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E. Etim: STOCHASTIC QUANTISATION AND THE NON-RELATIVISTIC MARKOFF FIELD.

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ABSTRACT.

A different method exists for quantising classical systems. It consists in considering the field q(t) as a stochastic process. This approach is equivalent to the traditional one, but it is less revolutionary. It is remarkable, nontheless, in its own way: it communicates between two worlds, the Euclidean and Minkowski space-times. This connection is investigated here in the non-relativistic framework. It turns out that the Euclidean Schrödinger equation and the Fokker-Planck equation in Minkowski space are associated with the same Markoff field.

1. - INTRODUCTION.

The non-causal behaviour of quantum systems can to some extent be explained by classical probability theory. The main idea is that their motion is modelled by a Markoff process. The kinematics is similar to that of the Einstein-Smoluchowski model of Brownian motion. Under these assumptions Nelson [1] gave, some time ago, a completely classical derivation of the Schrödinger equation. His approach is really an alternative method of quantising classical systems and is appropriately referred to as stochastic quantisation. Its implications for quantum field theory have been clarified, following Symanzik's introduction of random fields in Euclidean QFT [2], by recent investigations of Nelson [3-5] and Guerra [6]. One important result in this connection is that a Euclidean Markoff field can, a juste titrè, be considered as existing in Minkowski space-time. This result prompts a new interpretation of the relationship between the Langevin and Fokker-Planck (FP) equations which brings the latter into contact with quantum theory in a way which is more than formal. I hope to show in fact that if, for dissipative systems, one gives up current conservation in a prescribed manner and expresses Newton's law through the action of a combination of forward and backward derivatives different from but related to, that of Nelson [1], then one can derive a slightly modified, constant diffusion Fokker-Planck equation in much the same way as is done for the Schrödinger equation. The Hamiltonian in the one equation is the measure, and therefore non-unitary, transform of the Hamiltonian in the other. The measure is that of the space of solutions of the Fokker-Planck equation in Minkowski space-time. The same combination of forward and backward derivatives, in other words, the same stochastic process, is associated with the Euclidean Schrödinger equation. Given the Fokker-Planck Hamiltonian the relationship of this theory to the Langevin equations can be expressed in another way: the forward and backward derivatives are the operators in the Ehrenfest theorem with this Hamiltonian. The F P equations are therefore an essential part of stochastic quantisation.

The paper is organised as follows: in section 2 we review Nelson's stochastic quantisation; in section 3 we consider motion in a time-independent external field and then generalise in section 4 to time-dependent fields. Section 5 concludes the paper.

2. STOCHASTIC QUANTISATION.

Let

$$H(p,q) = \frac{p^2}{2m} + V(q) \tag{1}$$

be the Hamiltonian of a particle of mass m in an external field of potential $V(q)$ in a space of n dimensions: $q \equiv (q_1, q_2, \cdots q_n)$. The coordinates are Cartesian. The essential idea in Nelson's method of quantisation consists in the assumption that each coordinate variable $q_j(t)$ is a stochastic process of Markoff type. Since the corresponding sample functions $q_j(t; \cdot)$ are often non-differentiable functions of time t, it is necessary to distinguish between forward ($dt > 0$) and backward ($dt < 0$) increments of $q_j(t)$. This is done with the help of the Langevin equations

$$q_j(t \pm dt) - q_j(t) = V_j^{\pm}(t, q)\, dt + dr_j^{\pm}(t) \tag{2}$$

where, by assumption the random increments $dr_j^+(t)$ and $dr_j^-(t)$ have no memory of the history of $q_j(t)$, respectively, prior to and after the time t. For this reason they do not depend on $q_j(t)$. Eq. (2) models Brownian motion in configuration space if the $dr_j^{\pm}(t)$ are Wiener processes that is, if they are Gaussian distributed, with

$$\langle dr_j^{\pm}(t)\rangle = 0$$

$$\langle dr_i^{\pm}(t)\, dr_j^{\mp}(t+\Delta t)\rangle = 0 \tag{3}$$

$$\langle dr_i^{\pm}(t)\, dr_j^{\mp}(t+\Delta t)\rangle = \frac{\alpha}{m}\left|\Delta t\right|\delta_{ij}$$

The parameter α has the dimension of an action if $q(t)$ has the dimension of length. However $q(t)$ can also be the fluctuation of a thermodynamic variable in which case it must be supposed to be correctly normalised as to units if α is identified with Planck's constant divided by 2π. The functions $v_j^{\pm}(t, q)$ are the forward (+) and backward (-) velocities and are defined by the conditional averages

$$D_{\pm}\, q_j(t) = v_j^{\pm}(t, q) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \langle q_j(t \pm \Delta t) - q_j(t)\rangle \tag{4}$$

for a given state of the particle at time t. The dynamics is expressed through the action of the operators $D_{\pm}$, called forward and backward derivatives, on $v_j^{\pm}(t, q)$ in the following combinations

$$\frac{1}{2}(D_+ v_j^+ - D_- v_j^-) = 0 \tag{5. 1}$$

$$\frac{1}{2}(D_+ v_j^- + D_- v_j^+) = \frac{1}{m}\,\frac{\partial V(q)}{\partial q_j} \tag{5. 2}$$

Eqs. (5. 1) and (5. 2) express, respectively, current conservation and Newton's law. We need the explicit expressions of the operators $D_{\pm}$ in order to make this evident. To this end let A(t, q) be a stochastic process. Given $q_j(t)$, the average of the increments

$$A(t \pm dt, q(t \pm dt)) - A(t, q(t)) = \pm \frac{\partial A(t, q)}{\partial t} dt + \frac{\partial A(t, q)}{\partial q_j} (q_j(t \pm dt) -$$

$$- q_j(t)) + \frac{1}{2} \frac{\partial^2 A(t, q)}{\partial q_i \partial q_j} (q_i (t \pm dt) - q_i(t))(q_j(t \pm dt) - q_j(t)) + \cdots \quad (6)$$

may be computed using eqs. (2) and (3). Repeated indices in eq. (6) and elsewhere are to be summed over. On going to the limit $dt \to 0_+$ and recalling eq. (4) the $D_{\pm}$ are identified as

$$D_{\pm} = \pm \frac{\partial}{\partial t} + v_j^{\pm} \frac{\partial}{\partial q_j} + \frac{\alpha}{2m} \frac{\partial^2}{\partial q_j^2} \quad (7)$$

With the help of the new variables

$$u_j(t, q) = \frac{1}{2} (v_j^+ + v_j^-)$$
$$v_j(t, q) = \frac{1}{2} (v_j^+ - v_j^-) \quad (8)$$

eqs. (4) can be rewritten as

$$\frac{\partial u_j}{\partial t} = - \frac{\alpha}{2m} \frac{\partial^2 v_k}{\partial q_j \partial q_k} - \frac{\partial}{\partial q_j} (u_k v_k) \quad (9.1)$$

$$\frac{\partial v_j}{\partial t} + v_k \frac{\partial v_j}{\partial q_k} = - \frac{1}{m} \frac{\partial V}{\partial q_j} + u_k \frac{\partial u_j}{\partial q_k} + \frac{\alpha}{2m} \frac{\partial^2 u_j}{\partial q_k^2} \quad (9.2)$$

Eq. (9. 2) is in a recognisable form as a generalisation of Newton's law; motion is along a non-differentiable path, whence the non-vanishing of the stochastic velocity $u_j(t, q)$. Eq. (9. 1) becomes so if we introduce a density function $\varrho(t, q) \geqslant 0$ by

$$u_j(t, q) = \frac{\alpha}{2m} \frac{\partial}{\partial q_j} \ln \varrho(t, q) \tag{10}$$

and rewrite (9. 2) as

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial q_j} (\varrho v_j) = 0 \tag{11}$$

It is customany to rewrite eqs. (10) and (11) as two Fokker-Planck equations [7]

$$\pm \frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial q_j} \left[\varrho (v_j^{\pm} - u_j) \right] = 0 \tag{12}$$

These however are not a substitute for eqs. (5).

For completely random motion $v_j(t, q) = 0$ and by eq. (9. 1) $u_j(t, q)$ is independent of t. Eq. (9. 2) then becomes

$$\frac{\alpha}{2m} \frac{\partial^2 u_j(q)}{\partial q_k^2} + u_k(q) \frac{\partial u_j(q)}{\partial q_k} - \frac{1}{m} \frac{\partial V(q)}{\partial q_j} = 0 \tag{13}$$

Since $u_j(t, q)$ itself derives from a potential eq. (13) admits the first integral

$$\frac{1}{2} m u^2(q) + \frac{\alpha}{2} \frac{\partial u(q)}{\partial q} - V(q) = - E \tag{14}$$

where E is a constant. Eq. (14) reduces to the time-independent Schrödinger equation

$$\left(-\frac{\alpha^2}{2m}\nabla^2 + V(q)\right)\Psi(q) = E\,\Psi(q) \qquad (15)$$

if one substitutes

$$u_j(q) = \frac{\alpha}{m}\frac{\partial}{\partial q_j}\ln\,\Psi(q) \qquad (16)$$

in it. Hence when $u_j(q)$ evolves according to eq. (13) the wave function Ψ (q) does so following the Schrödinger equation. Conversely from the wave function Ψ (t, q) one can define velocity functions $u_j(t, q)$ and $v_j(t, q)$ which obey eqs. (9). To this extent the familiar quantisation scheme and the stochastic one are equivalent.

This method is not limited only to the quantisation of systems with conserved or time-dependent Hamiltonians whose states propagate in time. It is also applicable to systems relaxing in time with which it is intimately related.

3. DISSIPATIVE MOTION IN A TIME-INDEPENDENT EXTERNAL FIELD.

The conditions of motion of the Brownian particle in the last section were conveniently ideal. Probability was conserved. Brownian motion also models the dynamics of relaxing systems for which a similarly defined probability is not conserved. For these systems the probability current density $\hat{j}_\mu(t, q) \equiv (\hat{\varrho}(t, q), \hat{\varrho}\, v_j(t, q))$ satisfies

$$\frac{\partial \hat{\varrho}}{\partial t} + \frac{\partial (\hat{\varrho} v_j)}{\partial q_j} = \sigma(t, q) \tag{17}$$

We distinguish between this case and motion characterised only by uniform diffusion using slightly different symbols for the current densities. Besides $\hat{j}_\mu(t, q)$ the source term $\sigma(t, q)$ in eq. (7) has also to be specified for the theory to be completely defined. Given this option we refer the determination of $\sigma(t, q)$ to a comparison with the theory in the last section. We do so in two steps. Firstly we assume that $\sigma(t, q)$ derives from a time-independent driving force obtained by equating eq. (17) to (11). Subtracting the latter from the former the assumption amounts to

$$\sigma(t, q) = \hat{j}_k(t, q) \frac{\partial \ln \varphi^2(q)}{\partial q_k} = j_k(t, q) \frac{\partial \varphi^2(q)}{\partial q_k} \tag{18}$$

where $j_\mu(t, q) \equiv (\varrho(t, q), \varrho v_j(t, q))$ and

$$\varphi^2(q) = \hat{\varrho}(t, q)/\varrho(t, q) \tag{19}$$

We use $\varphi(q)$ to define a drift velocity

$$T_k(q) = \frac{\alpha}{m} \frac{\partial}{\partial q_k} \ln \varphi(q) \tag{20}$$

The problem is now reduced to determining $\varphi(q)$. Next we inquire as to the response of the particle to an external pertubation. This cannot be represented by eq. (5. 2) if $\sigma(t, q)$ is not zero. Guided however by eq. (5) we postulate

$$\frac{1}{2}(D_+ v_j^+ - D_- v_j^-) = 0 \qquad (21.1)$$

$$\frac{1}{2}(D_+ v_j^+ + D_- v_j^-) = \frac{1}{m} \frac{\partial V(q)}{\partial q_j} \qquad (21.2)$$

for the complete dynamics. Eqs. (21) describe two kinds of theories. In the first, the time coordinate t and the functions $v_j^{\pm}(t, q)$ in eqs. (5) and (21) are not the same but are related by the substitutions

$$\begin{aligned} t &\longrightarrow -it \\ v_j^{\pm} &\longrightarrow \frac{1}{2}(1 \pm i) v_j^+ + \frac{1}{2}(1 \mp i) v_j^- \end{aligned} \qquad (22)$$

which transform the one set of equations into the other. $\sigma(t, q)$ is consequently zero. Eqs. (21) are in this case equivalent to the Euclidean Schrödinger equation. This is also easily seen by writing (21.2) out explicitly

$$-\left(\frac{\partial v_j}{\partial t} + v_k \frac{\partial v_j}{\partial q_k}\right) = -\frac{1}{m} \frac{\partial V(q)}{\partial q_j} + u_k \frac{\partial u_j}{\partial q_j} + \frac{\alpha}{2m} \frac{\partial^2 u_j}{\partial q_k^2} \qquad (23)$$

Eqs. (21) are consequently the Euclidean version of eqs. (5). The second theory is in Minkowski space-time, the functions $v_j^{\pm}(t, q)$ are the same as in the Langevin equations but the dynamics is different: $\sigma(t, q) \neq 0$. The two theories are therefore described by the same stochastic process. Indeed if we use eqs. (19) and (20) and introduce new real wave functions by

$$T_j(q) + v_j^+(t, q) = \frac{\alpha}{m} \frac{\partial}{\partial q_j} \ln W^x(t, q) \qquad (24.1)$$

$$T_j(q) + v_j^-(t, q) = \frac{\alpha}{m} \frac{\partial}{\partial q_j} \ln W(t, q) \qquad (24.2)$$

then eqs. (21) become the Fokker-Planck equations

$$\alpha \frac{\partial W(t, q)}{\partial t} = \hat{H} W(t, q)$$

$$-\alpha \frac{\partial W^x(t, q)}{\partial t} = \hat{H} W^x(t, q)$$

where $\hat{H}$ is the transform by $\varphi(q)$ of the Schrödinger Hamiltonian, that is

$$\hat{H}(-ip, q) = -e^{\Lambda(q)} H(p, q)\, e^{-\Lambda(q)} = -E_o - \alpha \frac{\partial}{\partial q_j} T_j(q) + \frac{\alpha^2}{2m} \frac{\partial^2}{\partial q_j^2} \qquad (26)$$

with

$$\Lambda(q) = \frac{m}{\alpha} \int^q dx_j\, T_j(x) \qquad (27)$$

and

$$\frac{1}{2} m\, T^2(q) + \frac{\alpha}{2} \frac{\partial T_j(q)}{\partial q_j} - V(q) = -E_o \qquad (28)$$

We now let eq. (28) determine $\varphi(q)$ as an eigenvalue equation since substitution of (20) in it yields

$$H \varphi(q) = E_o \varphi(q) \qquad (29)$$

Using (26) we can also write (29) as

$$\hat{H}\varphi^2(q) = -E_o\varphi^2(q) \tag{30}$$

This is a remarkable relationship between the Schrödinger and the Fokker-Planck equations. Each stationary state $\varphi_1(q)$ of the former determines an F P Hamiltonian $\hat{H}$ and a complete set of eigen functions $W_k^{(1)}(q) = \varphi_1(q)\,\varphi_k(q)$, k= 0, 1, 2, $\cdots$ which are solutions of the equation

$$\hat{H}_1 W_k^{(1)}(q) = -E_k W_k^{(1)}(q);\; k = 0, 1, 2, \cdots \tag{31}$$

Their normalisation is

$$\int d^n q\; \varphi_1^{-2}(q)\left[W_k^{(1)}(q)\right]^2 = 1 \tag{32}$$

Under normal conditions the Schrödinger ground state $\varphi_o(q) \equiv \varphi(q)$ determines the equilibrium configuration into which the system relaxes, for, of all the states $W_k^{(1)}(q)$, k, 1 = 0, 1, 2, $\cdots$, only $W_o^{(o)}(q) = \varphi^2(q)$ has minimum energy and positive probability measure. We thus arrive at the conclusion: the Markoff field $q_j(t)$ associated with the Euclidian Schrödinger equation by stochastic quantisation is the same as that associated with the Fokker-Planck equation in Minkowski space. It is easy to check that $\hat{H}$ is self-adjoint in the scalar product

$$\langle W_1 | W_2 \rangle = \int d^n q\; \varphi^{-2}(q)\, W_1^{x}(t, q)\, W_2(t, q) \tag{33}$$

We can consequently formulate an Ehrenfest theorem

$$\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}\langle A(t \pm \Delta t) - A(t)\rangle = \pm \left(\langle \frac{\partial A}{\partial t}\rangle + \frac{1}{\alpha}\langle\left[A\;\hat{H}\right]\rangle\right) \tag{34}$$

and substituting for $\hat{H}$ from eq. (26) deduce

$$\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \langle A(t \pm \Delta t) - A(t) \rangle = \langle D \pm A \rangle \tag{35}$$

$\langle A(t) \rangle$ is defined by

$$\langle A(t) \rangle = \int d^n q \, \varphi^{-2}(q) \, W^x(t, q) \, A(t, q) \, W(t, q) \tag{36}$$

As far as the construction of the operators $D\pm$ is concerned the F P equations are therefore a valid substitute for the Langevin equations. Probability-wise these two equations have long been known to be equivalent.

The transformation in eq. (26) is a projection operation; it projects out the purely diffusion part of $\hat{H}$. This latter infact can be written as

$$\hat{H}(\vec{p}, q) = H_D(\vec{p}, q) - H(i\vec{p}, q) \tag{37}$$

$$\vec{p} = -\alpha \frac{\partial}{\partial q}$$

where H(p, q) is the Schrödinger Hamiltonian while

$$H_D(\vec{p}, q) = -E_o + V(q) + \vec{p} \cdot \vec{T}(q) = \frac{1}{2} m T^2 + \frac{1}{2}(\vec{p} \cdot \vec{T} + \vec{T} \cdot \vec{p}) \tag{38}$$

is responsible for drift. Note that

$$H_D(-\vec{T}) = e^{-\Lambda(q)} H_D(\vec{T}) \, e^{\Lambda(q)} \tag{39}$$

so that the character of H_D does not change under the transformation

(26). The classical Hamiltonian equations obtained from H_D i.e.

$$\frac{\partial H_D}{\partial p_j} = \frac{d q_j}{dt} = T_j(q) \tag{40}$$

$$-\frac{\partial H_D}{\partial q_j} = \frac{d\bar{p}_j}{dt} = -\bar{p}_k \frac{\partial T_k(q)}{\partial q_j} - \frac{\partial V(q)}{\partial q_j}$$

have indeed the form of equations of motion with friction [8].

4. MOTION IN A TIME-DEPENDENT EXTERNAL FIELD.

In a time-dependent field we again postulate eqs. (21). For a change we shall give the argument in reverse, i.e. we start from the Fokker-Planck equations

$$\alpha \frac{\partial W(t, q)}{\partial t} = \hat{H}(t, \bar{p}, q)\, W(t, q) \tag{41. 1}$$

$$-\alpha \frac{\partial W^x(t, q)}{\partial t} = \hat{H}^x(t, \bar{p}, q)\, W^x(t, q) \tag{41. 2}$$

where

$$\hat{H}(t, \bar{p}, q) = -E_o(t, q) + \vec{\bar{p}} \cdot \vec{T}(t, q) + \frac{\bar{p}^2}{2m} \tag{42. 1}$$

$$\hat{H}^x(t, \bar{p}, q) = -E_o^x(t, q) + \vec{\bar{p}} \cdot \vec{T}^x(t, q) + \frac{\bar{p}^2}{2m} \tag{42. 2}$$

are adjoints of each other in the scalar product (33), and deduce eqs. (21). That a time-independent measure, $\varphi^{-2}(q)$ in eq. (33), exists in this case, cannot be related to the properties of equilibrium unless the fields become asymptotically time-independent. Its existence is a constraint on the dependence of the drift velocity $T_j(t, q)$ on the external fields. We consider motion in an electromagnetic field. The equations

$$\hat{H}(t, \bar{p}, q)\varphi^2(q) = - E_o^x(t, q)\varphi^2(q) \tag{43.1}$$

$$\hat{H}^x(t, \bar{p}, q)\varphi^2(q) = - E_o(t, q)\varphi^2(q) \tag{43.2}$$

which detemine $\varphi(q)$ are generalisations of eq. (30). Substituting for the operators from eqs. (42), (43) reduce to the single equation

$$\frac{\partial A_j(t, q)}{\partial q_j} + 2 A_j \frac{\partial}{\partial q_j} \ln \varphi(q) = \frac{mc}{\alpha e}(E_o^x - E_o) \tag{44}$$

The vector $A_j(t, q)$ is defined by

$$\frac{e}{mc} A_j(t, q) = T_j(t, q) - \frac{\alpha}{m} \frac{\partial}{\partial q_j} \ln \varphi(q) = - T_j^x(t, q) + \\ + \frac{\alpha}{m} \frac{\partial}{\partial q_j} \ln \varphi(q) \tag{45}$$

whence

$$\frac{1}{2}(T_j + T_j^x) = \frac{\alpha}{m} \frac{\partial}{\partial q_j} \ln \varphi(q) \tag{46}$$

The parameters e and c have been introduced for later convenience. Using (46) the transformation (26) now gives

$$H(- i\vec{A}, \Phi) = - e^{-\Lambda(q)} \hat{H}(\vec{T}, E_o) e^{\Lambda(q)} = \frac{1}{2m}(- i\alpha\nabla + \frac{ie}{c}\vec{A})^2 + e\Phi(t, q) \tag{47}$$

the function Φ (t, q) is given by

$$\frac{1}{2} m T^2 + \frac{\alpha}{2} \frac{\partial T_j}{\partial q_j} - e\Phi = - E_o(t, q) \tag{48.1}$$

$$\frac{1}{2} m T^{x^2} + \frac{\alpha}{2} \frac{\partial T^x_j}{\partial q_j} - e\Phi = - E^x_o(t, q) \tag{48.2}$$

If we define the velocity functions $v^{\pm}_j$ (t, q) by

$$T^x_j(t, q) + v^+_j(t, q) = \frac{\alpha}{m} \frac{\partial}{\partial q_j} \ln W^x(t, q) \tag{49.1}$$

$$T_j(t, q) + v^-_j(t, q) = \frac{\alpha}{m} \frac{\partial}{\partial q_j} \ln W(t, q) \tag{49.2}$$

and recall, from eq. (45), that $v_j(t, q)$ is now not a gradient, then the F P equations can be reduced, after a little algebra, to the differential equations

$$\frac{1}{2}(D_+ v^+_j - D_- v^-_j) = 0 \tag{50.1}$$

$$\frac{1}{2}(D_+ v^+_j + D_- v^-_j) = \frac{1}{m} F_j(t, q) \tag{50.2}$$

where $F_j(t, q)$ is the Lorentz force

$$\begin{aligned} F_j(t, q) &= e E_j + \frac{e}{c}(\vec{v} \wedge \vec{B})_j \\ E_j(t, q) &= - \frac{\partial \Phi}{\partial q_j} - \frac{1}{c} \frac{\partial A_j}{\partial t} \\ B_j(t, q) &= (\nabla \wedge \vec{A})_j \end{aligned} \tag{51}$$

Eqs. (50) are also equivalent to the Euclidean Schrödinger equation. Hence we conclude as before, that the Euclidean Schrödinger and the Fokker-Planck equations are associated with the same Markoff process q(t). The functions $T_j(t, q)$ and $E_o(t, q)$ are gauge dependent. From eqs. (45), (47) and (51) their gauge transformations are

$$T'_j(t, q) = T_j(t, q) + \frac{e}{mc} \frac{\partial f(t, q)}{\partial q_j} \qquad (5\ .1)$$

$$E'_o(t, q) = E_o(t, q) - \frac{e}{c}(T_j + \frac{e}{2mc} \frac{\partial f}{\partial q_j}) \frac{\partial f}{\partial q_j} - \frac{e}{c}(\frac{\partial}{\partial t} + \frac{\alpha}{2m} \nabla^2) f \qquad (5\ .2)$$

Using (51) the gauge invariance of eqs. (49) can be established in the same way as for the Schrödinger equation. If together with (47) we also perform the transformations

$$\begin{aligned} &t \longrightarrow it \\ &(\vec{A}, \Phi) \longrightarrow (i\vec{A}, \Phi) \\ &W(it, q) = \varphi(q)\psi(t, q) \\ &W^x(it, q) = \varphi(q)\psi^*(t, q) \end{aligned} \qquad (53)$$

then eqs. (49) become Schrödinger equations in Minkowski space. From eqs. (42) and (47) the time-dependent drift Hamiltonian is

$$H_D(t, \vec{p}, q) = \frac{1}{2} m T_o^2 + \frac{1}{2}\left[(\vec{p} + \frac{e}{c}\vec{A})\vec{T}_o + \vec{T}_o(\vec{p} + \frac{e}{c}\vec{A})\right]$$

$$T_{oj}(q) = \frac{\alpha}{m} \frac{\partial}{\partial q_j} \ln \varphi(q) \qquad (54)$$

and the classical equations of motion obtained from it are

$$\frac{dq_j}{dt} = T_{oj}(q) \tag{55.1}$$

$$\frac{d\bar{p}_j}{dt} = - \bar{p}_k \frac{\partial T_{ok}}{\partial q_j} + e E_j + \frac{e}{c} (\vec{T}_o \wedge \vec{B})_j \tag{55.2}$$

5. DISCUSSION.

The considerations of the previous sections can be generalised in several directions. Only one such extension is considered here, namely non-uniform diffusion processes. For these we have the theorem [9, 10].

Theorem: let the tensor $g_{ij}(q)$ be Riemannian and let the n-beins $e_j^{\mu}(q)$ of its local Euclidean structure be such that the inverse matrix $e_{\mu}^{j}(q)$ satisfies

$$\oint dq_j \, e_{\mu}^{j}(q) = 0 \tag{56}$$

then the stochastic process

$$z_{\mu}(q) = \int^{q} dx_j \, e_{\mu}^{j}(x) \tag{57}$$

represents the Euclidean Schrödinger equation as well as the F P equation with Hamiltonian

$$\hat{H}(t, \bar{p}, q) = - E_o(t, q) + \bar{p}_j T_j(t, q) + \frac{1}{2m} \bar{p}_i \bar{p}_j g_{ij}(q)$$

$$\bar{p}_j = - \alpha \frac{\partial}{\partial q_j} \tag{58}$$

Proof: since $g_{ij}(q)$ is Riemannian we can write

$$g_{ij}(q) = e^{\mu}_{i}(q)\, e^{\mu}_{j}(q) \tag{59}$$

with

$$e^{\mu}_{j}(q)\, e^{j}_{\nu}(q) = \delta^{\mu}_{\nu}$$

(60)

$$e^{\mu}_{j}(q)\, e^{i}_{\mu}(q) = \delta^{i}_{j}$$

By eq. (56)

$$dz_{\mu}(q) = e^{j}_{\mu}(q)\, dq_{j} \tag{61}$$

is an exact differential. Hence making use of (61) to change variables in (57) we have the constant-diffusion Hamiltonian

$$\hat{H}(t, p', z) = - E'_{o}(t, z) + p'_{\mu} T'_{\mu}(t, z) + \frac{p'^{2}}{2m}$$

(62)

$$p'_{\mu} = -\alpha \frac{\partial}{\partial z_{\mu}}$$

acting on functions

$$W'(t, z) = e^{1/2}(q(z))\, W(t, q(z)) \tag{63}$$

where

$$T'_{\mu}(t, z) = T_{\mu}(t, q(z)) + \frac{\alpha}{2m} \left(e^{-1/2} \frac{\partial e^{1/2}}{\partial z_{\mu}} + \frac{\partial e^{1/2}}{\partial z_{\mu}} + \right.$$

$$\left. + \quad e^{\nu}_{j} \frac{\partial e^{j}_{\mu}}{\partial z_{\nu}} + 2\, e^{\mu}_{j} \frac{\partial e^{j}_{\nu}}{\partial z_{\nu}} \right) \tag{64.1}$$

$$E'_o(t, z) = E_o(t, q(z)) - \alpha(T_\mu(t, z) + \frac{\alpha}{2m}(e_j^\nu \frac{\partial e_\mu^j}{\partial z_\nu} +$$

$$+ e_j^\mu \frac{\partial e_\nu^j}{\partial z_\nu}) e^{-1/2} \frac{\partial e^{1/2}}{\partial z_\mu} - \frac{\alpha^2}{2m}((e_\mu^j \frac{\partial e_j^\nu}{\partial z_\mu})^2 +$$

$$+ (e^{-1/2} \frac{\partial e^{-1/2}}{\partial z_\mu})^2 + e^{-1/2} \frac{\partial e^{1/2}}{\partial z_\mu} e_j^\mu \frac{\partial e^j}{\partial z_\mu} +$$

$$+ \frac{\partial}{\partial z_\mu}(e^{-1/2} \frac{\partial e^{1/2}}{\partial z_\mu} + e_j^\mu \frac{\partial e_\nu^j}{\partial z_\nu}) - \frac{\partial e_\mu^j}{\partial z_\mu} \frac{\partial e_j^\nu}{\partial z_\nu}) \qquad (64.2)$$

and

$$T_\mu(t, q(z)) = e_\mu^j(q)\, T_j(t, q(z))$$

$$e(z) = \det \| e_j^\mu(q(z)) \| \qquad (65)$$

It was shown in sects. 2 and 3 that eq. (62) is associated with the same Markoff process as the Euclidean Schrödinger equation. The theorem is thus proven by transforming (58) into (62).

A further generalisation of eq. (58) is possible, namely to define the momentum $\bar{p}_j$ as the covariant derivative with respect to the metric $g_{ij}(q)$. Then $g_{ij}(q)$ is not required in the last term of (58). The stochastic process associated with the resultant equation also represents the Euclidean Schrödinger equation in the corresponding Riemannian manifold.

The method of quantisation considered in this paper is conceptually very simple. It does not require any drastic revision of classical notions. As far as observable consequences are concerned it is equivalent to traditional quantum mechanics and adds nothing new to the latter. Its main postulate, the assumption of universal Brownian motion underlying quantum phenomena, does however present some problem of physical interpretation. This postulate should perhaps be considered only as a model interpretation of the consequence of attributing quantum fields with stochastic properties. The most salient feature of these fields, to be abstracted from the results of this paper, is that one and the same Markoff field is associated with propagation in Euclidean space-time and relaxation in Minkowski space. Brownian motion has long been used to model the dynamics of certain types of these latter processes. That stochastic fields in the Euclidean and Minchowski space-time can be related in this way was first pointed out by Guerra and Ruggiero. We have exhibited this connection in a different, but very direct, way in the framework of non-relativistic quantum mecha nics. In the process it became clear that the relationship between the Schrödinger and Fokker-Planck equations is more than formal. The forward and backward derivatives of a stochastic process can be defined directly in terms of the Ehrenfest theorem using the Fokker-Planck operator. The F P equation plays therefore a central role in stochastic quantisation. The scheme derives its peculiarity from this fact.

REFERENCES.

[1] E. Nelson; Phys. Rev. 150, 1079 (1966).

[2] K. Symanzik; Rendiconto S. I. F. Varenna, Course VL, edited by R. Jost; Academic Press (New York, 1969).

[3] E. Nelson; Quantum Fields and Markoff Fields, in Amer. Math. Soc. Proceedings on Partial Differential Equations (Berkeley, 1971).

[4] E. Nelson; Journ. Funct. Anal. 12, 97 (1973).

[5] E. Nelson; Journ. Funct. Anal. 12, 211 (1973).

[6] F. Guerra and P. Ruggiero; Phys. Rev. Lett. 31, 1022 (1973).

[7] For the Fokker-Planck equation see R. L. Stratonovich; Topics in the Theory of Random Noice, Vol. 1; Gordon and Breach (New York, 1963).

[8] G. E. Uhlenbeck and L. S. Ornstein; Phys. Rev. 36, 823 (1930).

[9] R. Graham; Z. Phys. B26, 397 (1977).

[10] L. Garrido, D. Luriè and M. San Miguel. Phys. Lett. 67A, 243 (1978).