $\sigma(E, V_0)$ and the equality in (8) becomes impossible. There is thus a critical volume V_0 at which the density of all states of motion of all possible non-interacting N-clusters $\sigma(E, V_0)$ becomes equal to the dynamical density of mass states $\rho(m)$. V_0 is a crucial parameter which sets the mass scale. At such a volume eq. (8) is an extremely revealing and interesting equality between the many intrinsic degrees of freedom of a massive resonance as counted by the mass spectrum and the equally large number of dynamical states of motion which can be generated just by clustering.

Such equality presupposes that no hadron is elementary. Consequently if the input boson in example (c) above is composite one can write symolically

$$b \equiv bb, \quad bbb, \quad bbbb \; .... \; Nb, ... \; (N \rightarrow \infty)$$

Hence if for fixed N (N = 2, 3, 4 ....) one iterates eq. (7a) for each factor b the density of states of motion $\sigma(E, V_0)$ becomes

$$\sigma(E,V_o) = \sum_{N=2}^{\infty} \frac{1}{N!} \left[\frac{V_o}{(2\pi)^3}\right]^N \int \delta(E-\sum_{i=1}^{N} E_i)\,\delta^{(3)}(\sum_{i=1}^{N} \vec{p}_i) \times \prod_{i=1}^{N} \sigma(m_i,V_o)\,dm_i\,d^3\vec{p}_i \tag{9}$$

which on making use of eq.(8) can also be rewritten as

$$\rho(m) = \sum_{N=2}^{\infty} \frac{1}{N!} \left[\frac{V_o}{(2\pi)^3}\right]^N \int \delta(m-\sum_{i=1}^{N} E_i)\,\delta^{(3)}(\sum_{i=1}^{N} \vec{p}_i) \times \prod_{i=1}^{N} \rho(m_i)\,dm_i\,d^3\vec{p}_i \tag{1o}$$

Eq.(1o) is the mathematical formulation of the constitutional assumption of the feed-back. It was first translated into this form by Frautschi. It has been observed that eq.(1o) is a reformulation of Hagedorn's hadronic thermodynamics in the language of the microcanical ensemble. This is essentially correct if one is satisfied with taking Laplace transforms. But from the point of view of Hagedorn's original arguments for the equilibrium of hadronic matter it goes very much beyond thermodynamics. According to Hagedorn equilibrium of hadronic matter, taking place in the short time of about $1o^{-23}$ sec., is not brought about by a large number of collisions between particles, and therefore requires no relaxation time. It is an equilibrium between the enormous number of competing decay channels of an excited hadron. The probability weights of these decay channels are given directly by the S-matrix. There is thus no implication for the application of classical statistical thermodynamics nor the assumption that S-matrix elements should have no symmetry properties and no momentum dependence. Consequently in the approximation in which it is valid eq.(1o) represents not so much the Laplace transform of a partition function but an important information on the constitution of a hadron which the S-matrix must contain. We shall call it the cluster or Hagedorn-Frautschi representation.

Eq.(1o) can be solved exactly for $\rho(m)$. It was first done by Yellin. To exhibit the solution explicitly in covariant form let us make use of the fact that the critical volume V_0 in eqs. (8) and (1o) sets a mass scale in the theory to introduce a lowest mass discrete state in the mass spectrum at $m = m_0$ where for later convenience we write

$$\begin{aligned} V_0 &= \frac{4\pi}{3}(4\pi B)^{3/2} \\ B &= 1/4\pi m_0^2 \\ \delta_0(p^2-m_0^2) &= \theta(p_0)\,\delta(p^2-m_0^2) \end{aligned} \tag{11}$$

Eq. (1o) can now be rewritten in explicitly covariant form as

$$B\rho(p^2) = B\delta_0(p^2-m_0^2)+\sum_{N=2}^{\infty}\frac{1}{N!}\int\delta^{(4)}(p-\sum_{i=1}^{N}q_i)\times \\ \times\prod_{i=1}^{N}B\rho(q_i^2)\,d^4q_i \tag{1o}$$

The presence of the term $\delta_0(p^2-m_0^2)$ in the RHS of eq.(1o) is not strictly faithful to the idea that no hadron is elementary. However from the way it was introduced the "elementarity" of the particle with mass m_0 reflects the length scale to which one probes. It is not overtly inconsistent with the bootstrap philosophy, for what is "elementary" at one length scale may be "composite" when one probes to smaller distances.

Yellin's solution of eq. (1o') is

$$B\rho(p^2) = \sum_{N=1}^{\infty} g_N\, \Omega_N(p^2)\,;\; g_1 = 1 \tag{11}$$

where

$$\Omega_N(p^2) = \int \delta^{(4)}\Big(p - \sum_{i=1}^{N} p_i\Big) \prod_{i=1}^{N} B\delta_o(p_i^2 - m_o^2)\, d^4p_i \tag{12}$$

is the N-body phase space and the g_N are numerical coefficients which can be computed iteratively by substituting (11) into (1o') or more straightforwardly by using their Laplace transforms. We shall refer to eq. (11) as the granular or Yellin representation.

3. BOOTSTRAP FIELD THEORY

In the statistical bootstrap model (SBM) eq.(11) is nothing more than an interesting analytic solution of eq. (1o'). The point of view of bootstrap field theory (BFT) is radically different. According to it eqs.(1o') and (11) provide two different but physically equivalent ways of interpreting the complex structure of a hadron. For this reason it is argued that the two representations should be unitarily equivalent. In other words, if in the granular picture a hadron is described by the field $\varphi(x)$ and in the cluster picture by $\Phi(x)$ then there should exist a unitary transformation $\mathcal{U}$ such that

$$\Phi(x) = \mathcal{U}\, \varphi(x)\, \mathcal{U}^{\dagger} \tag{13}$$

Actually much less will be required; as U will be implemented, the fact that it is an isometry ($U^+U = 1$, $(UU^+)^2 = UU^+$) will suffice. Armed with U the SBM follows as a very special version of bootstrap field theory, obtained by making use of an interesting decomposition of phase space (totally ordered manifold)

$$\Omega_N(p^2) = \delta_{1N}\, B\,\delta_0(p^2 - m_0^2) + \frac{1}{d_N - 1}\sum_{\ell=2}^{N}\ \sum_{n_1+n_2+\cdots+n_\ell=N} \times \int \delta^{(4)}\Big(p - \sum_{i=1}^{\ell} q_i\Big) \prod_{i=1}^{\ell} \Omega_{n_i}(q_i^2)\, d^4 q_i \qquad (14)$$

where d_N is the number of partitions of N.

Eq.(14) is a bootstrap equation, indeed the bootstrap equation. The Frautschi bootstrap expresses the form invariance of (14) under the (order-preserving) defornations

$$\begin{aligned} \Omega_N(p^2) &\longrightarrow \hat{\Omega}_N(p^2) \\ \Omega_{n_i}(q_i^2) &\longrightarrow \hat{\Omega}_{n_i}(q_i^2) \end{aligned} \qquad (15)$$

i.e.

$$\hat{\Omega}_N(p^2) = \delta_{1N} B\,\delta_0(p^2 - m_0^2) + \sum_{\ell=2}^{N}\ \sum_{n_1+n_2+\cdots+n_\ell=N} \gamma_\ell(n_1, n_2 \cdots n_\ell) \times \int \delta^{(4)}\Big(p - \sum_{i=1}^{\ell} q_i\Big) \prod_{i=1}^{\ell} \hat{\Omega}_{n_i}(q_i^2)\, d^4 q_i \qquad (16)$$

in the special case of dilatations

$$\hat{\Omega}_N(p^2) = g_N \Omega_N(p^2)$$
$$\hat{\Omega}_{n_i}(q_i^2) = g_{n_i} \Omega_{n_i}(q_i^2) \tag{15'}$$

The problem represented by eqs.(14)-(16) belongs to Algebraic Topology. The dynamics in BFT is not in eqs.(1o') and (16). These equations are nothing more than an interesting way of expressing a particular normalization condition (quantization). The dynamics of the mass spectrum is completely specified by an abstract operation, called the cluster product, which is defined so as to simulate strong interactions.

3.1 CLUSTER PRODUCT

The three main features of strong interactions we wish to simulate are: -

(1) the state of two free pions is different from that of a ρ -meson and that of three free pions from that of ω. Symbolically

$$\begin{aligned} &\pi\pi \mp \rho \\ &\pi\pi\pi \mp \omega \mp \rho\pi \mp A_1 \end{aligned} \tag{17a}$$

etc.

Introducing formally a bracket symbol e.g.

$$\begin{aligned} \rho &= (\pi\pi) \\ \omega &= (\pi\pi\pi) \\ A_1 &= (\rho\pi) \end{aligned} \tag{17b}$$

etc.

to indicate that ρ is formed from two pions, ω from three, A_1 from ρ, π and so on, we can rewrite eq.(17a) as

$$\pi\pi \mp (\pi\pi)$$

$$\pi\pi\pi \mp (\pi\pi\pi) \mp (\pi\pi)\pi \mp ((\pi\pi)\pi)$$

etc. (17c)

The bracket operation is thus non-associative. Mathematically bracketing is a combinatorial problem, and as such has been known for a long time. Given a sample of size N e.g. the product

$$M_N = \pi_1 \pi_2 \cdots \cdots \pi_N \tag{17d}$$

the number of different ways of putting brackets between various factors in the product is called the bracketing coefficient b_N. The difference between one bracketing problem and another arises from the type of addtional constraints imposed on the formation of brackets e.g. bracketing only two numbers at a time, and that difference is reflected in b_N.

Bracketing is our mechanism for resonance formation starting with a given input. It consists in associating with each monomial M_N a Hilbert space H_N of degenerate states of dimension b_N. Each of the N factors in (17d) carries a basic unit of an additive "quantum number".

(2) The square of hadron masses are quantized i.e.

$$m^2 = m^2(N) \tag{18}$$

(3) The mass spectrum of hadrons grows exponentially; in other words the degeneracy at the mass level $m^2=m^2(N)$ grows exponentially with $\mathfrak{m}(N)$. An important property of the bracketing coefficient is that it too grows exponentially with some algebraic function of N. Hence if the mass spectrum is arranged to be equal to b_N the function (18) is known, at least asymptotically.

Consider then a set[+)]

$$\pi^{\pm}(p_1)\ ,\ \rho^{\pm}(p_2)\ ,\ \omega^{\pm}(p_3),\ \ldots\ldots\ldots \tag{19}$$

of creation (+) and annihilation (-) operators in momentum space. Their (generalized) Wick product is

$$W(\pi^{\pm},\rho^{\pm},\omega^{\pm},\ldots;p)=\int\delta^{(4)}(p-\sum_{i=1}p_i)\pi^{\pm}(p_1)\rho^{\pm}(p_2)\omega^{\pm}(p_3)\cdot \\ \times\prod_{i=1}d^4p_i \tag{20}$$

Introduce an abstract operation $\mathbb{C}$, which transforms Wick products into the creation and annihilation operators of particles.

+) $\pi^{\pm}$, $\rho^{\pm}$ do not stand for the corresponding particles in the Rosenfeld tables.

$$\mathbb{C}: W(\pi^{\pm}, \rho^{\pm}, \omega^{\pm}, \ldots; p) \longrightarrow (\pi^{\pm}, \rho^{\pm}, \omega^{\pm}, \ldots; p)$$

$$(\pi^{+}, \rho^{+}, \omega^{+}, \ldots; p) = (\pi^{-}, \rho^{-}, \omega^{-} \ldots; p)^{\dagger} \qquad (21)$$

and satisfies the following conditions:-

$\mathbb{C}$ 1 . For all single-particle operators e.g. $\pi^{\pm}(p)$, $\rho^{\pm}(p)$. $\omega^{\pm}(p)$etc.

$$(\pi^{\pm}; p) = \pi^{\pm}(p) \qquad (22)$$
$$(\rho^{\pm}; p) = \rho^{\pm}(p), \text{ etc}$$

Hence on the vacuum of the operators in (19)

$$\mathbb{C}|0\rangle = |0\rangle \qquad (23)$$

Because $(\pi^{\pm}, \rho^{\pm}, \omega^{\pm}, \ldots; p)$ is by definition a single--particle operator

$$((\pi^{\pm}, \rho^{\pm}, \omega^{\pm}, \ldots; p)) = (\pi^{\pm}, \rho^{\pm}, \omega^{\pm}, \ldots; p) \qquad (22')$$

implying

$$\mathbb{C}^{2} = \mathbb{C} \qquad (24)$$

$\mathbb{C}$ has therefore no inverse $\mathbb{C}^{-1}$. The decay of a resonance is therefore not the inverse of $\mathbb{C}$. It is much more complicated and will not be considered here.

$\mathbb{C}$ 2. For all local operators, $\pi^{\pm}$, $\rho^{\pm}$, $\omega^{\pm}$ .... irrespective of their number, $(\pi^{\pm}, \rho^{\pm}, \omega^{\pm}, \ldots; p)$ is local and the commutator

$$\left[(\pi^-, \rho^-, \omega^-, \ldots; p), (\pi^+, \rho^+, \omega^+, \ldots; q)\right] = \text{C-number} \neq 0 \tag{25}$$

is finite for all p^2.

The combination $\mathbb{C}W = W^*$ will be called a bracket or cluster or Wick star product in the set (19). The particles of which $(\pi^{\pm}, \rho^{\pm}, \omega^{\pm}, \ldots; p)$ are the creation and annihilation operators will be called clusters or simply resonances.

A few properties of W^* are that

(i) it is, as expected, non-associative

$$(\pi(\rho\omega)) \neq ((\pi\rho)\omega) \neq ((\omega\pi)\rho) \neq (\pi\rho\omega)$$

(ii) it is commutative

$$(\pi\rho) = (\rho\pi)$$

(iii) it is distributive over addition

$$(\pi\,[\rho+\omega]) = (\pi\rho) + (\pi\omega)$$

3.2 SBM OR PHASE SPACE BOOTSTRAP AS SPECIAL CASE OF BFT

Let

$$\varphi_0(x) = \int d^4p\, \delta_0(p^2 - m_0^2)\left(\bar{a}_0(p)e^{ipx} + \bar{a}_0^+(p)e^{-ipx}\right) \tag{26}$$

with

$$\left[\bar{a}_0(p), \bar{a}_0^+(q)\right] = 2E\delta^{(3)}(\vec{p} - \vec{q})\,(2\pi)^{-3} \tag{27}$$

be a free scalar field of mass m_0. Define the new operators

$$a_0(p) = \frac{\sqrt{2\pi^2}}{m_0}\,\delta_0(p^2 - m_0^2)\,\bar{a}_0(p)$$
$$a_0^+(p) = \frac{\sqrt{2\pi^2}}{m_0}\,\delta_0(p^2 - m_0^2)\,\bar{a}_0^+(p) \tag{28}$$

with commutation relation

$$\left[a_0(p), a_0^+(q)\right] = B\delta_0(p^2 - m_0^2)\,\delta^{(4)}(p-q) \tag{29}$$

For the W^* product of the a_0's we use the more convenient notation

$$a(p,N) = \mathbb{C}\int \delta^{(4)}\Big(p - \sum_{i=1}^{N} q_i\Big)\prod_{i=1}^{N} a_0(q_i)\,d^4q_i$$
$$a(p; n_1, n_2, \dots n_\ell) = \mathbb{C}\int \delta^{(4)}\Big(p - \sum_{i=1}^{\ell} q_i\Big)\prod_{i=1}^{\ell} a(q_i, n_i)\,d^4q_i \tag{3o}$$

According to conditions $\mathbb{C}1$ and $\mathbb{C}2$ the operators in (3o) are local and have commutators

$$[a(p,N)\,;\ a^+(q,N')] = \delta_{NN'}\,\delta^{(4)}(p-q)\,N!\,\Omega_N(p^2)$$

$$\left[a(p,n_1,n_2,\dots n_\ell),\, a^+(q, n_1', n_2', \dots n_{\ell'}')\right] =$$
$$= \delta_{\ell\ell'}\,\delta^{(4)}(p-q)\sum_{\wp}\Big(\prod_{i=1}^{\ell}\delta_{n_i n'_{\wp(i)}}\, n_i!\Big) \times$$
$$\times\ \Omega_{n_1+n_2+\dots+n_\ell}(p^2) \tag{31}$$

where $\wp \equiv$ permutation of (1,2,...l). Note that eq.(3o) is a rewriting of the physics contained in eq.(17b). Eq. (31) is then the most naive generalisation of eq.(29) and gives rise to the following mapping

$$H_N \equiv \left\{ a^+(p, n_1, n_2, \cdots n_\ell)\,;\; N = \sum_{i=1}^{\ell} n_i\,;\; \ell \geq 2 \right\} \longrightarrow \Omega_N(p^2) \quad (32a)$$

$$\frac{1}{\sqrt{N!}} J a(p,N) J^+ = \delta_{1N}\, a_0(p) + \frac{1}{\sqrt{d_N - 1}}\, \mathbb{C} \sum_{\ell=2}^{N} \frac{1}{\sqrt{\ell!}} \sum_{N = n_1 + n_2 + \cdots + n_\ell} \times$$

$$\times \int \delta^4 (p - \sum_{i=1}^{\ell} q_i) \prod_{i=1}^{\ell} \frac{1}{\sqrt{n_i!}}\, a(q_i, n_i)\, d^4 q_i \quad (32b)$$

between the Hilbert space H_N of dimension b_N and phase space, $b_N = (d_N - 1)^+$ is the number of different bracketings of first (.......) and second ((.....) (....) (...)) order for a sample of size N and J is an isometry ($J^+J=1$, $(JJ^+)^2 = JJ^+$).

The local space-time fields corresponding to the operators in (3o) are defined by

$$\varphi_N(x) = \frac{1}{\sqrt{2\pi^2}} \int d^4p \sqrt{p^2} \left(a(p,N)\, e^{ipx} + a^+(p,N)\, e^{-ipx} \right)$$

$$\varphi_1(x) \equiv \varphi_0(x) \quad (33)$$

with $a(p,1) = a_0(p)$, and similarly for $\varphi_{n_1 n_2 \cdots n_\ell}(x)$ in terms of $a(p,n,n_2 \ldots n_l)$. Consider next the operators defined by

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+ It is $d_N - 1$ and not d_N because $a(p,N) = a(p,\underbrace{1,1\ldots1}_{N})$ by definition.

$$A(p) = a_0(p) + \sum_{N=2}^{\infty} \frac{\alpha_N}{\sqrt{N!}} a(p,N) \tag{34}$$

$$A(p,\ell) = \mathbb{C} \int \delta^{(4)}\left(p - \sum_{i=1}^{\ell} q_i\right) \prod_{i=1}^{\ell} A(q_i)\, d^4q_i$$

$$= \sum_{n_1,n_2\ldots n_\ell=1}^{\infty} \frac{\alpha_{n_1}\alpha_{n_2}\ldots\alpha_{n_\ell}}{\sqrt{n_1!\,n_2!\ldots n_\ell!}} a(p,n_1 n_2\ldots n_\ell) \tag{35}$$

$$C(p) = a_0(p) + \sum_{\ell=2}^{\infty} \frac{\gamma_\ell}{\sqrt{\ell!}} A(p,\ell) \tag{36}$$

with space-time fields

$$\varphi(x) = \sum_{N=1}^{\infty} \frac{\alpha_N}{\sqrt{N!}} \varphi_N(x) \;; \quad \alpha_1 = 1 \tag{34'}$$

$$\Phi_\ell(x) = \sum_{n_1,n_2\ldots n_\ell=1}^{\infty} \frac{\alpha_{n_1}\alpha_{n_2}\ldots\alpha_{n_\ell}}{\sqrt{n_1!\,n_2!\ldots n_\ell!}} \varphi_{n_1 n_2\ldots n_\ell}(x) \tag{35'}$$

$$\Phi(x) = \varphi_0(x) + \sum_{\ell=2}^{\infty} \frac{\gamma_\ell}{\sqrt{\ell!}} \Phi_\ell(x) \tag{36'}$$

With no restrictions on the coefficients $\alpha \equiv \{\alpha_N\}$ and $\gamma \equiv \{\gamma_\ell\}$ these fields are not local. From eqs.(31), (34) - (36) the vacuum expectation value of their commutators are

$$\langle 0|[\varphi(x), \varphi(y)]|0\rangle = 4\pi i B \int ds\, \rho(s;\alpha)\Delta(x-y;s) \tag{37a}$$

$$\langle 0|[\Phi(x), \Phi(y)]|0\rangle = 4\pi i B \int ds\, \sigma(s;\alpha,\gamma)\Delta(x-y;s) \tag{37b}$$

where Δ(x-y,s) is the free field commutator for a particle of mass $\sqrt{s}$ and

$$B\rho(p^2;\alpha) = B\delta_0(p^2-m_0^2) + \sum_{N=2}^{\infty} |\alpha_N|^2 \Omega_N(p^2) \tag{38a}$$

$$B\sigma(p^2;\alpha,\gamma) = B\delta_0(p^2-m_0^2) + \sum_{\ell=2}^{\infty} |\gamma_\ell|^2 \int \delta^{(4)}(p-\sum_{i=1}^{\ell} q_i) \times \prod_{i=1}^{\ell} B\rho(q_i^2,\alpha)\, d^4q_i \tag{38b}$$

Eq.(13) is thus sufficient to obtain the bootstrap equation

$$B\rho(p^2;\alpha) = B\delta_0(p^2-m_0^2) + \sum_{\ell=2}^{\infty} |\gamma_\ell|^2 \int \delta^{(4)}(p-\sum_{i=1}^{\ell} q_i) \times \prod_{i=1}^{\ell} B\rho(q_i^2,\alpha)\, d^4q_i \tag{39}$$

and implies

$$\alpha = \alpha(\gamma) \tag{4o}$$

Eq.(39) determines (4o) only in modulus. In fact from

$$C(p) = U A(p) U^{+} \tag{41}$$

and

$$\langle 0|[C(p), C^{+}(q)]|0\rangle = \langle 0|[A(p), A^{+}(q)]|0\rangle$$

one gets

$$|\alpha_N|^2 = \delta_{1N} + \sum_{\ell=2}^{N} |\gamma_\ell|^2 \sum_{n_1+n_2+\cdots+n_\ell=N} |\alpha_{n_1}|^2 |\alpha_{n_2}|^2 \cdots |\alpha_{n_\ell}|^2 \tag{42}$$

which, with $g_N = |\alpha_N|^2$, $g_{n_i} = |\alpha_{n_i}|^2$,

$|\gamma_\ell|^2 = \gamma_\ell(n_1, n_2, \ldots n_\ell)$, is exactly what one would obtain by substituting (15') in (16). From eq.(41) and the commutation relations (31) it is easy to see that (42) follows from the operator equation

$$\frac{\alpha_N}{\sqrt{N!}} U a(p,N) U^{+} = \delta_{1N}\, a_0(p) + C \sum_{\ell=2}^{N} \frac{\gamma_\ell}{\sqrt{\ell!}} \sum_{n_1+n_2+\cdots+n_\ell=N} \times$$

$$\times \int \delta^4(p - \sum_{i=1}^{\ell} q_i) \prod_{i=1}^{\ell} \frac{\alpha_{n_i}}{\sqrt{n_i!}}\, a(q_i, n_i) d^4 q_i$$

Comparing (32b) and (43) we see that if phase space bootstrap (the Frautschi equation) is all one needs U can very well be just an isometry. Eq.(43) is then consistent with (32b) if

$$\alpha_N = \delta_{1N} + \sum_{\ell=2}^{N} \gamma_\ell \sum_{n_1+n_2+\cdots+n_\ell=N} \alpha_{n_1}\alpha_{n_2}\cdots\alpha_{n_\ell} \tag{44}$$

In turn (44) is consistent with (42) if

$$\mathrm{Re}(\alpha^*_{\mathbb{P}(N)}\alpha_{\mathbb{P}'(N)}) = 0; \ \mathbb{P}(N) \neq \mathbb{P}'(N) \tag{45}$$

where $\mathbb{P}(N)$ is a partition of N

$$\mathbb{P}(N) : N = n_1 + n_2 + \cdots + n_\ell ; \ \ell \geq 2$$

and

$$\alpha_{\mathbb{P}_\ell(N)} = \alpha_{n_1}\alpha_{n_2}\cdots\alpha_{n_\ell}\delta(N - \sum_{i=1}^{\ell} n_i)$$

$$\alpha_N = \sum_{\ell=2}^{N} \gamma_\ell \, \alpha_{\mathbb{P}_\ell(N)} \tag{46}$$

From eqs.(15), (16), (32b), (39), (42), (43) and (44) we see that many algebraic systems, e.g. functions, real and complex numbers, operators, the subspaces of a totally ordered topological space, obey bootstrap equations of the same general form. Thus one has only to verify the existence of a bootstrap relationship in a simple system (in our case the category of N-body (N=1,2...) phase space) and then reproduce it in other algebraic systems by suitable homomorphisms. Given this fact the Frautschi bootstrap equation is clearly nothing more than an interesting way of expressing a particular normalization condition (eq. (31)) for the states in H_N. Actually one need not go beyond eq.(34) to derive the Frautschi equation: for fixed N substitute $\hat{a}(q_i, n_i) = \alpha_{n_i}\, a(q_i, n_i)$ into eq.(32b) and change $1/(d_N - 1)$ to γ_ℓ then

Phase space

consists of phase spaces
each of which in turn

yields the bootstrap equation upon taking commutators and summing over N.

3.3 CORRELATION BETWEEN MASS AND SIZE OF CLUSTER

By definition, the cluster operators $a(p, n, n_2 \ldots n_l)$ ($n_i, \ell \geq 1$) are local and have finite commutators for all p^2. This definition actually stipulates a definite correlation between the mass $m_N = \sqrt{p^2}$ and the size $N = n_1 + n_2 + \cdots + n_l$ of a cluster. Consider, for simplicity, the case l=1 and take the limit $N \to \infty$ in eq.(31). Condition $\mathbb{C}$ 2 then implies the inequality

$$0 < \lim_{N\to\infty} \left[N!\, \Omega_N(m_N^2)\right] < \infty \tag{47}$$

and hence the said correlation. If for small m_0 and $m_N \gg m_0$, $\Omega_N(m_N^2)$ is approximated by

$$\Omega_N(m_N^2) \simeq \frac{\pi B^2}{2(N-1)!(N-2)!} \left(\frac{\pi B}{2} m_N^2\right)^{N-2} \tag{48}$$

then (47) is satisfied only if

$$m_N^2 \xrightarrow[N\to\infty]{} \left(\frac{2}{\pi B e}\right) N = \left(\frac{8\, m_o^2}{e}\right) N \tag{49}$$

where $e \simeq$ 2.78.... If the mass-size correlation is the same for all clusters then the square of cluster masses are quantized

$$m_N^2 = m_o^2 + b(N-1)\,; \quad b = 8 m_o^2 / e \tag{5o}$$

4. BFT IN GENERAL

BFT is concerned not with the structure of phase space but how to formulate dynamics in the Hilbert space

$$H = \bigoplus_{N=1}^{\infty} H_N$$

$$H_N \equiv \left\{ a^+(p, n_1, n_2, \cdots n_\ell)|0\rangle,\ N = \sum_{i=1}^{\ell} n_i\ ,\ \ell \geqslant 1 \right\} \tag{51}$$

The definition of the cluster products in eq.(3o) with only phase space correlation was done for the sake of simplicity and to get directly to the results of SBM. In general we have

$$a_{\lambda n_1 n_2 \cdots n_\ell}(p) = \frac{1}{\sqrt{n_1! n_2! \cdots n_\ell!}}\, C \int \delta^{(4)}\Big(p - \sum_{i=1}^{\ell} q_i\Big)\, \psi_{\lambda n_1 n_2 \cdots n_\ell}(p, q_1 \cdots q_\ell) \prod_{i=1}^{\ell} a(q_i n_i)\, d^4 q_i \tag{52}$$

in terms of wave functions $\psi_{\lambda n_1 n_2 \cdots n_\ell}(p, q_1 \cdots q_\ell)$ which may depend on additional quantum numbers represented collectively by λ. With (52) the normalization of the states in H_N is no longer given by (31). At this point we can incorporate the information that mass and size of clusters are correlated directly in the normalization of states and quantize as follows

$$\left[a_{\lambda n_1 n_2 \cdots n_\ell}(p), a^{+}_{\lambda' n_1' n_2' \cdots n'_{\ell'}}(q)\right] = \delta_{\lambda\lambda'}\,\delta_{\ell\ell'} \times \delta_o(p^2 - m^2(n_1 n_2 \cdots n_\ell))\,\delta^4(p-q) \tag{53}$$

Eq.(53) is true for $l = n_1 = 1, n_i = o$ $i \geqslant 2$ (eq.(29)) and is satisfied by (52) if

$$\int \delta^{(4)}\Big(p - \sum_{i=1}^{\ell} q_i\Big)\,\psi^{*}_{\lambda' n_1 n_2 \cdots n_\ell}(p, q_1, q_2 \cdots q_\ell)\,\psi_{\lambda n_1 n_2 \cdots n_\ell}(p, q_1, q_2 \cdots q_\ell) \times \prod_{i=1}^{\ell} \Omega_{n_i}(q_i^2)\,d^4 q_i = \delta_{\lambda\lambda'}\,B\,\delta_o(p^2 - m^2(n_1 n_2 \cdots n_\ell)) \tag{54}$$

Note that if $\psi_{\lambda N} \neq \psi_{\lambda \underbrace{11\cdots 1}_{N}}$ then $a_{\lambda N} \neq a_{\lambda \underbrace{11\cdots 1}_{N}}$ and H_N in eq.(51) has dimension d_N

The operators

$$A_\lambda(p) = a_{\lambda o}(p) + \sum_{N=2}^{\infty} \sum_{\ell=1}^{N} \sum_{n_1+n_2+\cdots+n_\ell = N} a_{\lambda n_1 n_2 \cdots n_\ell}(p) \tag{55}$$

are not local. Their mass spectrum is given by the vacuum expectation value of the commutator

$$\langle 0|[A_{\lambda}(p), A^{+}_{\lambda'}(q)]|0\rangle = \delta_{\lambda\lambda'}\delta^{4}(p-q)\, B\rho(p^{2}) \qquad (56)$$

with

$$\rho(p^{2}) = \delta_{0}(p^{2}-m_{0}^{2}) + \sum_{N=2}^{\infty} d_{N}\, \delta_{0}(p^{2}-m_{N}^{2}) \qquad (57)$$

Making use of $m_N^2 = M_0^2 + b\,(N-1)$ and the recursion relation

$$d_{N} = \frac{1}{N}\sum_{\ell=0}^{N-1} \sigma(N-\ell)\, d_{\ell} \qquad (58)$$

one gets

$$b\rho(p^{2}) = \sum_{N=0}^{\infty} d_{N}\, \omega_{N}(p^{2}) \qquad (59)$$

where

$$\omega_{N}(p^{2}) = \sigma\left(\frac{p^{2}-m_{0}^{2}}{b}+1-N\right)\Big/\left(\frac{p^{2}-m_{0}^{2}}{b}+1\right) \qquad (60)$$

and $\sigma(n)$ is the divisor function, that is the sum of all divisors of n. Eq.(59) is the "Yellin expansion" in this case and (58) the "bootstrap" equation analogous to

(42). Eq.(58) is a group theory (linear) decomposition with respect to the quantum number(N-1), that is the mass squared operator.

We are unfortunately not yet fully equipped to go beyond the mass spectrum and define the S-matrix. We still have to define an operation corresponding to decay and consider in some more detail the field theory properties of this non-local model. However what has been done is clear and can be concisely stated: Given the C^*-algebra generated by the creation and annihilation operators of a "particle" carrying a basic unit of an additive "quantum number" a multi-nary non-associative bracket product can be defined which maps the monomials

$$M_N = \underbrace{a_1 \, a_1 \, \ldots\ldots a_1}_{N\text{-times}} \qquad (17d)$$

into N-degenerate vector spaces. The degeneracy of the subspaces H_N consisting of products with at most two overall bracketings is exponential in $\sqrt{N}$ as $N \rightarrow \infty$. The "quantum number" N can be anything provided it is additive. In the statistical bootstrap model it is the number of particles in phase space. In the dual resonance model it is spin.

As far as the mass spectrum is concerned the difference between these two models is in the different normalization of the states.

In fact eq.(46)

$$\alpha_{\mathbb{P}_\ell(N)} = \alpha_{n_1} \alpha_{n_2} \cdots \alpha_{n_\ell} \delta(N - \sum_{i=1}^{\ell} n_i); \; \ell \geq 1 \qquad (46)$$

can also be used to generate H_N. In the dual model the α_{n_i} are commuting operators and the states $\alpha^+_{\mathbb{P}(N)}|0\rangle$, $\alpha^+_{\mathbb{P}'(N)}|0\rangle$ are orthogonal for $\mathbb{P}(N) \neq \mathbb{P}'(N)$. The need for bracketing does not arise because the states $\alpha^+_n|0\rangle$, $n \geq 1$ are assumed given from the beginning. If these states are normalized with respect to phase space and eq.(44) is imposed the statistical bootstrap model is obtained. The relationship between the two models ends here.

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D I S C U S S I O N

CHAIRMAN: Prof. E. Etim

Scientific Secretary: M.A. Ichola

DISCUSSION

- *LITTENBERG:*

How do you verify the form of $\rho(m)$ at high energies when you cannot identify individual resonances?

- *ETIM:*

It is not necessary to resolve individual resonances at very high energies in order to test the form of $\rho(m)$ there. In fact, the theoretical mass spectrum is a continuous function in the energy region where the resonances merge into a continuum.

- *WILKIE:*

The model is similar to the generalized Veneziano model and gives a spectrum of bosons of higher and higher spins. Is it possible to put in internal quantum numbers?

- *ETIM:*

It is not possible to give a proper treatment of angular momentum in this model, and this should be done before any attempt is made to include isospin.

- *FERBEL:*

I have often heard the statement you have just made -- namely that the mass spectrum grows exponentially. Could you explain what that means?

- *ETIM:*

That the spectrum grows exponentially! It means that if the mass is allowed to go to infinity the mass spectrum $\rho(m)$ behaves as an exponential exp (bm) where b is a constant.

- *FERBEL:*

I know that, but what does it mean and what does it imply?

- *ETIM:*

It is based on the bootstrap assumptions I drew attention to in the lecture. It implies that at very high energies, the number of open channels in a given hadronic reaction increases with energy and does so exponentially. Implications for cosmology have been discussed in various papers by Hagedorn, Frautschi, Huang, and Weinberg.