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G. Parisi: HAUSDORFF DIMENSIONS AND GAUGE THEORIES.

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ABSTRACT

Gauge theories are described as interacting random surfaces. It is conjectured that the Hausdorff dimensions of a random surface are 4; a consequence of this hypothesis is the simplicity of gauge theories in dimensions near or greater than 8.

A crucial problem in quantum field theory is to control the infrared behaviour in non abelian gauge theories, in particular to understand the properties of the conjectured phase transition[1,5] from the weak coupling phase (I), characterized by the Coulomb law, to the strong coupling phase (II) where "quarks" are confined. It is believed that in less than 4 dimensions ($D < 4$) phase II is realized for all the values of the bare coupling constant g_o and that the correlation lenght (the inverse of the mass gap) becomes infinite only for $g_o = 0$. In other words if the space-time dimensions are smaller than a lower critical dimension ($D < D_c^L$) the phase transition is located at $g_o = 0$: for gauge theories $D_c^L = 4$. When $D > 4$, there is a phase transition (a zero of the Callan-Symanzik $\beta(g)$ function) at $g_o^2 \sim D-4$: this transition is characterized by non trivial critical exponents, i. e. the Green functions at the critical point in the scaling region correspond to an interacting, non-free field theory. The theory is non renormalizable[6]: the expansion in $\varepsilon = D-4$ allows us to construct the theory only in phase I.

Let us investigate on the fate of this theory when D increases. It is a common belief that in enough high dimensions critical exponents always become trivial and that the appropriate free field (mean field) approximation is exact near the transition, i. e. there exists an higher critical dimension (D_c^H) such that if $D > D_c^H$ the theory is free in the infinite cutoff limit, $\Lambda \longrightarrow \infty$ (at the phase transition in the scaling region), if $D = D_c^H - \varepsilon$ the properties of the interacting theory can be computed using an expansion in powers of ε[9].

In this note we rationalize this argument and we suggest $D_c^H = 8$ in gauge theories. For pedagogical reasons we firstly discuss the case of a scalar interaction (a spin system) following an unpublished work of Des Cloizeaux[10,11].

Let us consider the high temperature expansion of a spin system on a lattice with continous O(N) symmetry and a nearest neighbour interaction: the high temperature expansion is a strong coupling expansion which corrisponds to treat the kinetical energy as a pertubation. The free energy $F(\beta)$(the vacuum to vacuum amplitude) can be expanded in powers of $\beta = 1/kT$:

$$F(\beta) = \sum_{o}^{\infty}{}_n F_n \beta^n \tag{1}$$

Each F_n is obtained by summing the contributions of all closed paths on the lattice containing n bonds, each path weighted according to the number and nature of the intersections of the path with itself[13]. A standard procedure to study the region near the transition consists in putting all the weight factors equal to 1, as zeroth order approximation, and to perform a pertubative expansion in the number of intersections. At the zeroth order the problem reduces to the study of a random walk: this case is equivalent to a free scalar field which becomes massless at the transition. The additional weights corresponding to n intersections in the same point, produce a $\Phi^{2(n-1)}$ interaction for the the scalar field associated to the random walk[14]. In this case $D_c^H = 4$ and $D_c^L = 2$, this last result being true only for enough large $N (N > 2)$.

Let us present a simple geometrical interpretation of the critical dimensions. At this end it is convenient to show explicitely how one can associate a measure on the path to a free scalar field; indeed the following chain of identities holds[12]:

$$\langle \Phi(x)\Phi(y) \rangle = \langle x \,|\, 1/(-\Delta + m^2) \,|\, y \rangle = \int_0^\infty dt \, \langle x \,|\, \exp(t\Delta - t m^2) \,|\, y \rangle =$$

$$= \int_0^\infty dt \, d[\omega]_{x,y}^t \exp\left[-\frac{1}{2} \int_0^t \left(\frac{d\omega}{d\tau}\right)^2 d\tau - m^2 t \right] = \int d\mu^{(m)} [\omega]_{x,y} \qquad (2)$$

The first functional integral is done over all the trajectories such that $\omega(o) = x$, $\omega(t) = y$. The measure $d\mu[\omega]_{x,y}^t$ does not depend from the parametrization of the trajectory as function of the internal time τ and it is a purely geometrical object; when $m = 0$, it corresponds to a random walk at the critical point in the scaling region[15].

Everyone knows that the $m = 0$ measure (the critical random walk) is concentrated on very rough trajectories; this qualitative statement may become quantitative if we use the concept of Hausdorff dimensions [16]. An object embedded in a D dimensional space, has H-dimensions d if the minimum number of D dimensional spheres of radius R needed to cover it, grows like R^{-d} when $R \rightarrow 0$. For smooth manifolds we recover the usual dimensions, but also non integer values of the dimensions are now allowed. It is easy to show that, if an object after a dilatation of a scale λ (for some $\lambda \neq 1$) is isomorfic to λ^d copies of itself, its H-dimension is d. Using this result one finds that the H-dimensions of the Cantor set [17] is $\ln_3 2 \simeq 0.63$ and that the measure associated to the critical random walk is concentrated on paths having H-dimensions 2.

We recall the well known rule for generical intersections:

$$d(A \cap B) = d(A) + d(B) - D \tag{3}$$

where a negative dimension means no generical intersections; if we apply eq. (3) to the case of two random walks, we find that their generical intersection has H-dimensions 4-D. The pertubative expansion we have described, is an expansion in the number of intersections; when this number becomes infinite, i. e. $D < 4$, infrared divergences are present: the interacting measure is no more concentrated on path having H-dimensions 2 and the system is characterized by non trivial exponents. When $D > 4$, the generical intersection is zero, so that we recover a free field in the zero mass limit.

When $D \leq 2$, we have serious difficulties to implement our program (the dimensions of an object are always smaller than the space dimensions). In this case two possibilities are open:

a) - the interacting system is "near" to a critical random walk, but the transition temperature is zero.

b) - the interacting system is very "far" from a critical random walk and the critical temperature is different from zero.

The first possibility is realized only for $N > 2$; indeed in the limit $N \to \infty$ the only effect of the interaction is a renormalization of the critical temperature [18]. Summarizing, one finds that for enough large N:

$$D_c^L = d_{RW}; \qquad D_c^H = 2\, d_{RW}; \qquad d_{RW} = 2 \qquad (4)$$

where d_{RW} is the H-dimension of a critical random walk.

What happens in gauge theories? On the lattice we can construct the high temperature (strong coupling) expansion [1, 2]. The basic concepts are plaquettes (surfaces) and not bonds (paths); each F_n of eq.(1) can be written as the sum on all closed surfaces constructed with n plaquattes with a weight depending on the type of intersections and on the group structure. Also in this case we can put all the weights equal to one as a zero th order approximation and take care of the effects of the intersections only pertubatively. In this way gauge theories are described by interacting random surfaces.

In order to realize this program we must have under control the zeroth order approximation, i. e. a random surface. While the definition of a random surface on the lattice is quite clear, the computation of the associated correlation functions is far from being trivial. It is possible that this step can be done using the knowledge gathered in the study of dual models [19, 20]; anynow what will happen at the next step, when we switch on the interaction, will crucially depend on the H-dimensions (d_{RS}); as in the scalar case we expect that in most cases:

$$D_c^L = d_{RS} \quad ; \qquad D_c^H = 2\, d_{RS} \tag{5}$$

According to the present folkore, in non abelian gauge theories $D_c^L = 4$; it is tempting to deduce that a random surface has H-dimensions 4 and that gauge theories can be expanded in $8 - \varepsilon$ dimensions. The fact that $d_{RS} = 2\, d_{RW}$ may esplain Migdal's observation on the similarities of spin systems in dimensions D and gauge theories in dimensions 2 D [4].

A direct euristical argument supporting $d_{RW} = 2$ and $d_{RS} = 4$, is the following. In an open random walk the mean distance $\langle r \rangle$ between the two ends points grows like $n^{1/2}$ (n being the number of the steps). Now, n is the "length " of the walk and it increases as the square of its linear dimensions $\langle r \rangle$; this fact suggests that in the continuum limit ($n \longrightarrow \infty$) $d_{RW} = 2$. A similar argument can be constructed for an open random surface with an open random boundary, in this case on finds:

$$\langle S \rangle \sim n^{1/2} \tag{6}$$

where S is the aerea enclosed by the boundary projected on a fixed plane, n being the number of plaquettes. Eq. (6) implies that the number of plaquettes increase as the fourth power of the linear dimension of the surface, suggesting that in the continuum limit the H-dimensions of an open random surface is 4. Unfortunately we need the H-dimensions of a closed random surface; in this case the corresponding argument would be much more involved, however it is reasonable to hope that the H-dimensions of closed and open random surfaces are the same.

In order to focalize the picture we propose, it is convenient to recall what happens in a spin system. In the continuum limit we can associate to it two different field theories, having the Lagrangians:

$$\frac{1}{2g} \sum_1^N i \, (\partial_\mu \Phi_i)^2 = \mathcal{L} \qquad \sum_1^N i \, \Phi_i^2 = 1 \tag{7a}$$

$$\frac{1}{2} \sum_1^N i \, (\partial_\mu \Phi_i)^2 + \frac{1}{2} \sum_1^N i \, (\Phi_i)^2 + \frac{1}{2N} \lambda \, (\sum_1^N i \, \Phi_i^2)^2 = \mathcal{L} \tag{7b}$$

Φ_i being an N component scalar field.

These two theories concide in the critical region ($\lambda \longrightarrow \infty$) where they are both asymptotically scale invariant [3, 5], while away from the critical region they have quite different proporties: the first formulation is related to the low temperature expansion and can be used mainly in the broken symmetry phase ($\langle \Phi \rangle \neq 0$) in $D_c^L + \varepsilon$ dimensions, the second formulation in related to the high temperature expansion and can be used in $D_c^H - \varepsilon$ dimensions in both phases.

It is evident that in gauge theory we know only the equivalent of the first formulation, while we are missing the equivalent of the second one. A reformulation of gauge theories using different variables is therefore mandatory if we want to construct the $8 - \varepsilon$ expansion. In more that 8 dimensions gauge theories are simple: in phase I they describe free non interacting gauge particles, in phase II they describe non interacting surfaces (it may be wiser, for the time being, not to yeld to the temptation of saying that in more that 8 dimensions gauge theories are equivalent to a dual model).

If the geometrical construction outlined in this note turns out to be too difficult, we have at our disposal an alternative possibility: we can reformulate the theory at the Lagrangian level by introducing a "prepotential" P such that the usual gauge potential A_μ can be written as:

$$A_\mu = P^+ \partial_\mu P \tag{8}$$

where P carries enough indices to make the whole construction non-trivial (e.g. A_μ must be not only a pure gauge field). An explicit example has been given by Bars[21]: he introduces D unitary matrices P_μ ($A_\mu = P^+ \partial_\mu P_\mu$). The introduction of a prepotential has nice features: the Lagrangian as function of the prepotential has 4 derivatives, the prepotential propagator has a behaviour $1/k^4$ at small momenta k, suggesting strong infrared singularities in less that 4 dimensions[22]; the similarities with the non linear σ model in dimensions D/2 are now striking[23]: it is possible that the $8 - \varepsilon$ expansion can be constructed using this formulation of gauge theories as starting point.

The two possibilities here described are not mutually self excluding; at the present moment it is not evident which is the best formalism to implement these ideas, but the situation seems quite promising.

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