

ISTITUTO NAZIONALE DI FISICA NUCLEARE  
Laboratori Nazionali di Frascati

LNF-78/61

G. Curci and M. Greco:  
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Estratto da: Phys. Letters 79B, 406 (1978).

Servizio Documentazione  
dei Laboratori Nazionali di Frascati  
Cas. Postale 13 - Frascati (Roma)

## MASS SINGULARITIES AND COHERENT STATES IN GAUGE THEORIES

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Received 18 July 1978

The problem of mass singularities is studied using the formalism of coherent states. The introduction of new states which account for both soft and hard radiation effects provides one with matrix elements which are free from infra-red and mass singularities at the leading logarithmic approximation. In QED our results coincide with those obtained from the renormalization group equations. The generalization to QCD agrees with perturbative calculations and provides a simple framework for evaluating the finite corrections to hard processes.

The problem of infra-red singularities in QED has been extensively studied in the literature, using mainly two different approaches. In the first and more conventional one [1] the infra-red structure of the on-shell form factors is inspected in perturbation theory, where various proofs have been given of the well-known exponentiation of the infra-red singularities. As is well known, in the calculation of inclusive cross sections for physical processes these divergences are exactly cancelled at all orders by those arising from real emission, in agreement with the Bloch–Nordsieck theorem [2], generalized by Kinoshita and Lee and Nauenberg [3].

In the second approach [4] the finiteness of observable cross sections has been the motivation for defining a finite  $S$  matrix, using the formalism of the coherent states. This is much closer to the physical reality because of the introduction in the theory of new states which are degenerate in the number of soft photons. The Kinoshita–Lee–Nauenberg theorem is then automatically satisfied and the new matrix elements are therefore finite.

The infra-red structure of non-abelian gauge theories has been investigated much the same as in QED. Using the first type of approach [5] and at the ap-

proximation where only leading powers of infra-red logarithms are kept, the on-shell form factors have been shown to satisfy the same infra-red differential equation found in QED, the only modification being the appearance of the effective coupling constant  $\bar{g}^2(\tau)$  for the pure Yang–Mills theory instead of  $g^2$  [6]. Furthermore explicit calculations [5] of observable quantities as inclusive cross sections, have also been found to be finite, in agreement with the Kinoshita–Lee–Nauenberg theorem.

These results have motivated the extension to non-abelian gauge theories of the coherent state formalism [7] which indeed provides one with matrix elements free from infra-red divergence for definite colour states, at the same leading logarithm approximations. This has been achieved by extending the concept of classical currents associated to the external particles to incorporate both the new properties of colour and the appearance of the effective coupling constant  $\bar{g}^2(\tau)$ .

In the present letter we further extend the formalism of coherent states to the problem of mass singularities in QED. The obvious motivation for this analysis is the very close connection between hard processes and mass singularities in quantum chromodynamics (QCD), as recently emphasized by various authors [8].

To this aim we enlarge the concept of soft photons by considering in addition to the usual low-energy photons ( $k \leq \Delta\omega < E$ ), also the hard photons ( $\Delta\omega \leq k < E$ ) which are emitted with an angle less than or equal to an arbitrarily small but fixed value  $\delta$ . In other words we include in the formalism all photons which have either small energies or high energies but small transverse momenta ( $k_{\perp}^2 \leq E^2\delta^2$ ).

Such an extension provides one with new matrix elements free of mass singularities at all orders of perturbation theory, in the leading logarithm approximation, in agreement with the Kinoshita–Lee–Nauenberg theorem. As a nice feature of the formalism the contribution of hard and collinear photons from any external leg of a diagram is easily obtained in exponentiated form. Our results are then simply generalized to QCD by virtue of the formalism developed in ref. [7] and provide a very simple tool for evaluating to all orders the finite corrections to hard processes.

To be more specific, let us consider electron scattering in an external field. After cancellation of the infrared divergences from real and virtual soft photons ( $k \leq \Delta\omega$ ) the inclusive cross section, at the lowest radiative correction, is given by ( $-q^2 \gg m^2$ )

$$\sigma_{\text{incl}} = \sigma_0 \left\{ 1 + 2 \frac{\alpha}{\pi} \left[ \left( \ln \frac{-q^2}{m^2} - 1 \right) \ln \frac{\Delta\omega}{E} + \frac{3}{4} \ln \frac{-q^2}{m^2} + O(1) \right] \right\}, \quad (1)$$

where  $\alpha = e^2/4\pi$  is the usual on-shell charge. The contribution of vacuum polarization is not included in eq. (1) and will be discussed later with the renormalization problem. As  $m \rightarrow 0$  the mass singularities of eq. (1) are exactly cancelled by adding the contributions of the states which are degenerate with the initial and final electron. More explicitly, the total hard cross section

$$\sigma_{\text{hard}} = \sigma [e(p - k) + \gamma(k) \rightarrow e(p')] + \sigma [e(p) \rightarrow e(p' - k') + \gamma(k')],$$

is given by

$$\sigma_{\text{hard}} = \sigma_0 \left\{ 2 \frac{\alpha}{\pi} \left[ \left( \ln \frac{E^2\delta^2}{m^2} - 1 \right) \left( \ln \frac{E}{\Delta\omega} - \frac{3}{4} \right) + O(1) \right] \right\}, \quad (2)$$

having limited the hard photon phase space as  $\Delta\omega < k$ ,

$k' < E$  and  $0 < \theta_k, \theta_{k'} < \delta$ . There, as is well known, the superinclusive cross section contains no mass singularities:

$$\sigma_{\text{super}} \equiv \sigma_{\text{incl}} + \sigma_{\text{hard}} = \sigma_0 \left\{ 1 + 2 \frac{\alpha}{\pi} \left[ \ln \frac{-q^2}{k_{\perp \text{max}}^2} \cdot \ln \frac{\Delta\omega}{E} + \frac{3}{4} \ln \frac{-q^2}{k_{\perp \text{max}}^2} + O(1) \right] \right\}, \quad (3)$$

with  $k_{\perp \text{max}}^2 = E^2\delta^2$ .

To improve eqs. (1) and (2) to all orders, to maximally logarithmic terms, we use the technique of the renormalization group [9] for the inclusive cross section [10] and then obtain the hard component by the Kinoshita–Lee–Nauenberg theorem. We find

$$\sigma_{\text{incl}} = \sigma_0 \exp \left( -2 \frac{\alpha}{\pi} \ln \frac{\Delta\omega}{E} \right) \exp \left[ \int_{\alpha/\pi}^{(\alpha/\pi)_{\text{incl}}} \frac{dx}{x} \frac{\gamma(x)}{\beta(x)} \right], \quad (4)$$

and

$$\sigma_{\text{hard}} = \sigma_0 \exp \left( 2 \frac{\alpha}{\pi} \ln \frac{\Delta\omega}{E} \right) \exp \left\{ - \int_{\alpha/\pi}^{(\alpha/\pi)_{\text{hard}}} \frac{dx}{x} \frac{\gamma(x)}{\beta(x)} \right\}, \quad (5)$$

where  $\gamma(x) = x(4 \ln(\Delta\omega/E) + 3)$  and  $\beta(x) = x\beta_1$  ( $\beta_1 = \frac{2}{3}$ ). In the above equations  $(\alpha/\pi)_{\text{incl}}$  and  $(\alpha/\pi)_{\text{hard}}$  are the running coupling constants for the inclusive and hard problem, namely the solution of the equations

$$\left\{ \frac{\partial}{\partial t_{\text{incl}}} - \beta \left( \frac{\alpha}{\pi} \right) \frac{\alpha}{\pi} \frac{\partial}{\partial \alpha/\pi} \right\} \left( \frac{\alpha}{\pi} \right)_{\text{incl}} = 0, \quad (6)$$

where  $t_{\text{incl}} = \frac{1}{2} \ln(-q^2/m^2)$  and  $t_{\text{hard}} = \frac{1}{2} \ln(E^2\delta^2/m^2)$ . We therefore obtain, for the superinclusive cross section

$$\sigma_{\text{super}} = \sigma_0 \exp \left\{ - \int_{(\alpha/\pi)_{\text{incl}}}^{(\alpha/\pi)_{\text{hard}}} \frac{dx}{x} \frac{\gamma(x)}{\beta(x)} \right\}. \quad (7)$$

To avoid mass singularities associated with on-shell charge renormalization we use as an expansion parameter  $(\alpha/\pi)_{\text{incl}}$ , which physically corresponds to renormalize the charge at the value  $q^2$ . Consistently we have omitted the vacuum polarization contributions.

Finally, expressing  $(\alpha/\pi)_{\text{hard}}$  in terms of  $(\alpha/\pi)_{\text{incl}}$   $\equiv \bar{\alpha}/\pi$ , we obtain

$$\sigma_{\text{super}} = \sigma_0 \exp \left\{ \frac{4}{\beta_1} \left( \ln \frac{\Delta\omega}{E} + \frac{3}{4} \right) \ln \left( 1 + \frac{\bar{\alpha}}{2\pi} \beta_1 \ln \frac{-q^2}{E^2 \delta^2} \right) \right\}. \quad (8)$$

We are now going to discuss the same problem from the approach of the coherent states. As is well known [4], for given initial and final states  $|i\rangle$  and  $|f\rangle$ , defining the corresponding coherent states  $|\tilde{i}\rangle$  and  $|\tilde{f}\rangle$  in terms of the classical currents associated to the external particles as

$$|\tilde{i}, f\rangle = e^{i\Lambda_{i,f}} |i, f\rangle, \quad (9)$$

with

$$\Lambda_{i,f} = \frac{1}{(2\pi)^4} \int d^4k j_{\mu}^{(i,f)}(k) A_{\mu}(-k), \quad (10)$$

the new  $S$ -matrix elements  $\langle \tilde{f} | S | \tilde{i} \rangle$  are finite, factorizable in the infra-red factors and directly comparable with the observable inclusive cross sections. More explicitly, one gets

$$\sigma_{\text{incl}} \propto |\langle \tilde{f} | S | \tilde{i} \rangle|^2 \quad (11)$$

$$= \exp \left\{ \int_{\lambda} \frac{d^3k}{2k} j_{\mu}(k) j_{\mu}^*(k) \right\} |\langle f | S | i \rangle_{\lambda}|^2,$$

where

$$j_{\mu}(k) = j_{\mu}^{(i)} + j_{\mu}^{(f)} = \sum_{\{i,f\}} \alpha \frac{ie}{(2\pi)^{3/2}} \frac{\eta_{\alpha} p_{\mu}^{(\alpha)}}{(p^{(\alpha)} k)}, \quad (12)$$

for  $k \leq \Delta\omega$ , and zero otherwise. In eq. (11) the parameter  $\lambda$  has been introduced consistently as the lower limit of the permitted energies of real and virtual photons and has to be finally set to zero. From eq. (11), one finds <sup>†1</sup>

$$\sigma_{\text{incl}} \quad (13) \\ = \sigma_0 \exp \left\{ 2 \frac{\alpha}{\pi} \left[ \left( \ln \frac{-q^2}{m^2} - 1 \right) \ln \frac{\Delta\omega}{E} + \frac{3}{4} \ln \frac{-q^2}{m^2} \right] \right\}.$$

<sup>†1</sup> So far the conservation of total energy ( $\Sigma k_n \leq \Delta\omega$ ) has not been taken into account. It can be shown (see the second reference of ref. [4]) that this only introduces a correcting factor of order  $\alpha^2$ .

Let us observe that the factor  $\frac{3}{4} \ln(-q^2/m^2)$  in the right-hand side of eq. (13) has been extracted from the asymptotic elastic form factor given in ref. [11] in the form

$$F^{\infty} \left( \ln \frac{-q^2}{m^2}, \alpha \right) \\ \sim \exp \left\{ -\frac{\alpha}{4\pi} \left[ n^2 \left( \frac{-q^2}{m^2} \right) - 3 \ln \left( \frac{-q^2}{m^2} \right) \right] + O(\alpha) \right\}.$$

Let us introduce now in the formalism the collective effect of hard and collinear photons which form, in the limit  $m \rightarrow 0$ , the set of states degenerate to the initial and final particles. Define the new states

$$|\tilde{\tilde{i}}, f\rangle = e^{i\Lambda_{i,f}^{\text{hard}}} |\tilde{i}, f\rangle, \quad (14)$$

where  $\Lambda_{i,f}^{\text{hard}}$  is obtained from eq. (10) with the substitution  $j_{\mu}^{(i,f)} \rightarrow j_{\mu}^{\text{hard}(i,f)}$ . The new currents  $j_{\mu}^{\text{hard}}(k)$  differ from the old ones  $j_{\mu}(k)$ , given by eq. (12), for the different domain of integration, i.e.,  $\Delta\omega < k < E$  and  $\theta_k \leq \delta$ , where  $\theta_k$  is the angle between  $\mathbf{k}$  and  $\mathbf{p}_a$ . In addition a factor  $\{1 + [1 - (k/E)]^2\}/2$  is introduced to correct the simple  $1/k$  spectrum which appears in any product  $j_{\alpha\mu}^{(k)} j_{\beta\mu}^{(k)}$ , to account for the hard character of the photons <sup>†2</sup>. Then we obtain

$$|\langle \tilde{\tilde{f}} | S | \tilde{\tilde{i}} \rangle|^2 \\ = \exp \left\{ \int \frac{d^3k}{2k} [j_{\mu}^{\text{hard}}(k) j_{\mu}^{\text{hard}}(k)^*] \right\} |\langle \tilde{f} | S | \tilde{i} \rangle|^2 \\ = \exp \left\{ \int \frac{d^3k}{2k} [j_{\mu}^{\text{hard}}(k) j_{\mu}^{\text{hard}}(k)^*] \right. \\ \left. + \int \frac{d^3k}{2k} [j_{\mu}(k) j_{\mu}^*(k)] \right\} |\langle f | S | i \rangle_{\lambda}|^2. \quad (15)$$

For electron scattering in an external potential we have

$$\int \frac{d^3k}{2k} j_{\mu}^{\text{hard}}(k) j_{\mu}^{\text{hard}}(k)^* \quad (16) \\ = 2 \frac{\alpha}{\pi} \left[ \left( \ln \frac{E^2 \delta^2}{m^2} - 1 \right) \left( \ln \frac{E}{\Delta\omega} - \frac{3}{4} \right) + O\left(\frac{\Delta\omega}{E}\right) \right].$$

<sup>†2</sup> A more formal derivation of this correction factor for hard photon and gluon emission can be easily found in the literature. See, for example, ref. [12].

Then we obtain for  $\sigma_{\text{super}} \propto |\langle \tilde{f} | S | \tilde{i} \rangle|^2$  the following result:

$$\sigma_{\text{super}} = \sigma_0 \exp \left\{ 2 \frac{\alpha}{\pi} \left[ \left( \ln \frac{-q^2}{m^2} - 1 \right) \ln \frac{\Delta\omega}{E} + \frac{3}{4} \ln \frac{-q^2}{m^2} \right] \right\} \\ \times \exp \left\{ 2 \frac{\alpha}{\pi} \left( \ln \frac{E^2 \delta^2}{m^2} - 1 \right) \left( \ln \frac{E}{\Delta\omega} - \frac{3}{4} \right) \right\}, \quad (17)$$

or

$$\sigma_{\text{super}} = \sigma_0 \exp \left\{ 2 \frac{\alpha}{\pi} \left[ \ln \frac{-q^2}{k_{1\text{max}}^2} \cdot \ln \frac{\Delta\omega}{E} + \frac{3}{4} \ln \frac{-q^2}{k_{1\text{max}}^2} \right] \right\}. \quad (18)$$

This result explicitly shows that the matrix elements of the new states are free of mass singularities, as stated in the beginning. Furthermore, comparing eqs. (13) and (18) it follows that the final result can be simply obtained from  $\sigma_{\text{incl}}$  by the substitution  $(1/\gamma) \equiv (m/E) \rightarrow \delta$ .

So far our results have been expressed in terms of  $\alpha$ , the physical electron charge which enters in the definition of the classical currents associated to the coherent state. However, the usual measuring procedure of the charge fails in the limit  $m \rightarrow 0$ , and one has to perform an off-shell renormalization. Furthermore, the emission and reabsorption of electron-positron pairs also become singular in the limit  $m \rightarrow 0$ , and have to be considered explicitly. Therefore one is naturally led to introduce an effective coupling constant in the definition of the classical currents which appears in the coherent states. This simple modification in the formalism automatically reproduces the results previously obtained from the renormalization group equations, as shown below.

The hard component is then given by

$$\int \frac{d^3k}{2k} j_{\mu}^{\text{hard}}(k) j_{\mu}^{\text{hard}}(k)^* \\ \approx \int_{m^2}^{k_{1\text{max}}^2} \frac{dk_{\perp}^2}{k_{\perp}^2} \int_{\Delta\omega/E}^1 \frac{dx}{x} [1 + (1-x)^2] \frac{\alpha_{\text{eff}}(M^2, k_{\perp})}{\pi} \quad (19) \\ \approx \frac{4}{\beta_1} \left( \ln \frac{\Delta\omega}{E} + \frac{3}{4} \right) \ln \left[ 1 - \frac{\alpha(M^2)}{2\pi} \beta_1 \ln \left( \frac{k_{1\text{max}}^2}{m^2} \right) \right],$$

where the effective coupling constant has been defined as

$$\alpha_{\text{eff}}(M^2, k_{\perp}) = \frac{\alpha(M^2)}{1 - (\alpha(M^2)/2\pi)\beta_1 \ln(k_{\perp}^2/M^2)}, \quad (20)$$

and  $\alpha(M^2)$  is the off-shell coupling constant. A similar treatment of the soft corrections, with  $-q^2 \ll k_{\perp}^2 < m^2$  then easily leads to eq. (8) instead of eq. (18).

The above analysis shows that the coherent state method gives the same results of the renormalization group equations, providing in addition a very simple physical description of the massless problem.

The generalization of the above results to QCD is straightforward, given the formalism developed in ref. [7] for the soft problem. In fact, for a quark in a singlet potential, the inclusive cross section was found,

$$\sigma_{\text{incl}} \propto |\langle \tilde{f} | S | \tilde{i} \rangle|^2 \\ = \exp \left\{ \int \frac{d^3k}{2k} [j_{\mu}^c(k) j_{\mu}^c(k)^*] \bar{g}_{\text{YM}}^2(k) \right\} \langle f | S | i \rangle, \quad (21)$$

where  $j_{\mu}^c(k)$  is the coloured generalization of eq. (12),  $\bar{g}_{\text{YM}}(k)$  is the effective coupling constant for pure Yang-Mills theory and the right-hand side of eq. (21) is free from infra-red singularities. At the leading logarithm approximation the finite corrections in eq. (21) coming after the cancellation of infra-red singularities can be written as

$$\exp \left\{ - \frac{c_F}{2\pi^2} \int_{\Delta\omega}^E \frac{dk}{k} \int_{m^2}^{-q^2} \frac{dk_{\perp}^2}{k_{\perp}^2} \bar{g}_{\text{YM}}^2(k_{\perp}) \right\}, \quad (22)$$

where we have explicitly introduced a dependence on  $k_{\perp}$  of the effective coupling constant. This dependence is not evident in the results of ref. [7], when only a leading  $\ln k$  behaviour of  $\bar{g}_{\text{YM}}(k)$  was found from the comparison with perturbative calculations. On the other hand a  $k_{\perp}$  dependence of  $\bar{g}$  is strongly suggested from the previous results in QED (eq. (20)) and forced by the requirement of cancelling the  $m^2$  singularities between the soft corrections (22) and the hard contributions, analogous to eq. (19), which have been shown [13] to depend explicitly on  $\bar{g}(k_{\perp})$ .

Furthermore, in the limit  $m \rightarrow 0$ , quark loops have to be taken also into account, in addition to pure gluon loops and the effective coupling constant for a pure Yang-Mills theory which appears in eq. (22) has to be replaced by the full  $\bar{g}(k_{\perp})$ .

The hard gluon corrections are obtained from the

obvious generalization of eqs. (15) and (19) where  $\beta_1$  is replaced by the complete  $\beta$  function (quarks + gluons). Then the  $m^2$  singularities exactly cancel between soft and hard contributions and the superinclusive cross section for the scattering of a quark-jet ( $\equiv |q\rangle + |q+g\rangle + \dots + |q+q\bar{q}\rangle + \dots$ ) in an external potential becomes, as in eq. (8),

$$\sigma_{\text{super}} = \sigma_0 \exp \left\{ -2c_F \int_{\Delta\omega/E}^1 \frac{dx}{x} \left[ \frac{1+(1-x)^2}{2} \right] \int_{E^2\delta^2}^{-q^2} \frac{dk_{\perp}^2}{k_{\perp}^2} \frac{\bar{\alpha}(k_{\perp})}{\pi} \right\} \quad (23)$$

$$= \sigma_0 \exp \left\{ 4 \frac{c_F}{\beta} \left( \ln \frac{\Delta\omega}{E} + \frac{3}{4} \right) \ln \left[ 1 + \frac{\bar{\alpha}(q^2)}{2\pi} \beta \ln \frac{-q^2}{E^2\delta^2} \right] \right\},$$

where  $\beta = -\frac{11}{6}N_c + \frac{2}{3}N_f/2$  and  $c_F = (N_c^2 - 1)/2N_c$ . In the case of  $e^+e^-$  annihilation into two jets, the result of Serman and Weinberg [14] is simply given by the first-order expansion of eq. (23) ( $-q^2 \rightarrow 4E^2$ ). The generalization to other hard processes will be discussed elsewhere.

To summarize, we have studied the problem of mass singularities within the formalism of coherent states. The introduction of the new states, which account both for soft and hard radiation effects, provides one with matrix elements which are free from infra-red and mass singularities. In QED our results coincide with those obtained using the renormalization group equations. The generalization to QCD agrees with low-order calculations and provides a very simple tool to evaluate the finite corrections to hard processes.

We are grateful to R. Meuldermans, Y. Srivastava and G. Veneziano for useful discussions. We wish also to thank G. Veneziano for a critical reading of the

manuscript. One of us (M.G.) gratefully acknowledges the warm hospitality enjoyed at the CERN Theoretical Physics Division, where part of this work was done.

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