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F. Palumbo: NONRELATIVISTIC SUPERSYMMETRY.

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ABSTRACT.

The essential concepts of supersymmetry are illustrated by means of relativistic examples. Nonrelativistic supersymmetry is introduced and some representations are discussed, showing the possible relevance to the physics of many-body systems.

# 1. - INTRODUCTION.

Relativistic supersymmetry [1, 2] is an extension of the relativistic space-time groups obtained by associating to the ordinary generators spinorial generators which obey anticommutation relations. The resulting algebraic structure is called a graded Lie algebra.

This construction is quite remarkable because it is a truly relativistic spin-containing symmetry whose irreducible representations combine a finite number of fermions and bosons. If the spinorial generators belong to some representation of an internal symmetry group, the resulting algebra provides a fusion between space-time and internal symmetry overcoming stated no-go theorems [3].

Local field theories which are supersymmetric can be constructed, and gauge invariance is consistent with supersymmetry. Moreover the higher degree of symmetry makes softer the divergences of relativistic quantum field theory.

In view of the appealing features of relativistic supersymmetry briefly mentioned above, it appears interesting to explore the possibility of constructing nonrelativistic supersymmetry. Such a construction is indeed possible and preliminary results show that it can be relevant to many-body systems.

In order to introduce the main concepts we will first see a relativistic example.

## 2. -RELATIVISTIC SUPERSYMMETRY.

The spinorial generators associated to the generators of the Poincarè group are Majorana spinors, namely selfconjugate spinors (they are real in the basis where the $\gamma$-matrices are real). These spinors, describing sefconjugate particles (no antiparticles), have only four real indendependent components, equivalent to two complex independent components. The simplest graded Lie algebra is obtained by associating to the Poincarè algebra the following commutation and anticommutation relations

$$\left[ Q_\alpha , M^{\mu\nu} \right] = i \, (\sigma^{\mu\nu} Q)_\alpha \tag{1}$$

$$\left[ Q_\alpha , P^\mu \right] = 0 \tag{2}$$

$$\left\{ Q_\alpha , Q_\beta \right\} = -2 \, (\gamma \cdot P)_{\alpha\beta} \, . \tag{3}$$

Eq. (1) tells that the generators $Q_\alpha$ are spinors. They do not commute with spin and therefore change bosons into fermions and vice-versa.

Eq. (2) tells that $Q_\alpha$ commutes with energy and momentum. This allows us to write eq. (3) in the rest frame

$$\left\{ Q_\alpha , Q_\beta \right\} = 2 \delta_{\alpha\beta} \, H, \tag{4}$$

which shows that the Hamiltonian is the square of the spinorial generator.

Since $Q_\alpha$ commutes with H, eigenstates occur in the following multiplets

$$|\Psi\rangle, \; Q_1 \, |\Psi\rangle, \; Q_1 Q_2 \, |\Psi\rangle, \; Q_1 Q_2 Q_3 \, |\Psi\rangle, \; Q_1 Q_2 Q_3 Q_4 \, |\Psi\rangle \tag{5}$$

There are no other states in each multiplet because the product of two identical components of Q only multiplyies $|\Phi\rangle$ by its energy eigenvalue.

Let us now discuss the representations. The minimum set of fields which constitutes a linear representation is the so called scalar or chiral multiplet introduced by Wess and Zumino[1]. It includes two scalars A(x) and F(x), two pseudoscalars B(x) and G(x) and a Majorana spinor $\psi$(x) transforming into each other according to the following equations

$$\delta_\eta A(x) = i\,\bar\eta\,\psi\,(x)$$

$$\delta_\eta B(x) = i\bar\eta\ \gamma_5\,\psi\,(x)$$

$$\delta_\eta \psi(x) = \frac{\partial}{\partial x^\mu}\Big[A(x) - \gamma_5\,B(x)\Big]\gamma^\mu\,\eta + \Big[F(x) + \gamma_5\,G(x)\Big]\eta \qquad (6)$$

$$\delta_\eta F(x) = i\,\bar\eta\,\gamma^\mu\,\frac{\partial}{\partial x^\mu}\,\psi\,(x)$$

$$\delta_\eta G(x) = i\,\bar\eta\ \gamma_5\,\gamma^\mu\,\frac{\partial}{\partial x^\mu}\,\psi\,(x).$$

In the above equations $\delta_\eta$ represents the supertransformation by the infinitesimal (spinor) parameter $\eta$. For quantized fields

$$\delta_\eta A(x) = \Big[\eta Q,\ A(x)\Big]. \qquad (7)$$

It can be shown that only A(x), B(x) and $\psi$(x) are physical fields, while F(x) and G(x) are auxiliary fields. This is according to the general requirement that the number of fermion and boson degrees of freedom must be the same in each multiplet, in order that bosons can be changed into fermions and vice versa by supertransformations.

A possible lagrangian density invariant under the supertransforma

tions (6) is

$$\mathcal{L} = -\frac{1}{2} \left[ (\frac{\partial}{\partial x^{\mu}} A)^2 + (\frac{\partial}{\partial x^{\mu}} B)^2 + i \bar{\psi} \gamma^{\mu} \frac{\partial}{\partial x^{\mu}} \psi + m^2 A^2 + m^2 B^2 + \right.$$

$$\left. + i m \bar{\psi} \psi \right] - gm A (A^2 + B^2) - \frac{1}{2} g^2 (A^2 + B^2)^2 - i g \bar{\psi} (A - \gamma_5 B) \psi \,, \quad (8)$$

where the auxiliary fields have been eliminated. It should be noted that the ratio of the Yukawa coupling to the scalar couplings is fixed by the requirement of invariance under eq. (6). This supersymmetric value of the ratio is a critical one. For greater values the vacuum is unstable, while it is otherwise stable[4].

It is very interesting that representations of the graded Poincaré group can also be given in terms of a fermion field only. This fermion field must therefore transform in itself under supertransformations. A nonlinear representation of this kind has been given by Volkov and Akulov[5]

$$\delta_{\eta} \lambda = \frac{1}{a} \eta + i a (\bar{\eta} \gamma^{\mu} \lambda) \frac{\partial}{\partial x^{\mu}} \lambda \quad , \tag{9}$$

a being a universal constant. A Lagrangian which is invariant under (9) is

$$\mathcal{L} = -\frac{1}{2 a^2} \det (g^{\nu}_{\mu} + i a^2 \bar{\lambda} \gamma^{\nu} \frac{\partial}{\partial x^{\mu}} \lambda) \quad . \tag{10}$$

Eqs. (9) and (10) are valid in any number of dimensions. Let us note for futher reference that in four dimensions the above Lagrangian contains two-, three- and four-body interactions, with strengths $a^4$, $a^6$ and $a^8$ respectively.

The Lagrangian (10) is not renormalizable. Nonlinear representations, however, are interesting in the discussion of spontaneous supersymmetry breaking, which must play a major role in supersymmetry. This is because all the particles combined in the same multiplet must have the same mass. Since no fermions and bosons having the same mass are known, with the possible exception of massless particles, supersymmetry must be broken. When this occurs spontaneously, a massless Goldstone spinor must appear, which is expected to belong to a nonlinear representation of supersymmetry. In order to understand how this can happen, we show how linear and nonlinear representations can be related to each other. We will do this for the Wess-Zumino and Volkov-Akulov model in two dimensions. In two dimensions the Wess-Zumino model contanis the fields $A, \psi$ and F only, transforming according to

$$\delta_\eta A = i \, \bar\eta \, \psi$$

$$\delta_\eta \psi = \gamma^\mu \frac{\partial}{\partial x^\mu} A_\eta + F_\eta \qquad\qquad (11)$$

$$\delta_\eta F = i \eta \gamma^\mu \frac{\partial}{\partial x^\mu} \psi \quad .$$

The composite fields

$$A = \frac{1}{2} \bar\lambda \lambda$$

$$\psi = \frac{1}{a} \lambda - \frac{1}{2} a \, \bar\lambda \lambda \gamma^\mu \frac{\partial}{\partial x^\mu} \lambda \qquad\qquad (12)$$

$$F = \frac{1}{a^2} + i \bar\lambda \lambda + \frac{a^2}{2} \bar\lambda \lambda \frac{\partial}{\partial x^\mu} \bar\lambda \gamma_5 \frac{\partial}{\partial x^\nu} \lambda \, \varepsilon^{\mu\nu}$$

transform[6] according to eqs. (11) if $\lambda$ transforms according to eq. (9). Therefore F has the necessary form for spontaneous symmetry breaking according to standard definitions[7], and $\lambda$ is the Goldstone spinor.

Formulas analogous to (12) can be given for the four-dimensional case[6].

Let us note for later reference that if the number of $\psi$- fermions is conserved, the number of $\lambda$- fermions must not necessarily be conserved.

## 3. - NONRELATIVISTIC FIELD THEORY.

Now we introduce the notations and definitions necessary to discuss nonrelativistic supersymmetry.

A system will be said Galilean if its state vectors (more properly rays) form a representation of the Galilei group. The algebra of the generators of this group is

$$
\begin{bmatrix} J_i, P_j \end{bmatrix} = i\, \varepsilon_{ijk}\, P_k
$$

$$
\begin{bmatrix} J_i, J_j \end{bmatrix} = i\, \varepsilon_{ijk}\, J_k
$$

$$
\begin{bmatrix} J_i, H \end{bmatrix} = \begin{bmatrix} P_i, H \end{bmatrix} = \begin{bmatrix} P_i, P_j \end{bmatrix} = 0
$$

(13)

$$
\begin{bmatrix} K_i, H \end{bmatrix} = i\, P_i
$$

$$
\begin{bmatrix} J_j, K_j \end{bmatrix} = i\, \varepsilon_{ijk}\, K_k
$$

$$
\begin{bmatrix} K_i, P_j \end{bmatrix} = \delta_{ij}\, m
$$

(14)

$$
\begin{bmatrix} K_i, K_j \end{bmatrix} = 0
$$

In the above equations $P_i$ and $J_i$ are the i-th components of the total momentum and angular momentum resp., $K_i$ is the generator of Galilean boosts in the i-th direction   and H is the Hamiltonian.

Eqs. (13) define the subalgebra of rotations and space-time translations. The representations of the group of rotations and space-time translations, rather than the representations of the full Galilei group, are often used to describe infinite many-body systems. Well known examples are the electron gas coupled to fonons in crystals, or nucleons coupled to pions in nuclear matter. The justification for this is that in infinite systems Galilean invariance is always spontaneously broken, because a Galilei transformation produces an infinite change of energy and therefore is not unitarily implementable. If, however, one wants to study the spontaneaus breaking of Galilean invariance[8], or if one wants to study finite systems like nuclei, Galilean invariance is necessary.

In order to proceed further we must distinguish the case i) in which we have a theory of selfinteracting fermions from the case ii) in which we have a theory of fermions and bosons.

In case i) the general form of a translation invariant Hamiltonian is

$$
\begin{aligned}
H = &\int dx \int dy \left\{ \lambda_\alpha^+ (x) \, t_{\alpha\beta} (y) \lambda_\beta (x + y) + \left[ D_{\alpha\beta} (y) \lambda_\alpha^+ (x) \lambda_\beta^+ (x + y) + h.\,c. \right] \right\} \\
&+ \int dx \int dy \int dx' \int dy' \left\{ V_{\alpha\beta\gamma\delta} (y, y', x - x') \lambda_\alpha^+ (x) \lambda_\alpha^+ (x + y) \lambda_\delta (x' + y') \lambda_\gamma (x') \right. \\
&+ \left[ U_{\alpha\beta\gamma\delta} (y, y', x' - x') \lambda_\alpha^+ (x) \lambda_\beta^+ (x + y) \lambda_\delta^+ (x' + y') \lambda_\gamma^+ (x') + h.\,c. \right] \quad + \quad (15) \\
&+ \left[ W_{\alpha\beta\delta} (y, y', x' - x') \lambda_\alpha^+ (x) \lambda_\beta^+ (x + y) \lambda_\gamma^+ (x') \lambda_\delta (x' + y') + h.\,c. \right] + \cdots \right\}
\end{aligned}
$$

where $\lambda_\alpha$ (x) is the fermion field satisfying canonical anticommutation relations, and we introduce the convenction of sum over repeated indices. The dots stand for many-body interactions which can be present, like in the nucleon-nucleon interaction.

We will say that the above Hamiltonian is local if all the functions appearing in it factorise into $\delta$-functions of their arguments, for instance

$$V_{\alpha\beta\gamma\delta} (y, y', x - x') = V_{\alpha\beta\gamma\delta} \, \delta(y) \, \delta (y') \, \delta (x - x'). \tag{16}$$

We use this terminology to be close to the language of relativistic field theory, though in the usual many-body language the potential $V_{\alpha\beta\gamma\delta}(y, y', x - x')$ is said to be local if

$$V_{\alpha\beta\gamma\delta}(y, y', x - x') = V_{\alpha\beta\gamma\delta}(y) \, \delta (y - y') \, \delta (x - x').$$

Let us now determine under which conditions H is Galilei invariant. Under a Galilei transformation the fermion field transforms according to

$$\lambda_\alpha (x, t) \longrightarrow D_{\beta\alpha}^{1/2} \, \lambda_\beta (x - vt, t) \exp \left\{ i \left[ m_\lambda \, v x + \frac{m_\lambda \, v^2}{2} t \right] \right\} \tag{17}$$

where $m_\lambda$ is the fermion mass, v is the relative velocity of the Galilean frames and $D_{\beta\alpha}^{1/2}$ is the two-dimensional ray representation of the rotation group. As is well known, invariance under the transformation (17) requires mass conserving interactions; (in the present case D = U = W = 0), and determines the dependence of V from x - x' to be

$$V_{\alpha\beta\gamma\delta}(y, y', x - x') = V_{\alpha\beta\gamma\delta}(y, y')\, \delta \left[ 2(x - x') + y - y' \right].\tag{18}$$

In case ii) let us assume for the sake of definiteness to have a single fermion field $\psi(x)$ and two scalar fields $A(x)$ and $B(x)$. The general form of a translation - invariant Hamiltonian involving these field is

$$
\begin{aligned}
H = \int dx \int dy & \left[ t_{\psi\alpha\beta}(y)\psi_\alpha^+(x)\psi_\beta(x + y) + t_A(y)\, A^+(x)\, A(x + y) \right. \\
& \left. + t_B(y)\, B^+(x)\, B(x + y) \right] + \int dx \int dy \int dz \left\{ \psi_\alpha^+(x)\,\psi_\beta(x + y) \right. \\
& \left[ G^{(1)}_{\alpha\beta}(y, z)\, A(x + z) + G^{(2)}_{\alpha\beta}(y, z)\, B(x + z) + \text{h. c.} \right\} + \\
& + \int dx \int dy \int dx' \int dy' \left\{ G^{(3)}_{\alpha\beta\gamma\delta}(y, y', x - x')\psi_\alpha^+(x)\psi_\beta^+(x + y) \right. \\
& \cdot\, \psi_\delta(x' + y')\,\psi_\gamma(x') + G^{(4)}(y, y', x - x')\, A^+(x)\, A^+(x + y)\, A(x' + y')\, A(x') \\
& + G^{(5)}(y, y', x - x')\, A^+(x)\, B^+(x' + y')\, A(x' + y')\, B(x') \\
& + G^{(6)}_{\alpha\beta}(y, y', x - x')\psi_\alpha^+(x)\, A^+(x + y)\,\psi_\alpha(x' + y')\, A(y') + \\
& \left. + \left[ G^{(7)}_{\alpha\beta}\,\psi_\alpha^+(x)\psi_\beta^+(x + y)\, B(x' + y')\, A(y') + \text{h. c.} \right] \right\} + \cdots
\end{aligned}\tag{19}
$$

The theory is said to be local if formulas analogous to eq. (16) hold. Again Galilean invariance requires mass conservation. This implyies that trilinear couplings must be absent unless $m_A = m_B = 0$, and that $G^{(7)}_{\alpha\beta} = 0$ unless $2m_\psi = m_A + m_B$, $m_\psi$, $m_A$ and $m_B$ being the masses of the fermion and of the bosons. Moreover the dependence on $x - x'$ in the functions $G^{(i)}$ must factor out into a $\delta(2x - 2x' + y - y')$ in analogy with eq. (18).

We are finally in the position to discuss the graduation of the group of rotations and space-time translations and of the full Galilei group.

## 4. - NONRELATIVISTIC SUPERSYMMETRY.

We will consider 4 graded algebras.

ALGEBRA I.

Is the simplest algebra containing only the Hamiltonian and the spinorial generator $Q_\alpha$

$$\{Q_\alpha , Q_\beta\} = 0$$

$$\{Q_\alpha , Q_\beta^+\} = H$$

$$[Q_\alpha , H] = 0 \tag{20}$$

It generates the graded time-translations group.

ALGEBRA II.

Contains $J_i$, $P_i$,H and the spinorial generator $Q_\alpha$ obeying eqs. (13) and

$$[Q_\alpha , P_i] = 0$$

$$[Q_\alpha , J_i] = \frac{1}{2} (\sigma_i Q)_\alpha$$

$$[Q_\alpha , H] = 0 \tag{21}$$

$$\{Q_\alpha , Q_\beta\} = 0$$

$$\{Q_\alpha , Q_\beta^+\} = \delta_{\alpha\beta} H ,$$

where $\sigma_k$ are the Pauli matrices.

It generates the graded rotations and space-time translations group.

ALGEBRA III

Contains the algebra of the Galilei group and the spinorial generator $R_\alpha$ obeying eqs. (13), (14) and

$$\left[ R_\alpha , J_i \right] = \frac{1}{2} (\sigma_i R)_\alpha$$

$$\left[ R_\alpha , P_i \right] = 0$$

$$\left[ R_\alpha , H \right] = 0 \qquad\qquad (22)$$

$$\left[ R_\alpha , K_i \right] = 0$$

$$\left\{ R_\alpha , R_\beta \right\} = 0$$

$$\left\{ R_\alpha , R_\beta^+ \right\} = \delta_{\alpha\beta}\, m\ .$$

It generates the simplest graded Galilei group[9], and is obtained by contraction (velocity of ligth $\rightarrow \infty$) of the graded Poincarè group discussed in sect 2. It should be noted that unlike the relativistic case the spinorial generator is not related to the Hamiltonian, but to the mass, which plays the role of a central charge. Central charges can appear also in relativistic algebras, where it is not possible however, to have only central charges anticommuting spinorial generators.

ALGEBRA IV.

Contains the generators of the Galilei group and two spinorial[9] generators $Q_\alpha$ and $R_\alpha$[10] satisfying eqs. (13), (14), (22) and

$$\left[ Q_\alpha , K_i \right] = -\frac{i}{2} \sigma_{i\alpha\beta} R_\beta$$

$$\left[ Q_\alpha , J_i \right] = \frac{1}{2} \sigma_{i\alpha\beta} Q_\beta$$

$$\left\{ R_\alpha , Q_\beta^+ \right\} = -\frac{1}{2} \sigma_{i\alpha\beta} P_i - c\, \delta_{\alpha\beta}\, m \qquad (23)$$

$$\left\{ Q_\alpha , Q_\beta \right\} = 0$$

$$\left\{ Q_\alpha , Q_\beta^+ \right\} = \delta_\alpha{}^\beta H + c\, (\vec{\sigma}, \vec{P})\, \delta_{\alpha\beta}\ ,$$

where c is an arbitrary constant. It is to be noted that in order to re-
late the spinorial generator $Q_\alpha$ to the Hamiltonian as in the relativi-
stic case, a second spinorial generator is needed. In fact $Q_\alpha$ cannot
be assumed to commute with $K_i$ , otherwise the last equation above
would be in contrast with the first of eqs. (14). Therefore $K_i$ must
not commute with $Q_\alpha$ , and their commutator must generate the se-
cond spinorial generator $R_\alpha$ .

Let us now review the representations of the above algebras.

Algebra I has been considered by Nicolai[11] who gave a nonlinear re
presentation of it in terms of a fermion field describing a spin sy-
stem on a lattice. The peculiarity of this theory is the vanishing of
certain correlation functions. Since this subject is rather far from
the subject of the present congress, we will not discuss it any futher.

Algebra II has been considered by the author to test his conjec-
ture of a relation between supersymmetry and superconductivity[12].
This conjecture is based on the following remark.

As we have seen supersymmetry has been realized both in terms
of boson and fermion fields. Moreover we have seen that the boson
and fermion fields of the linear representation can be expressed as
composite fields in terms of the fermion field of the nonlinear one.
The Hamiltonian commutes with the spinorial generator which chan-
ges bosons into fermions and vice-versa. Now also the theory of su
perconductivity has been formulated both as a theory of fermions
(electrons) and bosons (fonons), and as a theory of self interacting
fermions. The superconducting state results from a dynamical equi-
librium where composite bosons (formed by two electrons) are con-
verted into electrons and vice-versa.

In order to check the above conjecture a nonlocal nonlinear representation of algebra II has been constructed. In order to get a Hamiltonian quadratic and quartic in the fermion field, the spinorial generator has been taken linear and cubic. The most general expression for it taking into account translational invariance is

$$Q_\alpha = \int dx \left[ a_\alpha^{\,\beta} \lambda_\beta^+(x)\, b_\alpha^{\,\beta} \lambda_\beta(x) \right] + \int dx_1 \int dx_2 \int dx_3 \left\{ \lambda_\beta^+(x_1)\, \lambda_\gamma^+(x_2) \right.$$

$$\left[ \lambda_\delta(x_3)\, F_\alpha^{(1)\beta\gamma\delta}(x_i - x_j) + \lambda_\delta^+(x_3)\, F_\alpha^{(2)\beta\gamma\delta}(x_i - x_j) \right] \qquad (24)$$

$$+ \left[ \lambda_\delta^+(x_3)\, F_\alpha^{(3)\beta\gamma\delta}(x_i - x_j) + \lambda_\delta(x_3)\, F_\alpha^{(4)\beta\gamma\delta}(x_i - x_j) \right] \lambda_\gamma(x_2)\lambda_\beta(x_1) \right\}$$

where $a_\alpha^{\,\beta}$ and $b_\alpha^{\,\beta}$ are constant matrices and $F_\alpha^{(k)\beta\gamma\delta}$ are arbitrary functions. Confining to the case that the $F_\alpha^{(k)\beta\gamma\delta}$ are functions of a single argument, a Hamiltonian of the general form (15) is obtained, where $V_{\alpha\beta\gamma\delta}$, $U_{\alpha\beta\gamma\delta}$ and $W_{\alpha\beta\gamma\delta}$ are independent of $x - x'$, and are expressed in terms of the $F_\alpha^{(k)\beta\gamma\delta}$. It turns out that eqs. (13) and (21) can be satisfyed only if the ground state is a BCS state, in agreement with the mentioned conjecture. Apart from the constraints arising from the above condition (that the ground state be a BCS state) and rotational invariance, the $F_\alpha^{(k)\beta\gamma\delta}$ are arbitrary. In particular it is possible to choose them in such a way that in eqs. (15)

$$D = U = W = 0$$

$$t_{\alpha\beta}(y) = 8\,\Omega\, a\, t^{(1)}(y)\delta_{\alpha\beta}$$

$$\qquad (25)$$

$$V_{\alpha\beta\gamma\delta}(y, y', x - x') = + 2\,\Omega^2\,(\sigma_2)_{\alpha\beta}\,(\sigma_2)_{\gamma\delta}\, f^{(1)}(y)\, f^{(1)}(y') -$$

$$- 8\,\delta_{\alpha\gamma}\,\delta_{\beta\delta}\,\Omega^2\, t^{(1)}(y)\, t^{(1)}(y'),$$

where $t^{(1)}(y)$ and $f^{(1)}(y)$ are square integrable functions otherwise arbitrary, a is an arbitrary constant and $\Omega$ the volume of the quantization box of the field $\lambda_\alpha(x)$. Assuming in momentum space

$$t^{(1)}(p) = + \frac{1}{16 \, a \, m_\lambda \Omega} \, p^2 \, \vartheta(P - p), \tag{26}$$

where $m_\lambda$ is the fermion mass, P a cut-off parameter and $\vartheta(x)$ the step function, the BCS Hamiltonian itself can be approximated to arbitrary accuracy by the Hamiltonian (15) by taking a large enough[13].

Let us finally come to the representations of the Galilei group. They have been derived only for algebra III by Puzalowski, who gave all the local linear representations. We will confine ourselves here with the smallest multiplet, which contains two scalars A (x) and B(x), and a Pauli spinor $\psi(x)$ whose transformation laws are

$$\delta_\eta A(x) = \left[ R_\alpha^+ \eta_\alpha + \eta_\alpha^* R_\alpha \, , \, A(x) \right] = \sqrt{m} \, \eta_\alpha \, \varepsilon_{\alpha\beta} \, \psi_\beta(x)$$

$$\delta_\eta B(x) = \sqrt{m} \, \eta_\alpha^* \, \psi_\alpha(x) \tag{27}$$

$$\delta_\eta \psi_\alpha(x) = - \sqrt{m} \left[ B(x) \, \eta_\alpha + A(x) \, \varepsilon_{\alpha\beta} \, \eta_\beta \right] ,$$

where $\varepsilon_{12} = - \varepsilon_{21} = 1$. As previously noted the Hamiltonian in the present case is not determined by the rappresentation. Puzalowski[9], however, has also found a Hamiltonian invariant under eqs. (27). It is of the general form (19) with

$$G_{\alpha\beta}^{(1)} = G_{\alpha\beta}^{(2)} = 0$$

$$t_{\psi\alpha\beta}(y) = - \frac{1}{2 \, m} \, \delta(y) \, \Delta_y \, \delta_{\alpha\beta} \tag{28}$$

$$t_A(y) = t_B(y) = - \frac{1}{2 \, m} \, \delta(y) \, \Delta_y$$

$$G^{(3)}_{\alpha\beta\gamma\delta}(y, y', x - x') =$$

$$G^{(4)}(y, y', x - x') =$$

$$G^{(5)}(y, y', x - x') = \qquad G(y, y', x - x') \cdot \begin{cases} -\dfrac{1}{2}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\delta\gamma} \\[4pt] 2 \\[4pt] 2 \\[4pt] 4\delta_{\alpha\beta} \\[4pt] -\varepsilon_{\alpha\beta} \end{cases} \qquad (29)$$

$$G^{(6)}_{\alpha\beta}(y, y', x - x') =$$

$$G^{(7)}_{\alpha\beta}(y, y', x - x') =$$

where

$$G(y, y', x - x') = g\,\delta(y)\,\delta(y')\,\delta(x - x') \qquad (30)$$

g being an arbitrary constant and m the common fermion-boson mass. This Hamiltonian is hardly relevant to phenomenology because the fermion number is not conserved, like in the Wess-Zumino model of which it is the nonrelativistic counterpart. However it is interesting to investigate whether it shares the peculiar vacuum property of the Wess-Zumino model (separating a class of interactions with respect to which the vacuum is stable from a class of interactions with respect to which the vacuum is unstable [4]) and whether it can be generalized to a nonlocal one. Only Hamiltonians nonlocal in the sense of the previous section can in fact be phenomenologically relevant. To finish with, therefore, we would also like to know how vacuum properties depend on locality.

Let us start from extension to nonlocality. It is easy to cheek [14] that invariance under transformations (27) is mantained if the function $G(y, y', x - x')$ of eq. (29) has either of the following expressions

$$G(y, y', x - x') = g(y, y') \, \delta (2x - 2x' + y - y') \tag{31}$$

$$G(y, y', x - x') = g(y, y') \; , \tag{32}$$

with the condition that

$$\text{if } g(y, y') = g(y) \, \delta (y - y'), \quad \text{then} \quad g(y) = g \, \delta(y) \; . \tag{33}$$

In the case of eq. (31) the Hamiltonian remains invariant with re spect to Galilei transformations, while in the case of eq. (32) it does not, so that we have only a representation of the group of rotations and space-time translations. Summarizing we have three classes of functions $G(y, y', x - x')$ given by eqs. (30), (31) and (32) resp..

Let us now come to vacuum properties. To this purpose let us consider the Hamiltonian

$$H = - \frac{1}{2m} \int dx \left[ \psi_\alpha^+(x) \, \Delta \psi_\alpha (x) + A^+(x) \, \Delta A(x) + B^+(x) \, \Delta B(x) \right]$$

$$+ \int dx \int dy \int dx' \int dy' \; G(y, y', x - x') \Big\{ 4 A^+(x) A^+(x+y) A(x'+y') A(x') +$$

$$+ \; 4 A^+(x) B^+(x+y) A(x'+y') A(x') + 8 \psi_\alpha^+(x) A^+(x+y) \psi_\alpha (x'+y') A(x') -$$

$$\tag{34}$$

$$- \; 2 \psi_\alpha^+(x) \varepsilon_{\alpha\beta} \psi_\beta^+(x+y) B(x'+y') A(x') + 2 B^+(x) A^+(x+y) \psi_\alpha (x'+y') \varepsilon_{\alpha\beta} \psi_\beta (x')$$

$$- \; (1-h) \psi_\alpha^+(x) \varepsilon_{\alpha\beta} \psi_\beta^+(x+y) \psi_\gamma (x'+y') \varepsilon_{\gamma\delta} \psi_\delta (x') \Big\} \quad ,$$

where h is a parameter. For h = 0 and G given by eq. (30), or (31) or (32) this Hamiltonian is supersymmetric. Moreover, for any value of h and any form of $G(y, y', x - x')$ it is invariant under the gauge transformation

$$A(x) \longrightarrow c^{i\alpha} A(x)$$

$$B(x) \longrightarrow c^{i\beta} B(x) \tag{35}$$

$$\psi(x) \longrightarrow c^{i(\alpha + \beta)} \psi(x) \quad,$$

where $\alpha$ and $\beta$ are arbitrary phases. This invariance is related to the conservation of the total number of particles

$$N = \int dx \left[ \psi_\alpha^+(x)\psi_\alpha(x) + A^+(x)A(x) + B^+(x)B(x) \right] \quad. \tag{36}$$

In the case where $G(y, y', x - x')$ is given by eq. (32) the ground state can be determined exactely[14] for any value of h. In such a case, in fact, we can apply Haag's theorem[15] stating that

$$D = \frac{1}{\Omega} \int dx \int dy \; f(y)\psi_\alpha(x)\varepsilon_{\alpha\beta}\psi_\beta(x + y) \tag{37}$$

is a classical quantity in the limit $\Omega \longrightarrow \infty$, $\Omega$ being the volume of the quantization box of the fields and $f(y)$ a square integrable function. As a consequence the ground state is determined by the requirement of selfconsisteney of the motion of fermions in the average boson field and of bosons in the average "quasi-boson" field. Let us assume for simplicity $g(y, y')$ to be separable

$$g(y, y') = \frac{1}{\Omega} f(y) f(y) \quad , \tag{38}$$

and let us define

$$F = \int dy\, f(y) \quad . \tag{39}$$

The normalized ground state for the boson sector can be shown[14] to be

$$\left| \Psi_{Boson} \right\rangle = \lim_{\tau \to 0} e^{-\frac{1}{2} \Omega \left[ \tau^2 + \frac{D^2}{4\,\tau^2 F^2} \right]} e^{\tau \sqrt{\Omega}\, a_o^+}$$

$$e^{-\frac{D}{2\tau F} \sqrt{\Omega}\, b_o^+} \left| \Psi_o \right\rangle \quad , \tag{40}$$

where

$$a_o = \sqrt{\Omega} \int dx\, A(x)$$

$$b_o = \sqrt{\Omega} \int dx\, B(x) \tag{41}$$

and $\left| \Psi_o \right\rangle$ is the vacuum for the fields $A(x)$, $B(x)$ and $\psi_\alpha(x)$. The ground state for the fermion sector is $\left| \Psi_o \right\rangle$ or the BCS state according to whether the gap parameter D is vanishing or not. Now the total potential energy in the ground state is $- h\, D^2$. For $h \leqslant 0$, $D = 0$, $\left| \Psi_{Bosons} \right\rangle = \left| \Psi_o \right\rangle$, and the vacuum is stable. For $h > 0$, on

the other hand, $D \neq 0$ and the vacuum is unstable with respect to crea tion of a condesate of Cooper pairs of fermions and a Bose-Einstein condensate of B - bosons, while A - bosons are absent. For $h > 0$ the gauge invariance of eqs. (35) is spontaneously broken.

When $G(y, y', x - x')$ is given by eq. (30) the ground state cannot be determined exactely, but the same vacuum property can be established by a variational procedure. This vacuum property is therefo re independent of locality.

The fact that supersymmetry is not spontaneausly broken (for $h = 0$ the vacuum is stable), could have been guessed starting from a different point. When supersymmetry is spontaneausly broken, the Goldstone spinor is expected to transform according to a nonlinear representation of the graded group. But it can be shown[14] that the algebra III does not admit nonlinear representations, and therefore there can be no Goldstone spinor.

It is perhaps worth while to note, in connection with the conjectured relation between supersymmetry and superconductivity, that the supersymmetric value $h = 0$ separates the class of Hamiltonians with vacuum stability from the class of Hamiltonians with supercon ducting ground state.

5. - CONCLUDING REMARKS.

The breef review of the recent progress in nonrelativistic super symmetry shows that it can be relevant to many-body systems. The linear representations which have been found do not oppear to be phy

sically relevant, because of fermion non conservation and of the absence of spontaneous supersymmetry breaking, implying that the masses of fermions and bosons remain equal. However we have seen that it is possible to generalize to non-local models and that in the case studied the peculiarity of the relativistic model is shared by its nonrelativistic counterpart both in the local and non-local version. This supports the hope that also algebras II and IV will have both local and nonlocal representations, whose properties will bear some relation to the properties of relativistic models.

It remains to find such representations and to see whether they give rise to spontaneous supersymmetry breaking. In the affirmative case the linear representations can be replaced by the nonlinear ones describing selfinteracting fermions. The conespon ding Hamiltonians may well not conseve the number of fermions[12], and will generally contain many-body interactions.

REFERENCES AND FOOTNOTES. -

(1) - J. Wess and B. Zumíno, Nucl. Phys. $\underline{70B}$, 39 (1974).

(2) - P. Fayet and S. Ferrara, Phys. Rep. $\underline{32}$, 250 (1977).

(3) - L. O'Raifertaigh, Phys. Rev. Lett. , $\underline{14}$ , 575 (1965); Phys. Rev. , $\underline{139}$, B 1052 (1965).

(4) - T. L. Cutright and G. I. Ghandour, Phys. Lett. $\underline{59B}$, 387 (1975).

(5) - D. V. Volkov and V. P. Akulov, Phys. Lett. $\underline{46B}$, 109 (1973).

(6) - M. Roček, Phys. Rev. Lett. $\underline{41}$, 451 (1978).

(7) - A. Salam and J. Strathdee, Phys. Lett. $\underline{49B}$, 465 (1974); S. Browne, Phys. Lett. $\underline{59B}$, 253 (1975).

(8) - J. A. Swieca, Commun. Math. Phys. $\underline{4}$, 1 (1967).

(9) - R. Puzalowski, Karlsruhe preprint (1978).

(10) - The role of $R_\alpha$ and $Q_\alpha$ is changed here with resp. to ref. (9).

(11) - H. Nicolai, J. Phys. A $\underline{9}$, 1497 (1976); Nuovo Cim. $\underline{41}$, 1 (1977).

(12) - F. Palumbo, Phys. Lett. $\underline{A}$ , submitted to.

(13) - The actual limit a $\longrightarrow \infty$ cannot be taken, due to an additive constant which in this limit is divergent. For details see ref. (12).

(14) - F. Palumbo, to be published.

(15) - R. Haag, Nuovo Cim. $\underline{25}$, 287 (1962).