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CLASSICAL SOLUTIONS IN TWO-DIMENSIONAL SUPERSYMMETRIC FIELD THEORIES

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Classical solutions of some supersymmetric field theories in two dimensions are investigated. These solutions are constructed by solving first-order differential equations in superspace, which are the supersymmetric extension of the analogous equations used in the purely bosonic sector.

1. Introduction

Non-perturbative solutions of the classical equations of motion have been constructed explicitly in a number of field theories. In Minkowski space the time-independent solutions of the classical equations of motion correspond to new particles of the physical spectrum [1], while in euclidean space they are useful to get approximate expressions for Green functions of the field theory [2]. Their stability is ensured by the existence of topological currents which are conserved independently of the equations of motion. Classical solutions of the fermion field in the background field of the instanton have also been extensively studied [3,4].

It has been pointed out [5–7] that in a number of field theories the minima of the action can be computed more easily by solving first-order differential equations instead of the more complicated second-order Euler-Lagrange equations. In general, a solution of the first-order equations corresponds to an N-instanton solution with non-interacting instantons.

It has also been recognized [4,8] that if one considers a supersymmetric theory one can use the supersymmetry of the Lagrangian in order to construct non-trivial classical solutions for the fermion field in the field of the bosonic classical solution.

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The supersymmetry of the Lagrangian has also been used in the case of the Yang-Mills theory with the fermions in the adjoint representation, to get a relation between the bosonic and the fermionic zero modes [9].

In this paper we consider a number of supersymmetric field theories in two dimensions and we write first-order differential equations, which are the supersymmetric generalization of the corresponding equations with only boson fields. These equations imply that the first-order differential equations for the bosons are not changed by the presence of the fermions and they give, for the latter, algebraic relations which in some cases are very useful to find explicitly the classical solution of the fermions in the field of the instanton.

In sect. 2 we discuss the classical solutions of the supersymmetric extension of the two-dimensional euclidean non-linear σ-model. Sect. 3 is devoted to the supersymmetric euclidean Higgs model. Finally in sect. 4 we consider the supersymmetric extension of the soliton solutions in two dimensions with an arbitrary potential.

2. Euclidean supersymmetric σ-model in two dimensions

The supersymmetric version of the two-dimensional non-linear σ-model is given by the following action:

$$S = \int d^2x\, d^2\theta\, \tfrac{1}{4} D\phi^i \gamma_5 D\phi^i \tag{1}$$

with the additional constraint

$$\phi^i\phi^i = 1\,, \qquad i = 1, 2, 3\,. \tag{2}$$

ϕ^i is the following superfield:

$$\phi^i(x, \theta) = A^i(x) + i\theta\chi^i(x) + \tfrac{1}{2}\, i\, \theta\gamma_5\theta\, F^i(x)\,, \tag{3}$$

which transforms according to the vector representation of O(3). The covariant derivative is given by

$$D_\alpha = \frac{\partial}{\partial\theta_\alpha} + i(\gamma^\mu\theta)_\alpha\, \partial_\mu\,. \tag{4}$$

In two dimensions the euclidean γ matrices can be chosen to be real and they are given by

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_5 = \gamma_0\gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5}$$

This is different from the four-dimensional case where a real representation for the γ matrices [9] does not exist. The euclidean invariant scalar product between two Majorana spinors φ and χ is given by $\varphi_\alpha\chi_\alpha$.

The action (1) is invariant under the following supersymmetry transformations:

$$\delta\phi^i = \epsilon\left(\frac{\partial}{\partial\theta} - i\gamma^\mu\theta\partial_\mu\right)\phi^i \,. \tag{6}$$

Starting from the action (1) and taking into account the constraint (2), one gets the following Euler-Lagrange equations:

$$(D\gamma_5 D)\,\phi_i = -\phi_i(D\phi_j\gamma_5 D\phi_j) \,. \tag{7}$$

In the case of the conventional non-linear σ-model in two dimensions, Belavin and Polyakov [6] have found the minima of the action by solving the first-order differential equation

$$D_\mu A_i(x) \mp \epsilon_{ijk}\,\epsilon_{\mu\nu}A_j\,\partial_\nu\,A_k = 0 \,. \tag{8}$$

The supersymmetric generalization of this first-order equation (8) is given by

$$D_\alpha\phi_i(x,\theta) \mp \epsilon_{ijk}\,\phi_j(x,\theta)\,(\gamma_5 D)_\alpha\,\phi_k(x,\theta) = 0 \,. \tag{9}$$

Eq. (9) implies the equation of motion (7), as can easily be seen by multiplying (9) by $D\gamma_5$ and using the relation $D_\alpha D_\alpha = 0$.

In terms of the component fields, from the term which is θ-independent one gets the following equation (we choose the minus sign in eq. (9)):

$$\chi_i = \epsilon_{ijk}\,A_j\gamma_5\chi_k \,. \tag{10}$$

The terms linear in θ give the following equations:

$$\epsilon_{ijk}A_jF_k = 0 \,, \tag{11a}$$

$$F_i = \tfrac{1}{2}i\,\epsilon_{ijk}(\chi_j\chi_k) \,, \tag{11b}$$

$$\partial_\mu A_i = \epsilon_{ijk}\epsilon_{\mu\nu}\left[A_j\partial_\nu A_k + \frac{i}{2}\,\chi_j\gamma_\nu\chi_k\right] . \tag{11c}$$

Finally, the quadratic term in θ gives

$$\gamma^\mu\gamma_5\partial_\mu\chi_i = \epsilon_{ijk}\,[A_j\gamma^\mu\partial_\mu\chi_k + \gamma^\mu\chi_j\partial_\mu A_k + 2F_j\gamma_5\chi_k] \,. \tag{12}$$

The constraint (2) gives the following equations:

$$A^iA_i = 1 \,, \tag{13a}$$

$$A_i\chi_i = 0 \,, \tag{13b}$$

$$A_iF_i - \tfrac{1}{2}i\,\chi_i\gamma_5\chi_i = 0 \,. \tag{13c}$$

Using eq. (10) in (11c) one gets for the last term of (11c) the following expression:

$$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\chi_j\gamma_\mu\chi_m A_l \,, \tag{14}$$

which is vanishing as a consequence of (13b) and of the identity $\chi_i\gamma_\mu\chi_i = 0$.

Therefore the presence of fermions in the action (1) does not modify the first-order equation (8) for the bosons. It is important to note that the first-order equation (9) implies for the fermions an algebraic equation (10) together with a first-order differential equation (12). This is analogous to the case of the four-dimensional supersymmetric Yang-Mills theory where eqs. (10), (8) and (12) are replaced, respectively, by [9]

$$(1-\gamma_5)\psi = 0\,,$$
$$F_{\mu\nu} = \widetilde{F}_{\mu\nu}\,,$$
$$\gamma^\mu D_\mu \psi = 0\,. \tag{15}$$

The solution of the system of eqs. (10), (11) and (12) can be obtained using the supersymmetry of the action (1). In terms of the component fields, the transformations (6) give

$$\delta A^i = i\epsilon\chi^i,$$
$$\delta\chi^i = (\gamma^\mu \partial_\mu A^i + \gamma_5 F^i)\epsilon\,,$$
$$\delta F^i = i\epsilon\gamma_5\gamma^\mu\partial_\mu\chi^i\,. \tag{16}$$

It is then easy to see that the following expression for χ^i:

$$\chi^i = \gamma^\mu \partial_\mu A^i \chi\,, \tag{17}$$

satisfies eq. (10) provided that A^i is a solution of eq. (8). If we then insert (17) into eq. (12) we get that the spinor χ must satisfy the conformal supersymmetry condition

$$\gamma^\nu\gamma^\mu\partial_\nu\chi = 0\,. \tag{18}$$

The auxiliary field F_i can then be computed inserting (17) into eq. (11b). The presence of an arbitraryfunction χ in the solution of the equations of motion is a consequence of the invariance of action (1) under conformal supersymmetry in two dimensions. It is now easy to check that the solutions to eqs. (10)–(12) give a vanishing contribution to the spinor current and the energy-momentum tensor. They are given, respectively, by

$$j^\mu_\alpha = (\gamma^\nu\gamma^\mu\partial_\nu A^i\chi^i)_\alpha \tag{19}$$

and

$$\theta_{\mu\nu} = \partial_\mu A^i \partial_\nu A^i - \tfrac{1}{2}\delta_{\mu\nu}\partial_\rho A^i\partial_\rho A^i$$
$$+ \tfrac{1}{4}i[\chi^i(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)\chi^i - \delta_{\mu\nu}\chi^i\gamma^\mu\partial_\mu\chi^i)]\,. \tag{20}$$

Inserting the solution (17) into (19), one gets

$$j_{\mu\alpha} = 2\gamma^\nu[\partial_\nu A^i\partial_\mu A^i - \tfrac{1}{2}\delta_{\nu\mu}(\partial_\rho A^i)^2]\chi$$
$$= 2\gamma^\nu\theta^{\text{boson}}_{\mu\nu}\chi\,, \tag{21}$$

which is vanishing because $\theta_{\mu\nu}^{\text{boson}} = 0$ as a consequence of eq. (8). Analogously one can also prove that the fermionic contribution to the energymomentum tensor (20) is also vanishing as a consequence of (17) and (18).

If we perform the integration over the variables θ, the Lagrangian corresponding to the action (1) with the constraint (2) is given by

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu A_i)^2 + \tfrac{1}{2} i\, \chi_i \gamma^\mu \partial_\mu \chi_i + \tfrac{1}{2} F_i^2 + F_\lambda (A_i^2 - 1)$$
$$+ 2A_\lambda (F_i A_i - \tfrac{1}{2} i\, \chi_i \gamma_5 \chi_i) + 2i \chi_\lambda \chi_i A_i \;, \tag{22}$$

where A_λ, χ_λ and F_λ are three Lagrange multipliers corresponding to the three constraints (13). The equation of motion for the auxiliary field F_i is given by

$$F_i + 2A_\lambda A_i = 0 \;. \tag{23}$$

Multiplying (23) by A_i and using the constraints (13a) and (13c) one gets

$$A_\lambda = -\tfrac{1}{4} i\, \chi_i \gamma_5 \chi_i \;, \qquad F_i = \tfrac{1}{2} i\, \chi_j \gamma_5 \chi_j A_i \;, \tag{24}$$

which can be inserted back into (22) to get a Lagrangian without the auxiliary fields F_i and A_λ,

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu A_i)^2 + \tfrac{1}{2} i\, \chi_i \gamma^\mu \partial_\mu \chi_i - \tfrac{1}{8}(\chi_i \gamma_5 \chi_i)^2 + F_\lambda (A_i^2 - 1) - 2i A_i \chi_i \chi_\lambda \;. \tag{25}$$

It is interesting to note that the elimination of the auxiliary fields has introduced into the Lagrangian (19) a Thirring-like four-fermion interaction. Eqs. (10–(12) can also be solved using the supersymmetric generalization of the procedure used in ref. [6]. It is convenient to introduce the complex variable $z = \frac{1}{2}(x_0 + ix_1)$ together with the superfields

$$\phi_\pm = \sqrt{\tfrac{1}{2}}(\phi_1 \pm i\phi_2) \qquad \text{and} \qquad \phi_3$$

and the covariant derivative

$$D_+ = (1 + i\gamma_5) D \;. \tag{26}$$

In terms of these new quantities eqs. (9) become

$$D_+\phi_+ = \phi_3 D_+\phi_+ - \phi_+ D_+\phi_3 \;, \tag{27a}$$

$$D_+\phi_- = -\phi_3 D_+\phi_- + \phi_- D_+\phi_3 \;, \tag{27b}$$

$$D_+\phi_3 = -\phi_+ D_+\phi_+ + \phi_- D_+\phi_- \;. \tag{27c}$$

If we now introduce the superfield

$$\Psi(z, z^*, \theta_+, \theta_-) = \frac{\phi_+}{1 - \phi_3} \;, \tag{28}$$

it is then easy to prove that eqs. (21) imply that

$$D_+\Psi(z, z^*, \theta_+, \theta_-) = 0 \;, \tag{29}$$

where

$$\theta_\pm = (1 \pm i\gamma_5)\theta \,. \tag{30}$$

If we expand Ψ in terms of the component fields one gets

$$\Psi(z, z^*, \theta_+, \theta_-) = \mathcal{A}(z, z^*) + i\theta_+\psi_+(z, z^*) + i\theta_-\psi_-(z, z^*) + i\theta_+\theta_- \quad (z, z^*) \tag{31}$$

and we use the fact that

$$D_+ = 2\left[(1 + i\gamma_5)\frac{\partial}{\partial\theta_-} + \gamma_1\theta_- \frac{\partial}{\partial z^*}\right], \tag{32}$$

then eq. (23) implies that

$$\psi_- = 0 = \mathcal{F} \,,$$
$$\frac{\partial}{\partial z^*}\,\psi_+ = \frac{\partial}{\partial z^*}\,\mathcal{A} = 0 \,. \tag{33}$$

Therefore the equations of motion (10–(13) imply that the superfield $\Psi(z, z^*, \theta_+\theta_-)$ is only a function of z and θ_+, i.e.

$$\Psi(z, \theta_+) = \mathcal{A}(z) + i\theta_+\psi(z) \,. \tag{34}$$

This is the supersymmetric extension of the analogous result of ref. [6] for the field $\mathcal{A}(z)$ (see eq. (12) of ref. [6]).

The supersymmetric invariant expression for the topological number is given by

$$N = \frac{1}{8\pi}\int d^2x\, d^2\theta\, D\phi_i\, \epsilon_{ijk}\, \phi_j D\phi_k \,. \tag{35}$$

If the equation of motion (9) is satisfied, one gets that (35) is proportional to the action (1):

$$S = 8\pi N \,, \tag{36}$$

and it gets contribution only from the boson field A_i.

3. Euclidean supersymmetric Higgs model in two dimensions

In this section we consider the two-dimensional euclidean version of the Higgs model [5]. This model, as far as the bosonic sector is concerned, is known to have instanton solutions. Moreover, its solutions can be interpreted as three-dimensional static solutions [10] (vortex lines).

The supersymmetric euclidean vector multiplet [11] is given by the following spinor superfield:

$$V_\alpha(x, \theta) = \xi_\alpha + (\gamma^\mu\theta)_\alpha B_\mu + \theta_\alpha M + (\gamma_5\theta)_\alpha L + \tfrac{1}{2} i\, \theta\gamma_5\theta\rho_\alpha \,. \tag{37}$$

The (Abelian) gauge transformation is defined as

$$\delta V_\alpha = iD_\alpha \Lambda \,, \tag{38}$$

Λ being a (real) scalar superfield.

In the Wess-Zumino [12] gauge $\xi_\alpha = L = 0, M$ and ζ_α are gauge-invariant fields, while B_μ undergoes the usual gauge transformation

$$\delta B_\mu = \partial_\mu A \,, \tag{39}$$

A being the first component of the superfield $\Lambda(x, \theta) = A + \ldots$. The properly normalized superfield Lagrangian density for the vector multiplet is

$$\tfrac{1}{16} DV D\gamma_5 D DV \,. \tag{40}$$

Note that the gauge-invariant superfield DV is the supersymmetric generalization of the field strength. The Higgs superfield is given by a doublet S^i which transforms as a vector under the O(2) gauge group

$$\delta S^i = g\Lambda S^j \epsilon^{ij} \,, \tag{41}$$

g being the gauge coupling constant and Λ a superfield gauge parameter. The supercovariant derivative is defined as

$$\nabla_\alpha^{ij} = D_\alpha \delta^{ij} + ig\, V_\alpha \epsilon^{ij} \tag{42}$$

and it has the property of being covariant under supersymmetry as well as local gauge transformations.

The Higgs Lagrangian is constructed in terms of the supergauge field V_α, the super-Higgs field S_i, and an additional superfield S, which is an O(2) singlet but whose introduction is needed in order to have spontaneous symmetry breaking. The final action is given by

$$I = \int d^2x\, d^2\theta \,[\tfrac{1}{16} DV D\gamma_5 D DV + \tfrac{1}{4} DS\gamma_5 DS + \tfrac{1}{4}(\nabla S)^i \gamma_5 (\nabla S)^i + i\lambda S^{i2} S + i\eta S] \,. \tag{43}$$

We note that the above action is invariant under (real) euclidean supersymmetry for any value of λ and g (and η).

In terms of the component fields

$$S^i = A^i + i\theta\psi^i + \tfrac{1}{2} i\, \theta\gamma_5\theta F^i \,,$$
$$S = N + i\theta\chi + \tfrac{1}{2} i\, \theta\gamma_5\theta D \,, \tag{44}$$

the complete bosonic potential is

$$-\tfrac{1}{2}(\lambda A_i^2 + \eta)^2 + \tfrac{1}{2} g^2 M^2 A_i^2 - 2\lambda^2 N^2 A_i^2 \,. \tag{45}$$

Moreover, the kinetic terms of the bosonic fields are

$$-\tfrac{1}{4} F_{\mu\nu}^2 - \tfrac{1}{2}(\partial_\mu N)^2 + \tfrac{1}{2}(\partial_\mu M)^2 - \tfrac{1}{2}(D_\mu A_i)^2 \,, \tag{46}$$

which shows that the scalar field M, contained in V_α has opposite sign with respect to the ordinary bosonic fields. We note also that for $N = M = 0$ the bosonic Lagrangian just reproduces the usual Higgs Lagrangian.

The Euler-Lagrangian equations are

$$\tfrac{1}{4} D_\alpha D\gamma_5 DDV = ig\,\epsilon^{ij}\, S^i\gamma_5\, \nabla_\alpha S^j\,,$$

$$\tfrac{1}{2} D\gamma_5 DS = i\lambda S^{i2} + i\eta\,,$$

$$\tfrac{1}{2}(\nabla\gamma_5 \nabla S)^i = 2i\lambda S^i S\,. \tag{47}$$

These equations can be solved by means of the first-order equations

$$\gamma_5 \nabla_\alpha S^i = \epsilon^{ij}(\nabla_\alpha S)^j\,, \tag{48}$$

$$\tfrac{1}{2} D_\alpha V_\alpha = S \tag{49}$$

(or the same ansatz with the $-$ sign) provided

$$g^2 = 4\lambda^2\,.$$

These first-order equations are just the supersymmetric extensions of the first-order equations of motion of the Higgs model. Note in fact that eq. (48) implies

$$\gamma_5 \psi^i = \epsilon^{ij}\psi^j\,, \tag{50}$$

$$D_\mu A^i = -\epsilon_{\mu\nu}\,\epsilon_{ij} D_\upsilon A_j\,, \tag{51}$$

while eq. (49) implies

$$M = N\,, \qquad 2\gamma_5 \zeta = \chi \tag{52}$$

and

$$F_{\mu\nu} = -\epsilon_{\mu\nu}(\lambda A_i^2 + \eta)\,, \qquad (g = 2\lambda)\,. \tag{53}$$

Eqs. (51) and (53) are the usual first-order equations of motion of the pure Higgs model [5].

The value $g^2 = 4\lambda^2$, for which the equations can be solved by imposing first-order differential equations, has a particular meaning because only for this value does the action have a bigger symmetry ⋆. In fact, when $g = 2\lambda$, the action is invariant under extended euclidean supersymmetry with internal symmetry O(2). [Complex Euclidean supersymmetry.] V_α and S belong to the same irraducible (real) multiplet of complex euclidean supersymmetry

$$V = C + i\theta^i\eta^i + i(\theta^i\gamma_5\theta^j - \tfrac{1}{2}\delta^{ij}\theta^l\gamma_5\theta^l)\, T_{ij}$$

⋆ From a physical point of view, the value $g^2 = 4\lambda^2$ is the transition point between superconductivity of first and second type.

$$+ i\theta^i\theta^j\epsilon^{ij}M + i\theta^i\gamma_\mu\theta^j\epsilon^{ij}B_\mu + i\theta^i\gamma_5\theta^i N$$

$$+ \theta^j\gamma_5\theta^j\theta^i\lambda^i + \theta^i\gamma_5\theta^i\theta^j\gamma_5\theta^j D\,, \qquad (54)$$

in which $C = \eta^i = T_{ij} = 0$ in the Wess-Zumino gauge and then (M, B_μ, λ_2) and (N, λ_1, D) are, respectively, the component fields of the two real multiplets V_α and S of the simple euclidean supersymmetry previously introduced. The Higgs field S^i becomes a complex scalar multiplet of complex supersymmetry

$$S_1 + iS_2 = S = A + i\theta\chi + \tfrac{1}{2}i\,\theta\theta F\,, \qquad (55)$$

in which

$$\theta = \theta^1 + i\theta^2\,, \qquad A = A^1 + iA^2\,, \qquad \chi = \chi^1 + i\chi^2\,, \qquad F = F^1 + iF^2\,.$$

The transformation laws of superfields are

$$\delta\phi(x, \theta^i) = \phi(x - i\epsilon^i\gamma\theta^i, \theta^i + \epsilon^i)\,, \qquad (56)$$

so that one can easily check the previous identification. For instance,

$$\delta M = \tfrac{1}{2}i\,\epsilon^i\gamma_5\lambda^j\epsilon^{ij}\,, \qquad \delta N = -\tfrac{1}{2}i\,\epsilon^i\lambda^i\,,$$

$$\delta B_\mu = \tfrac{1}{2}i\,\epsilon^{ji}\epsilon^i\gamma_5\gamma_\mu\lambda^j\,. \qquad (57)$$

It is easy to show that the M and N scalar fields in V have opposite kinetic term in complex euclidean supersymmetry. This not surprising in view of the fact that the O(2) euclidean supersymmetry can be obtained by reduction of Minkowski four-dimensional supersymmetry in which z, t are treated as internal coordinates [13]. M and N are just the space and the time components of a four-dimensional Minkowski vector *. This is also analogous to the four-dimensional euclidean supersymmetry investigated in ref. [9], in which the two scalar fields also have opposite kinetic terms because of a similar phenomenon.

We conclude the discussion on the Higgs model by again noting the fact that the first-order equations for the bosons are not modified by the fermions and that the supersymmetric extension of the first-order equations implies the Weyl condition for the fermions together with their first-order equations in the field of the bosonic classical solution.

4. Static classical solutions in two-dimensional Minkowski space with supersymmetry

So far we have considered instanton solutions, i.e. classical solutions in euclidean two-dimensional field theories. We now turn to soliton solutions, i.e. classical, time-independent solutions in two-dimensional Minkowski space. These solutions may be re-

* From this point of view, the previous Lagrangian is nothing but the dimensional reduction [13] of the supersymmetric Minkowski four-dimensional Higgs model [14].

garded as one-dimensional instantons just as two-dimensional instantons are related to three-dimensional vortex solutions.

We consider a bosonic field theory with potential

$$U(\phi) = -\tfrac{1}{2} V'(\phi)^2 \tag{58}$$

and Lagrangian

$$\mathcal{L}(\phi) = -\tfrac{1}{2}(\partial_\mu \phi)^2 + U(\phi) . \tag{59}$$

If we add to it a fermionic part, then the complete Lagrangian

$$\mathcal{L}(\phi, \chi) = -\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} i \bar{\chi} \partial\!\!\!/ \chi + U(\phi) - \tfrac{1}{2} i \bar{\chi}\chi Z(\phi) \tag{60}$$

is supersymmetric provided

$$Z(\phi) = V''(\phi) . \tag{61}$$

As a Lagrangian density in superspace it corresponds to

$$\mathcal{L}(S) = \tfrac{1}{4} \bar{D}SDS - iV(s) , \tag{62}$$

with

$$S = \phi + i\bar{\theta}\chi + \tfrac{1}{2} i \bar{\theta}\theta F . \tag{63}$$

It is straightforward to show that $\delta\mathcal{L}$ is a total divergence under the variations

$$\begin{aligned} &\delta\phi = i\bar{\epsilon}\chi , \\ &\delta\chi = \partial\!\!\!/\phi\epsilon - V'(\phi)\,\epsilon , \end{aligned} \tag{64}$$

ϵ being a (constant) anticommuting spinor.

The field equations are

$$\begin{aligned} &\Box\phi = V'(\phi)\, V''(\phi) + \tfrac{1}{2} i \bar{\chi}\chi\, V'''(\phi) , \\ &(\partial\!\!\!/ + V''(\phi))\,\chi = 0 . \end{aligned} \tag{65}$$

A soliton solution is a solution of the first-order equation

$$\phi' = \pm V'(\phi) , \qquad \text{in which } \phi' = \frac{\mathrm{d}}{\mathrm{d}x}\phi(x) . \tag{66}$$

As we consider static solutions, then

$$\begin{aligned} &(1 \pm \gamma_1)\,\chi = \chi_\pm = 0 , \\ &\chi'_\pm = \pm V''(\phi)\,\chi_\pm . \end{aligned} \tag{67}$$

Supersymmetry gives

$$\begin{aligned} &\delta\phi = \tfrac{1}{2} i\, \bar{\epsilon}_\pm \chi_\mp , \\ &\delta\chi_\mp = \mp 2\phi' \epsilon_\mp . \end{aligned} \tag{68}$$

It is trivial to show that

$$\chi = (\partial\!\!\!/\phi - V'(\phi))\,\epsilon \tag{69}$$

is an exact solution of the coupled system of eqs. (65) with ϕ still a solution of the equation

$$\phi' = \pm\, V'(\phi)\,. \tag{70}$$

We also point out that the fermionic constraint in eq. (67) as well as the bosonic first-order equations are given by the first-order supersymmetric constraint

$$(1 \pm \gamma_1)\, D\phi = 0\,, \tag{71}$$

which gives the equations

$$\chi_\pm = 0\,, \qquad F \pm \phi' = 0\,, \qquad \text{i.e. } \phi' = \pm\, V'(\phi)\,. \tag{72}$$

The above considerations apply in particular to the sine-Gordon potential $V'(\phi) = A\cos(\omega\phi + \beta)$ and to the ϕ^4 theory with $V'(\phi) = (\phi^2 - m^2)$. In this way we have found exact fermionic solutions in the soliton field. They are given by eq. (69), with ϕ given by a solution of eq. (70).

5. Conclusions

In the present paper we have investigated a wide class of classical solutions of two-dimensional field theories. With the requirement of supersymmetry the first-order differential equations for the bosons have been extended to the fermionic sector. These supersymmetric first-order relations do not alter the bosonic solution, while they give an algebraic equation for the fermion, a kind of Weyl condition, together with their first-order differential equation.

In our case, namely the euclidean σ-model, because of superconformal symmetry, these relations completely determine the fermionic solution in terms of an anticommuting spinor, analytic in the variable $z = x_0 + ix_1$. This solution is the supersymmetric extension of the corresponding bosonic solution found in ref. [6].

In the case of the Higgs model [5] the critical value $g^2 = 4\lambda^2$, for which the equations of motion can be solved in first-order, has been shown to be related to a higher euclidean supersymmetry; i.e. to the extended euclidean supersymmetry with O(2) as internal symmetry. The rather curious fact that kinetic terms of bosons do not occur with the same sign can be understood simply with the dimensional reduction from four-dimensional Minkowski supersymmetry, according to the analysis of ref. [13]. This phenomenon is quite analogous to the four-dimensional case [9], which can also be understood from dimensional reduction from Minkowski six-dimensional supersymmetry [13].

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Note added in proof

It has been pointed out to us that a supersymmetric version of the sine-Gordon equation has been also constructed by J. Hruby, Dubna preprint.

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