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M. Greco, F. Palumbo, G. Pancheri-Srivastava and Y. Srivastava: COHERENT STATE APPROACH TO THE INFRA-RED BEHAVIOUR OF NON-ABELIAN GAUGE THEORIES.

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M. Greco, F. Palumbo, G. Pancheri-Srivastava<sup>(\*)</sup> and Y. Srivastava<sup>(\*)</sup>:  
COHERENT STATE APPROACH TO THE INFRA-RED BEHAVIOUR OF  
NON-ABELIAN GAUGE THEORIES.

ABSTRACT.

The infrared behaviour of matrix elements between coherent states of definite color is studied in non abelian gauge theories. The matrix elements are shown to be finite to the lowest non trivial order, and to all orders if the soft meson formula for real gluons holds to all orders. Factorization occurs in fixed angle regime.

In this letter we study the infrared (IR) behaviour of non-abelian gauge theories (NAGT) using the coherent state formalism. This method has been applied<sup>(1)</sup> in QED providing one with a definition of matrix elements which are (i) free from IR divergences, (ii) factorizable and directly comparable with the experimental cross-sections. This approach is exactly equivalent to the standard one of summing cross-sections for real and virtual photon emission<sup>(1, 2)</sup>.

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Our extension to NAGT is based on the (assumed) validity, to all orders, of the so-called soft meson formula<sup>(3)</sup>. This formula has been proved to all orders in the leading logarithms for virtual gluons<sup>(4)</sup>.

It has also been checked for single-gluon emission in different processes<sup>(5)</sup>, and for double-gluon emission both in quark scattering in external colorless potential<sup>(6)</sup> and in quark-quark scattering<sup>(7)</sup>.

In all cases the leading divergences cancel out in the inclusive cross sections after color average over both initial and final states, in agreement with the Lee and Nauenberg theorem<sup>(8)</sup>. Due to the color average, however, nothing can be said about cross sections for states of definite color. The coherent state formalism, on the other hand, is such as to provide one with matrix elements which are free from IR divergences for any initial or final color states. This formalism, therefore, turns out to be not equivalent to the standard one of summing cross-sections for real and virtual gluon emission, in contrast to QED.

The factorization properties of the new matrix elements are also different from QED. In fact, factorization occurs only in the fixed angle regime (see below for definitions), as also found for the inclusive color averaged cross section<sup>(5-7)</sup>.

Our results rely on the possibility of defining an effective coupling constant  $\bar{g}(k)$  (see eq. (3)) which makes our treatment meaningful. How to calculate  $\bar{g}(k)$  to all orders is of course an open question. For an analysis see reference (9).

We shall not discuss in this letter the connection between our results and physically measurable quantities.

For given initial and final states  $|i\rangle$  and  $|f\rangle$ , we define the corresponding coherent states  $|\tilde{i}\rangle$  and  $|\tilde{f}\rangle$ :

$$|\tilde{i}\rangle = e^{iA_i} |i\rangle, \quad |\tilde{f}\rangle = e^{iA_f} |f\rangle \quad (1)$$

with

$$A = \frac{1}{(2\pi)^4} \int d^4k j_\mu^c(k) A_\mu^c(-k). \quad (2)$$

In eq. (2)  $A_\mu^c(x)$  is the quantized gluon field of color index  $c$ , and  $j_\mu^c(k)$  is the "classical" current

$$j_\mu^c(k) = \frac{i}{(2\pi)^{3/2}} \bar{g}(k) \sum_a \frac{\eta_a p_{a\mu}}{(p_a^0 k)} t^c(a) \equiv \sum_a j_{a\mu}(k) t^c(a), \quad (3)$$

where  $a$  is the label of particles in initial and final states,  $\eta_a = +1$  for incoming particles and outgoing antiparticles and  $\eta_a = -1$  otherwise. The operators  $t^c(a)$  are the appropriate generators of the color group for the  $a$ -th particle [  $t^c(a)$  must be replaced by  $t^{c*}(a)$  for outgoing particles ] and  $\bar{g}(k)$  is an effective coupling constant.

Let us define new S-matrix elements

$$\bar{M} = \langle \tilde{f} | S | \tilde{i} \rangle = \langle f | e^{-iA_f} S e^{iA_i} | i \rangle \equiv \langle f | \bar{S} | i \rangle. \quad (4)$$

In terms of a perturbative expansion,

$$\bar{M} = \sum_{r,s,t} \langle f | \frac{(-iA_f)^r}{r!} T \left( \frac{(iA_i)^s}{s!} \right) \frac{(iA_i)^t}{t!} | i \rangle \equiv \sum_{n=n_0}^{\infty} \bar{M}_n, \quad (5)$$

with

$$\begin{aligned} \bar{M}_n &= \sum_{k=n_0}^{\infty} \sum_{\ell=n_0}^k \langle f | \frac{(-iA_f)^{k-\ell}}{(k-\ell)!} T \left( \frac{(iA_i)^\ell}{\ell!} \right) \frac{(iA_i)^{n-k}}{(n-k)!} | i \rangle \\ &= \sum_{k=n_0}^{\infty} \sum_{\ell=n_0}^k \langle f | \frac{(-iA_f)^{k-\ell}}{(k-\ell)!} S_\ell \frac{(iA_i)^{n-k}}{(n-k)!} | i \rangle, \end{aligned} \quad (6)$$

where  $n_0$  corresponds to the lowest order non-vanishing transition amplitude. Let us concentrate on  $\bar{M}_{n_0+2}$ , the first radiatively corrected matrix element. One of the three terms in  $\bar{M}_{n_0+2}$  is given by

$$\langle f | S_{n_0+1}(iA_i) | i \rangle = \frac{i}{(2\pi)^4} \int d^4k \sum_{\substack{\alpha \in \{i\} \\ \sigma}} j_{\alpha\mu}(k) \langle f | S_{n_0+1} A_\mu^c(-k) | i, \lambda_\alpha \rightarrow \sigma \rangle \langle \sigma | t^c | \lambda_\alpha \rangle. \quad (7)$$

where  $\lambda_\alpha$  is the color index of the  $\alpha$ -th particle in the initial state  $i$ .

The matrix element appearing in eq. (7) is evaluated by using the soft meson formula<sup>(3, 4)</sup> for the emission of a soft gauge vector from any on-shell process

$$\begin{aligned} \langle f | T_\nu^a(k) | i \rangle &= \sum_{\gamma \in \{i\}} \bar{g}(k) \eta_\gamma \frac{p_{\gamma\nu}}{(p_\gamma \cdot k)} \langle \lambda | t^a | \lambda_\gamma \rangle \langle f | S | i, \lambda_\gamma \rightarrow \lambda \rangle \\ &+ \sum_{\gamma \in \{f\}} \bar{g}(k) \eta_\gamma \frac{p_{\gamma\nu}}{(p_\gamma \cdot k)} \langle \lambda_\gamma | t^a | \lambda \rangle \langle f, \lambda_\gamma \rightarrow \lambda | S | i \rangle. \end{aligned} \quad (8)$$

Then eq. (7) becomes

$$\begin{aligned} \langle f | S_{n_0+1}(i\Lambda_i) | i \rangle &= \int d^4k \delta(k^2) \theta(k_0) \left\{ \sum_{\alpha \in \{i\}} j_{\alpha\mu}(k) \langle \sigma | t^c | \lambda_\alpha \rangle \right\} \\ &\cdot \left\{ \sum_{\alpha' \in \{i\}} j_{\alpha'}^\mu(-k) \langle \sigma' | t^c | \lambda_{\alpha'}(\sigma) \rangle \langle f | S_{n_0} | i, \begin{bmatrix} \lambda_\alpha \rightarrow \sigma \\ \lambda_{\alpha'}(\sigma) \rightarrow \sigma' \end{bmatrix} \rangle \right\} \quad (9) \\ &+ \sum_{\beta' \in \{f\}} j_{\beta'}^\mu(-k) \langle \lambda_{\beta'} | t^c | \varrho' \rangle \langle f, \lambda_{\beta'} \rightarrow \varrho' | S_{n_0} | i, \lambda_\alpha \rightarrow \sigma \rangle, \end{aligned}$$

where in the state  $|\lambda_{\alpha'}(\sigma)\rangle$  the color index is  $\lambda_{\alpha'}$  for  $\alpha' \neq \alpha$  and  $\sigma$  otherwise. A similar evaluation of the other terms in  $\bar{M}_{n_0+2}$  leads to the result

$$\bar{M}_{n_0+2} = \langle f | S_{n_0+2} | i \rangle + \langle f | R_{n_0+2} | i \rangle, \quad (10)$$

with

$$\begin{aligned} \langle f | R_{n_0+2} | i \rangle &= \frac{1}{2} \int d^4k \delta(k^2) \theta(k_0) * \\ &* \left\{ \sum_{\alpha, \alpha' \in \{i\}} j_{\alpha\mu}(k) j_{\alpha'}^\mu(-k) \langle \sigma | t^c | \lambda_\alpha \rangle \langle \sigma' | t^c | \lambda_{\alpha'}(\sigma) \rangle \langle f | S_{n_0} | i, \begin{bmatrix} \lambda_\alpha \rightarrow \sigma \\ \lambda_{\alpha'}(\sigma) \rightarrow \sigma' \end{bmatrix} \rangle \right\} + \\ &+ \sum_{\alpha \in \{i\}} j_{\alpha\mu}(k) j_{\beta'}^\mu(-k) \langle \sigma | t^c | \lambda_\alpha \rangle \langle \lambda_{\beta'} | t^c | \varrho' \rangle \langle f, \lambda_{\beta'} \rightarrow \varrho' | S_{n_0} | i, \lambda_\alpha \rightarrow \sigma \rangle + \end{aligned} \quad (11)$$

$$\begin{aligned}
 & + \sum_{\substack{\beta \in \{f\} \\ \alpha \in \{i\} \\ \varrho, \sigma'}} j_{\beta\mu}^{\mu}(k) j_{\alpha'}^{\mu}(-k) \langle \sigma' | t^c | \lambda_{\alpha'} \rangle \langle \lambda_{\beta} | t^c | \varrho \rangle \langle f, \lambda_{\beta} \rightarrow \varrho | S_{n_0} | i, \lambda_{\alpha'} \rightarrow \sigma' \rangle + \\
 & + \sum_{\substack{\beta, \beta' \in \{f\} \\ \varrho, \varrho'}} j_{\beta\mu}^{\mu}(k) j_{\beta'}^{\mu}(-k) \langle \lambda_{\beta} | t^c | \varrho \rangle \langle \lambda_{\beta'} | t^c | \varrho' \rangle \langle f, \left[ \begin{array}{c} \lambda_{\beta} \rightarrow \varrho \\ \lambda_{\beta'} \rightarrow \varrho' \end{array} \right] | S_{n_0} | i \rangle \left. \right\} .
 \end{aligned} \tag{11}$$

In shorthand notation

$$\begin{aligned}
 \langle f | R_{n_0+2} | i \rangle & = \frac{1}{2} \int d^4 k \delta(k^2) \theta(k_0) \langle f | \left\{ S_{n_0} j_{in\mu}^c(k) j_{in}^{c\mu}(-k) + \right. \\
 & \left. + 2 j_{out\mu}^c(k) S_{n_0} j_{in}^{c\mu}(-k) + j_{out\mu}^c(k) j_{out}^{c\mu}(-k) S_{n_0} \right\} | i \rangle .
 \end{aligned} \tag{12}$$

On the other hand, for the pure virtual gluon contribution  $\langle f | S_{n_0+2} | i \rangle$  one gets the same result, but for the replacement of  $\delta(k^2) \theta(k_0)$  by  $(-i/2\pi) \frac{1}{k^2}$ .

One concludes therefore that  $\overline{M}_{n_0+2}$  is finite<sup>(\*)</sup> and has the following form

$$\begin{aligned}
 \overline{M}_{n_0+2} & = \frac{1}{2} \int d^4 k \left[ \delta(k^2) \theta(k_0) - \frac{i}{2\pi} \frac{1}{k^2} \right] . \\
 \langle f | \left\{ j_{out\mu}^c(k) j_{out}^{c\mu}(-k) \overline{S}_{n_0} + 2 j_{out\mu}^c(k) \overline{S}_{n_0} j_{in}^{c\mu}(-k) + \right. \\
 & \left. \overline{S}_{n_0} j_{in\mu}^c(k) j_{in}^{c\mu}(-k) \right\} | i \rangle .
 \end{aligned} \tag{13}$$

We are now going to relate the matrix element  $R_{n_0+2}$  to the probability of emission of one real gluon. From eq. (8), rewritten in short as

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(\*) As stated at the beginning, we assume that  $g^{-2}(k)$  is such that our formulae are meaningful...

$$\langle f | T_\mu^c(k) | i \rangle = \langle f | j_{\text{out}\mu}^c(k) S_{n_0} | i \rangle + \langle f | S_{n_0} j_{\text{in}\mu}^c(k) | i \rangle, \quad (14)$$

we have

$$\begin{aligned} & \int d^4k \delta(k^2) \theta(k_0) \langle f | T_\mu^c(k) | i \rangle \langle f | T^{c\mu}(k) | i \rangle^* = \\ & = \int d^4k \delta(k^2) \theta(k_0) \left\{ \langle f | j_{\text{out}}^c(k) S_{n_0} | i \rangle \langle i | S_{n_0}^+ j_{\text{out}}^c(-k) | f \rangle + \right. \\ & + \langle f | j_{\text{out}}^c(k) S_{n_0} | i \rangle \langle i | j_{\text{in}}^c(-k) S_{n_0}^+ | f \rangle + \\ & + \langle f | S_{n_0} j_{\text{in}}^c(k) | i \rangle \langle i | S_{n_0}^+ j_{\text{out}}^c(-k) | f \rangle + \\ & \left. + \langle f | S_{n_0} j_{\text{in}}^c(k) | i \rangle \langle i | j_{\text{in}}^c(-k) S_{n_0}^+ | f \rangle \right\}. \end{aligned} \quad (15)$$

On the other hand, from eq. (5) we have

$$\begin{aligned} |\overline{M}|^2 &= |\overline{M}_{n_0} + \overline{M}_{n_0+2}|^2 = \left| \langle f | S_{n_0} | i \rangle \right|^2 + \\ & + \left\{ \langle f | S_{n_0} | i \rangle \langle f | S_{n_0+2} | i \rangle^* + \langle f | S_{n_0} | i \rangle \langle f | R_{n_0+2} | i \rangle^* + \right. \\ & \left. + \text{c. c.} \right\} + \text{higher orders}. \end{aligned} \quad (16)$$

From eqs. (12) and (16) it is clear that the term  $R_{n_0+2}$  in  $\overline{M}$  is equivalent to the single gluon bremsstrahlung only upon average over color of the external particles.

A recursion formula can be established for the matrix element  $\overline{M}$ ,

$$\begin{aligned} \langle f | \overline{S} | i \rangle_{n+2} &= \frac{1}{2} \int d^4k \left[ \delta(k^2) \theta(k_0) - \frac{i}{2\pi} \frac{1}{k^2} \right] \times \\ & \times \left\langle f \left| j_{\text{out}\mu}^c(k) j_{\text{out}}^{c\mu}(-k) \overline{S}_n + j_{\text{out}\mu}^c(k) \overline{S}_n j_{\text{in}}^{c\mu}(-k) + \overline{S}_n j_{\text{in}\mu}^c(k) j_{\text{in}}^{c\mu}(-k) \right| i \right\rangle, \end{aligned} \quad (17)$$

which is based on the extension of the soft-meson formula for virtual

gluon to  $\bar{S}$ . Eq. (13) is the first step towards eq. (17), which proves the finiteness of  $\bar{M}$ . As for the lowest order calculation, the equivalence of the present calculation to the n-th order inclusive cross section holds only after average over initial and final color states, in contrast to the Abelian case.

The virtual gluon part of eq. (17) has been derived in references (3) and (4) in the form of a differential equation.

In the fixed angle regime, when all the invariant squared energies and momentum transfers become large, the r. h. s. of eq. (17) gets factorized and  $\bar{M}$  exponentiates as in the Abelian case. One finds

$$\langle f | \bar{S} | i \rangle_{n+2} = \frac{1}{2} B \left( \sum_{\alpha} C_{\alpha} \right) \langle f | \bar{S} | i \rangle_n, \quad (18)$$

where  $C_{\alpha}$  is the eigenvalue of the Casimir operator for the  $\alpha$ -th particle, the sum runs over both the initial and final particles, and

$$B = B_{\alpha\beta} = \int d^4k \left[ \delta(k^2) \theta(k^0) - \frac{i}{2\pi} \frac{1}{k^2} \right] \frac{(p_{\alpha} p_{\beta})}{(p_{\alpha} k)(p_{\beta} k)} \frac{\bar{g}^2(k)}{(2\pi)^3} \quad (19)$$

Notice that in the same regime, exponentiation occurs for the real bremsstrahlung only for the color averaged cross-sections. Details of this result shall be presented elsewhere.

So far nothing has been said about  $\bar{g}(k)$ , but for the requirement that our formulae make sense. A suggestion which comes from the perturbative calculations by Frenkel et al. <sup>(6)</sup> and from the general results for virtual gluons by Kinoshita and Ukawa <sup>(4)</sup> is that  $\bar{g}(k)$  is related to the lowest order approximation to the effective coupling constant for pure NAGT.

Summarizing, we have constructed a coherent state formalism for NAGT, showing that matrix elements between coherent states of definite color are finite and factorized in the fixed angle regime.



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