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R. Benzi, G. Martinelli and G. Parisi:
HIGH TEMPERATURE EXPANSION WITHOUT LATTICE.

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ABSTRACT. -

In this paper we show how perform a strong coupling expansion in quantum field theory at very high orders in the kinetic term. We have chosen the simplest model of QFT, i. e. the $\lambda\Phi^4$ theory. The methods we have used can be easily extended to other quantum field theories like the O(N) model or the gauge theories.

1. - INTRODUCTION. -

The principal aim of this work is to explain in details how to perform the high temperature expansion of a relativistic quantum field theory (QFT) without the troubles one usually has on the lattice[(1)].

To explain our methods we have chosen the simplest model of QFT, i. e. a $\lambda\Phi^4$ theory. If one wants to study a field theory one has to introduce a cutoff in momentum space to avoid ultraviolett divergences. An equivalent way to introduce the cutoff is to define the theory on a eucledian lattice of spacing $a = 1/\Lambda$. On the lattice the generating functional becomes the partition function of a system of interacting spins. The continuum space-time theory is obtained in the limit $a \to 0$. It is well known that we can recover the limit $a \to 0$ at the critical temperature, the temperature where the systems undergoes a second order phase transition[(2)]. Then the problem is to study the critical behaviour of the spins system, the transition being characterized by the fact that correlation function become singular at zero external momenta. Therefore we are interested in the long wave length behavior.

One way to compute the correlation function is to do an expansion of the generating functional in powers of β (β being the inverse of the temperature), that is to perform an high temperature expansion[(3)]. The expansion in powers of β is an expansion in the kinetic term (strong coupling theory). Howevere there are troubles connected with such an expansion if one has to work on a lattice. For instance if one wants to compute the two point correlation function (the two point Green function in the field theory language) at a gives order, say β^n, one has to find how many paths, with n steps, connect two given points. This turn out to be a hard problem and of course harder and harder, higher and higher is the dimension of the space. Here we show the existence of a model that allows us to performe an high temperature expansion overcoming most of the difficulties of the lattice. In our model the n^{th} order of the expansion is the sum of certain number

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of Feynman diagrams which are easier to compute than in the usual "lattice" theory. Moreover a very nice feature is that there is no to compute different diagrams in different dimensions: that is Feynman diagrams can be calculated once for a generic dimension D and D can be treated as an external parameter(4).

Our model, respect to the usual lattice, has the nice property to be invariant under eucledian rotations.

In Sections 2, 3, 4 we explain in details our model. InSection 5 we test the power of this method by comparing our results with the results found with the standard methods for the Ising model and for the $\lambda\Phi^4$ theory.

2. - REVIEW OF THE STRONG COUPLING EXPANSION. -

A strong coupling expansion is equivalent to a high temperature expansion in statistical mechanics(5). Let us start with the generating functional of Green functions for the $\lambda\Phi^4$ theory, S (J), in an euclidean D-dimensional space:

$$S(J) = \text{const} \int d\left[\Phi\right] \exp\left[-\int d^D x \left(\frac{1}{2}\partial_\mu \Phi(x) \partial^\mu \Phi(x) + \frac{m_o^2}{2}\Phi^2(x) + \frac{\lambda_o}{4!}\Phi^4(x) + J(x)\Phi(x)\right)\right] \tag{1}$$

$\int d\left[\Phi\right]$ indicates the functional integral; Φ, m_o and λ_o are the bare fields, mass and coupling constant.

To study the theory one has to introduce a cutoff Λ in momentum space to avoid ultraviolet divergences. An equivalent way to eliminate the divergences is to define the theory on lattice of spacing $a = 1/\Lambda$.

On the lattice the generating functional becomes:

$$S(J) = \text{const} \int \prod_i d\Phi_i \exp\left[\frac{1}{2}\beta \sum_{ij} \Phi_i V_{ij} \Phi_j - \frac{m^2}{2}\sum_i \Phi_i^2 a^D - \frac{\lambda_o}{4!}\sum_i \Phi_i^4 a^D + \sum_i J_i \Phi_i a^D\right] \tag{2}$$

where i is a label which indicates the lattice site, β, V_{ij} and m^2 are functions of the lattice spacing a and of the bare mass m_o. βV_{ij} is a first neighbour interaction. β can be considered as the inverse of the temperature (a part for an irrelevant multiplicative constant), of a spin system.

S(J) now is just the partition function of a system of interacting continuous spin on a lattice. We are of course interested to study the theory in the limit $a \to 0$. We can recover this limit at the critical value of the temperature, $\beta = \beta_c$, where the system undergoes a second order phase transition (i.e. the correlation length of the two point Green function, ξ, goes to infinity). The problem is then to study the critical behaviour of the system at the transition point: this can be done expanding the generating functional in powers of β. This is an expansion in the kinetic term i. e. a strong coupling expansion. To every order in β it corresponds the sum of a certain number of Feynman diagrams on the lattice. The combinatories of such diagrams becomes very complicated for increasing the power of β and the complexity increases faster for higher dimensionality of the space in which one has to work.

We are going to show in the next Section how to arrive at one model (equivalent at the critical point to the field theory on a lattice), which permits a high temperature expansion at very high orders in β without the troubles of the lattice.

3. A SIMPLE MODEL. THE GAUSSIAN EXAMPLE. -

We start from the expression (2). V_{ij} is a first neighbour interaction of the form:

$$V_{ij} = \text{const} \quad \delta_{\vec{i},\, \vec{i}+\vec{\mu}} \tag{3}$$

where δ_{ij} is the standard Kronecker delta, $\vec{i}$ is the label of the lattice site and $\vec{\mu}$ is the unit vector in one direction of the space. In momentum space the term $\beta \sum_{ij} \Phi_i V_{ij} \Phi_j$ becomes:

$$\beta \int_{\mathcal{D}} d^D p \, \Phi(\vec{p}) V'(\vec{p}) \Phi(-\vec{p}) \tag{4}$$

where $V'(\vec{p})$= const $\sum_{k=1}^{D} \cos p_k a$ (p_k is the component of the D-dimensional vector p along the k direction). $\mathcal{D}$ is the first Brioullin zone (in the case of cubic lattice it is the region $-\frac{\pi}{a} \leqslant p_k \leqslant \frac{\pi}{a}$).
Because we are interested in the long wavelength behaviour (small momenta) of the theory (for $\beta \sim \beta_c$), we can replace $V'(\vec{p})$ with the expression:

$$V(\vec{p}) = \text{const. } \exp. -\frac{(p^2 R^2)}{4} \tag{5}$$

For small p, taking $\frac{R^2}{4} = \frac{a^2}{2D}$, we have:

$$V'(p) \underset{p \to 0}{\sim} V(p) \underset{p \to 0}{\sim} 1 - \frac{p^2 a^2}{2D} \tag{6}$$

The kinetic term becomes then(*):

$$\beta \int d^D p \left[\Phi(p) \exp - \left(\frac{p^2 R^2}{4}\right) \Phi(-p) \right] \tag{7}$$

According to the Kadanoff[6] principle of universality we expect the system whose spins interact as in eq. (7) belongs to the same class of universality as the first neighbour interaction system previously considered, so that its critical behavior should be the same. If we consider R as an independent parameter, the theory has now a cutoff for $R \neq 0$, even in the limit $a \to 0$. If we want to return to the QFT without cutoff, we have to send also $R \to 0$. Since however R playing the same role (in the continuum theory) that a plays in the lattice theory, we can study, according to the previous discussion, instead of the limit $R \to 0$, the limit $\xi \to \infty$, or, equivalently, the limit $\beta \to \beta_c$.

First of all we want to arrive at the expression of S(J) in the limit a = 0. Let us write the eq.(2) as

$$S(J) = \text{const} \left\{ \exp \left[\beta \sum_{i,k} \frac{\delta}{\delta J_i} V_{ij} \frac{\delta}{\delta J_k} \right] \cdot \prod_i \int d\Phi_i \exp \left[-\frac{m^2}{2} \sum_i \Phi_i^2 a^D - \frac{\lambda_0}{4!} \sum_i \Phi_i^4 a^D + \sum_i J_i \Phi_i a^D \right] \right\} \tag{8}$$

The same expression can be written in the following way:

(*) In this Section all inessential multiplicative constants has been inserted in the definition of .

$$S(J) = \text{const}\left\{\exp\left[\beta \sum_{i,k} \frac{\delta}{\delta J_i^{ad}} V'_{ik} \frac{\delta}{\delta J_k^{ad}}\right] \cdot \exp\left[\sum_i H(J_i^{ad})\right]\right\} = \text{const}\left\{\exp\left[\beta \sum_{i,k} \cdot \frac{\delta}{\delta J_i^{ad}} V'_{ik} \frac{\delta}{\delta J_k^{ad}}\right] \exp\left[\sum_{i,n} \frac{H_{2n}}{(2n)!} (J_i^{ad})^{2n}\right]\right\} \tag{9}$$

where

$$H(J_i^{ad}) = \ln \int d\,\Phi_i^{ad} (\exp\left\{-\frac{m_{ad}^2}{2} \sum_i (\Phi_i^{ad})^2 - \frac{\lambda^{ad}}{4!} \sum_i (\Phi_i^{ad})^4 + \sum_i J_i^{ad} \Phi_i^{ad}\right\}) \tag{10}$$

The different parameters have been redifined so as to become dimensioneless:

$$m_{ad} = ma$$

$$\lambda_{ad} = \lambda_o a^{D-4}$$

$$\Phi_i^{ad} = \Phi_i a^{D-2/2}$$

$$J_i^{ad} = J_i a^{D+2/2}$$

$$\beta \longrightarrow a^{D+2} \beta$$

$$H_{2n} = \left[\frac{\delta^{2n}}{(\delta J_i^{ad})^{2n}} H(J^i_{ad})\right]_{J_i^{ad} = 0} .$$

The H_{2n} do not depend on the lattice site nor on a. Apart for a redefinition of the temperature β, everything is now independent of a, then it is possible to eliminate the lattice. Because now every thing is independent respect to a and the effective parameter is R, a possibility is to send a $\longrightarrow 0$ assuming that at every lattice site there are $(\frac{a}{R})^D$ spins expression (9) becomes:

$$S(J) = \text{const}\left[\exp\left\{\frac{\beta a^{2D}}{2R^{2D}} \sum_{ij} \frac{\delta}{\delta J_i^{ad}} V'_{ij} \frac{\delta}{\delta J_i^{ad}}\right\} \exp\left\{(\frac{a}{R})^D \sum_i H(J_i^{ad})\right\}\right] \tag{11}$$

In the limit a $\longrightarrow 0$ we have:

$$S(J) = \text{const} \times \left[\exp\left\{\frac{\beta}{2R^{2D}} \int d^Dx\, d^Dy\, (\frac{\delta}{\delta J(x)} V(x-y) \frac{\delta}{\delta J(y)})\right\} \cdot \exp\left\{\int \frac{d^Dx}{R^D} H(J(x))\right\}\right] \tag{12}$$

where $$H(J(x)) = \sum_n H_{2n} (J(x))^{2n} / (2n)! \quad \text{(o)}$$

$$V(x-y) = R^{-D} \exp - (x-y)^2/R^2 \tag{13}$$

Expression (12) will be our starting point. The following analysis will show that in this theory the high temperature expansion may be carried on the relatively high orders faster than in other models. The question whether this theory is actually equivalent to the original Φ^4 one will remain open although our numerical results suggest a positive answer.

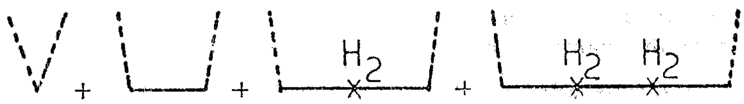
The role of the lattice spacing in now palyed by R (for a= 0). It is interesting to show in the simplest case of the gaussian model, which is exactly soluble, that the two limit $R \longrightarrow 0$ or $\beta \longrightarrow \beta_c$ are equivalent. In this case, being $\lambda_{ad}=0$ all the H_{2n} are zero except H_2.

The two point correlation function in momentum space, $G_2(p^2, \beta)$ has a pole for $p^2=0$ for $\beta = \beta_c$. We can find the critical temperature from the expansion of $G_2(p^2, \beta)$ in powers of β. If, in eq. (12); we expand

$$\exp \left\{ \frac{\beta}{2} \int d^D x \, d^D y \left(\frac{\delta}{\delta J(x)} V(x-y) \frac{\delta}{\delta J(y)} \right) \right\}$$

in powers of β, at every order in β there will be a set of diagrams resulting from the field contractions. In the gaussian case, being all the $H_{2n}=0$ ($n > 2$) only a small class of diagrams survives. If we represent V(x-y) as a line between the points x and y (Fig. 1) the only non zero diagrams are those where there are two lines arriving at each vertex. The diagrams for $G_2(p^2, \beta)$ are shown in Fig.2.

FIG. 1



-------- are external lines

FIG. 2

The sum of all these diagrams gives:

$$G_2(p^2, \beta) = \sum_{n=0}^{\infty} \beta^n H_2^{n+1} \left(\frac{e^{-p^2 R^2/4}}{2^{D/2}} \right)^n = \frac{H_2}{1 - \frac{\beta H_2}{2^{D/2}} e^{-p^2 R^2/4}} \tag{14}$$

The two point correlation function has a pole for $p^2=0$ at the value of the temperature $\beta_c = 2^{D/2}/H_2$. For small value of p we write:

$$G_2(p^2 \sim 0, \beta \sim \beta_c) \sim \frac{1}{\frac{(\beta_c - \beta)}{AR^2} + p^2} \tag{15}$$

(o) H_{2n} are the same as in the discrete lattice case.

where

$$A = \frac{\beta_c^2 H_2}{4\, 2^{D/2}}$$

Then the renormalized mass will be:

$$m^2_{ren} = (\beta_c - \beta)/AR^2 \tag{16}$$

Let us write eq. (16) as

$$m^2_{ren} R^2 = (\beta_c - \beta)/A \tag{17}$$

keeping m^2_{ren} fixed, we see that $\beta \longrightarrow \beta_c$ or $R \longrightarrow 0$ are perfectly equivalent. We see also that, for $\beta \sim \beta_c$, the correlation length $\xi = 1/m_{ren}$ has the correct behavior, i. e. $\xi \sim 1/(\beta_c - \beta)^{1/2}$.

4. - THE $\lambda\Phi^4$ THEORY AND THE ISING MODEL: FEYNMAN DIAGRAMS. -

The model we have defined in expression (12) has nice properties. Instead of a cubic lattice we have a continuum space and the system is rotational invariant. The numerical calculus of the diagrams is very easy (all the diagrams have integration which are gaussian) and the multiplicity of each diagrams is much simpler than in the lattice case.

Here we want briefly to describe how to construct the high temperature expansion for the $\lambda\Phi^4$ theory and the Ising model, how to calculate the multiplicities of the diagrams and how to find all diagrams (at the seventh order in β we have about one thousand diagrams!).

We start to calculate the two point correlation function $G_2(x-y)$:

$$G_2(x-y) = \left. \frac{\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} S(J)}{S(J)} \right|_{J=0} \tag{18}$$

As in the gaussian case, in expression (12) we can expand the first on the r. h. s. in power of β:

$$\begin{aligned} G_2(x-y) = & \Bigg[\Big(1 + \frac{\beta}{2} \int d^D x_1\, d^D x_2 \big(\frac{\delta}{\delta J(x_1)} V(x_1 - x_2) \frac{\delta}{\delta J(x_2)} \big) + \\ & + \frac{\beta^2}{2!} \big(\int d^D x_1\, d^D x_2 \frac{\delta}{\delta J(x_1)} V(x_1 - x_2) \frac{\delta}{\delta J(x_2)} \big)^2 + \cdots \Big) \cdot \\ & \cdot \big(\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} \big) \exp \big(\int \frac{d^D x}{R^D} \frac{H_{2n} (J(x))^{2n}}{(2n)!} \big) \Bigg] \Bigg|_{J=0} \end{aligned} \tag{19}$$

Now only the term such that at the same point there have as many functional derivates as powers of $J(x)$ survive. For example at the second order in β we will have:

$$G_2(x-y)_{\text{ord}\,\beta^2} = \left[\left(\frac{\beta^2}{2!}\int dx_1 dx_2 dx_3 dx_4 \left(\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)}\cdot\right.\right.\right.$$

$$\left.\left.\left.\cdot V(x_1-x_2)V(x_3-x_4)\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}\right)\exp\left\{\int\frac{H_{2n}J^{2n}(x)}{(2n)!}\frac{d^D x}{R^D}\right\}\right]\right|_{J=0} =$$

$$= \left[\left\{\frac{\beta^2}{2!}\int d^D x_1\, d^D x_2\, d^D x_3\, d^D x_4 \,(V(x_1-x_2)V(x_3-x_4)\cdot\right.\right. \qquad (20)$$

$$\cdot \frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)})\frac{\delta}{\delta J(y)}\frac{\delta}{\delta J(x)}\left(\left[\int(H_2\frac{J^2(\tilde{x})}{2!}\frac{d^D\tilde{x}}{R^D}\right]^3 +\right.$$

$$\left.\left.\left.+\left[\int(\frac{H_2 J^2(x')}{2!}\frac{d^D x'}{R^D}\right]\left[\int\frac{H_4 J^4(x'')}{4!}\frac{d^D x''}{R^D}\right]+\left[\int\frac{H_6 J^6(x')}{6!}\frac{d^D x}{R^D}\right]\right)\right\}\right]_{J=0} =$$

$$= 4\beta^2\left\{H_2^3\int V(x-x')\,V(x'-y)\,d^D x' + \frac{1}{2}H_2H_4\int\left[V(x-x')\right]^2 d^D x' +\right.$$

$$\left.+\frac{H_2H_4}{2}V(0)\,V(x-y)+\frac{H_6(V(0))^2}{8}\right\}^{(*)} \; .$$

Representing, as in the preceding Section, V(x-y) as a line between x and y, all the diagrams up to the third order can be represented as in Fig. 3. We can see from the first order in β the main fea-

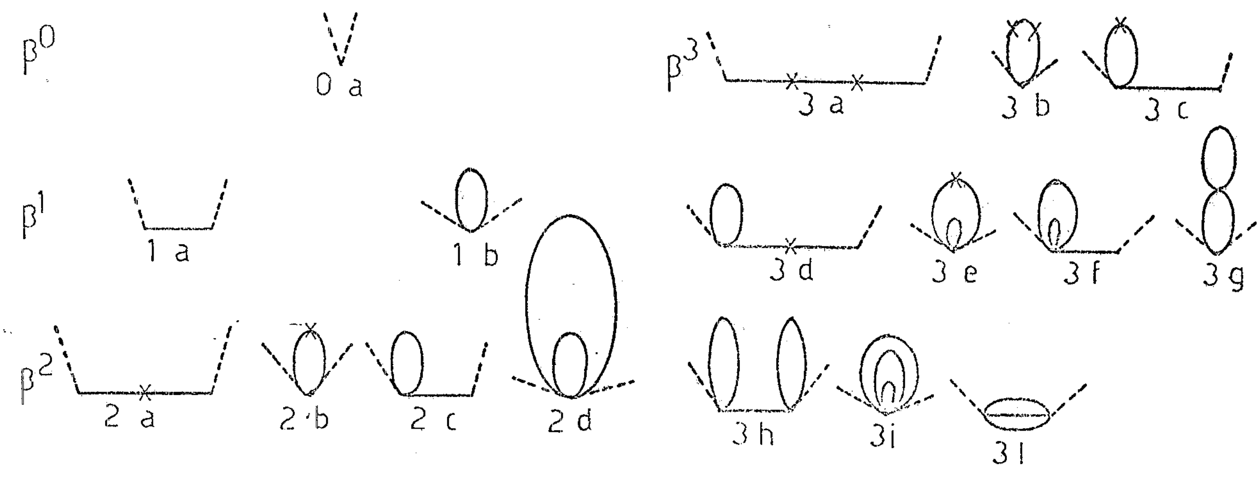


FIG. 3

(*) Because the fields J(x) are dimensionless the factors $\frac{1}{R^D}$ disappear.

tures of the high temperature expansion in our model. First of all let us define the quantity $\pi(p^2)$, the two point irriducible vertex. The β expansion for $\pi(p^2)$ is shown in Fig. 4. All diagrams for

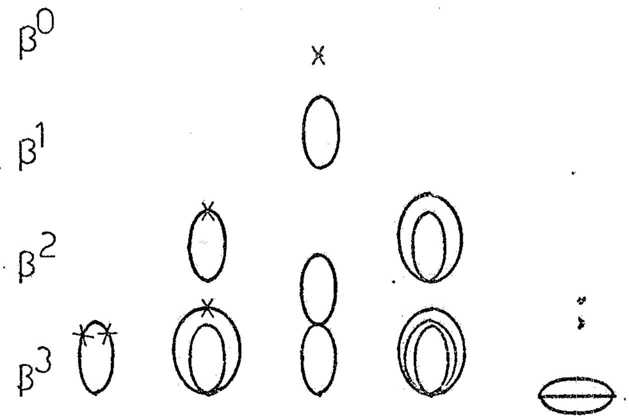


FIG. 4

$G_2(p^2, \beta)$ can be expressed in terms of $\pi(p^2)$. For example, the diagram 3-d of Fig. 3 is obtained combining $\pi(p^2)$ and $V(p^2)$ as indicated in Fig. 5, where we take one $\pi(p^2)$ at first order and the

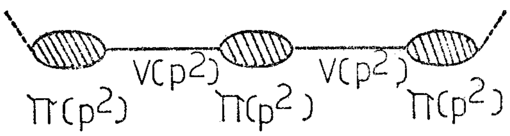


FIG. 5

other two at the zero, order in β.

It is then necessary to study only the expansion for $\pi(p^2)$. From the first few diagrams of Fig. 4 we can see how to construct the order n knowing the preceding orders. We give here these simple rules:

a) put on all the diagrams of the n^{th} order a mass insertion (Fig. 6); if there are more lines on whi-

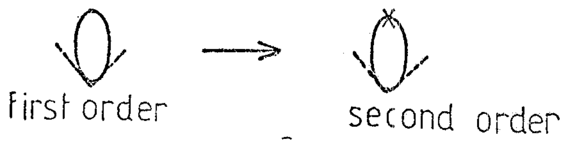


FIG. 6

ch is possible to put a mass insertion, insert it in alla the possible inequivalent ways (Fig. 7);

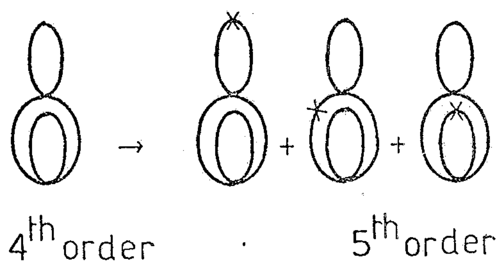


FIG. 7

b) in all the diagrams that do not have more then a mass insertion insert a tadpole in every vertex (Figg. 8 and 9)(if we insert tadpoles in diagrams with more insertion we had some double counting);

FIG. 8

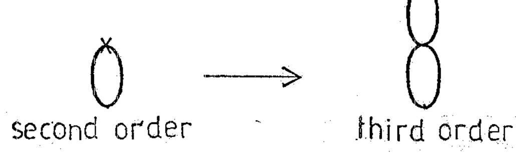


FIG. 9

c) introduce "new" topological diagrams (Fig. 10) i. e. those diagrams that can not be obtained through the operations) a) or b);

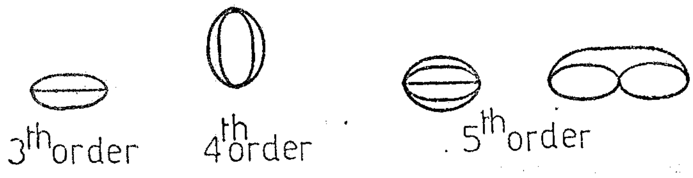


FIG. 10

d) combine all the new diagrams of m^{th} order ($m \geq 3$) with the diagrams of order p (such that m+p=n) to make the n^{th} order (Fig. 11).

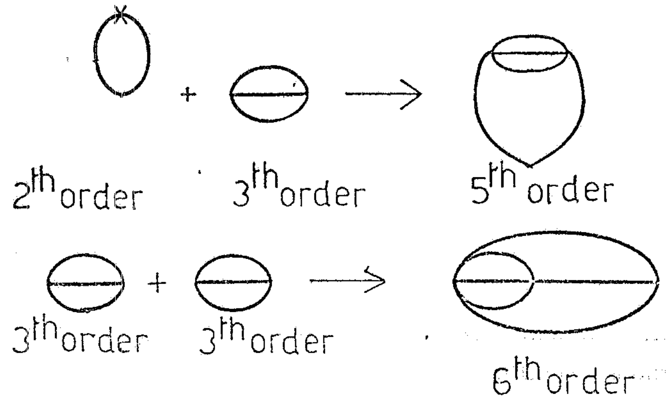


FIG. 11

The main difficulty arises because of the "new" diagrams that cannot be found by rules a) and b). We do not explain the rules for the numerical values of the diagrams, because they are very simple.

We want only say some words about the multiplicities of the diagrams. In the cubic lattice, for the two point correlation function at the order n, it is necessary to calculate in how many ways it is possible to connect two points of the lattice in n steps. In our model, we have to calculate only in how many ways it is possible to contract the fields to obtain the same Feynman diagram (this problem is just the same as in quantum field theory and is easier than the lattice analogous). Simple rules can be given. If there are n identical lines insert a factor $1/n!$ (e. g. for the diagrams 3-1 there is a factor $1/3!$); for every tadpole insert a factor 1/2 (e. g. the diagram 3-1 has a factor $1/3! \times 1/2^3$ where $1/3!$ comes from the identity of three lines, and $1/2^3$ from the three closed lines). All other factors come out from the k^{th} power of the two point irriducible vertes $\pi(p^2)$. There is a very simple control that shows if we count all the diagrams with the right multiplicity. For the Ising model, when the dimension of the space is zero, the sum of all the diagrams of a given order $\beta^N (N \geq 1)$ must be zero; i. e. $G_2(p^2=0, \beta)=1$.(Remember that the numerical values of the diagrams are rational numbers at the power D/2 multiplied powers od R^D so we can change freely D without calculate the same diagrams once for every dimensions as in the lattice case).

Using the techniques we have explained in this Section, seven orders have been calculated for the Φ^4 theory and the Ising model. For the higher (V^o, VI^o, VII^o) the computer has been used to reconstruct the topology and the numerical value of the more common diagrams.

The next Section will be devoted to explain the analysis that we have done on the numerical results of our high temperature expansion.

5. - NUMERICAL ANALYSIS OF THE SERIES. -

From the analysis of the high temperature expansion of the two point correlation function and of the correlation lenght is possible to find the value of the critical temperature β_c and the values of the critical components $\gamma = \frac{2 - 2A_\phi}{2 - 2A_{\phi^2}}$ and $\nu = \frac{1}{2 - A_{\phi^2}}$ (A_ϕ and A_{ϕ^2} are the anomalous dimensions of the fields). Let us consider first the two point correlation function.

For $\beta \sim \beta_c$ $G_2(p^2, \beta)\Big|_{p^2=0}$ is expected to behave as:

$$\lim_{\beta \to \beta_c} G_2(0,\beta) \sim \frac{1}{(\beta_c - \beta)^{\gamma}} \qquad (21)$$

Writing $G_2(0,\beta)$ as:

$$G_2(0,\beta) = \sum_{n=0}^{\infty} A_n \beta^n \qquad (22)$$

we have then[(7)]:

$$\lim_{n \to \infty} \frac{A_{n-1}}{A_n} = \beta_c$$
$$\lim_{n \to \infty} n\left(\frac{A_n}{A_{n-1}} \beta_c - 1\right) = \gamma - 1 \qquad (23)$$

At finite orders in n the relations (23) are approximately true apart factors of order 1/n. Numerical methods (Neville tables[(7)]) have been used to extrapolate the value of β_c and γ from finite orders in β. If we define $x_n^o = \frac{A_n}{A_{n-1}}$ we have that $x_n^o \sim \frac{1}{\beta_c} + O(\frac{1}{n})$. We can define the linear extrapolation for x_n^o:

$$x'_n = n_n^o - (n-1)\, x_{n-1}^o \qquad (24)$$

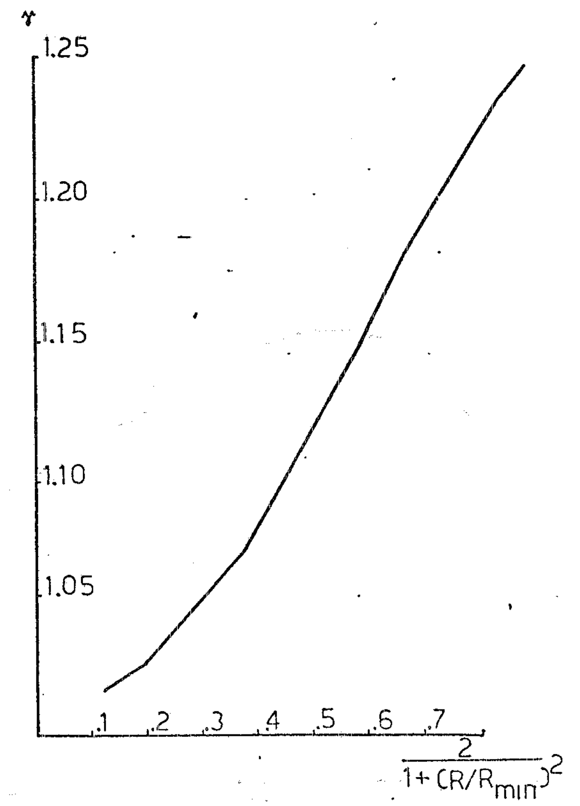
We have now that $x_n^1 - \frac{1}{\beta_c} \sim O(\frac{1}{n^2})$. More in general the m^{th} extrapolant will be defined as:

$$x_n^m = \left[n\, x_n^{m-1} - (n-m)\, x_{n-1}^{m-1} \right] / m \qquad (25)$$

The relation (25), for $m \geq 2$, can be succesfully used only if the ferromagnetic pole is very strong and we have many orders in β. Here we give an example of the Neville table for the calculus of the critical temperature for D=3 and R^2=1. 9.

n	x_n^o	x_n^1	x_n^2
2	26. 55462		
3	26. 24934	25. 63879	
4	26. 06463	25. 51050	25. 38221
5	25. 93989	25. 44093	25. 33658
6	25. 84962	25. 39827	25. 31296
7	25. 78105	25. 36961	25. 29797

In our model, $G_2(p^2, \beta)$ depends also on the parameter R that gives the magnitude of the momentum cutoff ($p_{max} \sim 1/R$). Then the coefficients A_n of the high temperature expansion depend on R. We expect (from the universality principle) that in the limit $n \longrightarrow \infty$ γ will not depend on R. The R dependence arises because we must stop at a finite order in β and gives an annoying dependence of the critical exponent from R. Not for all the values of R we can obtain resonable results. In the limit $R \longrightarrow \infty$ we return to the gaussian model because the loop corrections respect to the diagrams shown in Fig. 2 are at least of order 1/R. For $R \rightarrow 0$ the antiferromagnetic singularity at ngative β is shifted near the origin ($\beta = 0$) so that (23) loose their validity; we must istimate the value of the critical exponent in an intermediate region between too small and too large R, where the extimated critical exponent depends weakly on R. In this region more sofisticated numerical techniques can be used to minimize the effects of this unwanted R dependence. We have used the Padè appoximants to analyse our numerical results. In Fig. 12 here we present the value of the critical exponent γ obtained with the Padè approximant (23) plotted against R in three dimensions for the Ising model.



We have considered the function:

$$\frac{d}{d\beta} \ln G(\beta, 0) = L(\beta)$$

For $\beta \rightarrow \beta_c$ it behaves as $\frac{\gamma}{\beta_c - \beta}(1+O(\beta_c - \beta)^{\alpha})$. We construct the Padè approximants for $L(\beta)$ to find the residuum of the pole nearest to the origin on the positive axis.

The value of β_c given by Padè approximants is consistent with that extracted from the Neville table.

We have also studied the behavior of the series for the correlation lenght ξ:

$$\xi^2 = - \left. \frac{\frac{d^2}{dp^2} G_2(p^2, \beta)}{G_2(p^2, \beta)} \right|_{p^2=0} \qquad (26)$$

FIG. 12

Near the critical temperature ξ goes as $\xi \sim (\beta_c - \beta)^{-\nu}$. The values of ν plotted against R in three dimensions for the Ising model are shown in Fig. 13. From the analysis we have done, our extimates of the critical exponents γ and ν are:

$$\gamma \simeq 1.20, \qquad \nu \simeq 0.60$$

When D=2 the series for the critical exponent are not very easy to extrapolate because of oscillations and poles in the Padè approximants. However using the critical temperature extrapolated by the Padè analysis to compute the Neville tables, we obtained for γ a value near 1.70-1.80. For ν there are not such troubles and Fig. 14 shows the value of ν plotted against R. In this case we obtained $\nu = 1.04$.

In the same way as done in the case of the Ising model, we have considered the $\lambda\Phi^4$ theory. Because of the universality principle the critical exponents should not depend on the value of the coupling constant unless $\lambda \ll 1$. Indeed we must obtain for $\lambda = 0$ that critical exponents of the gaussian

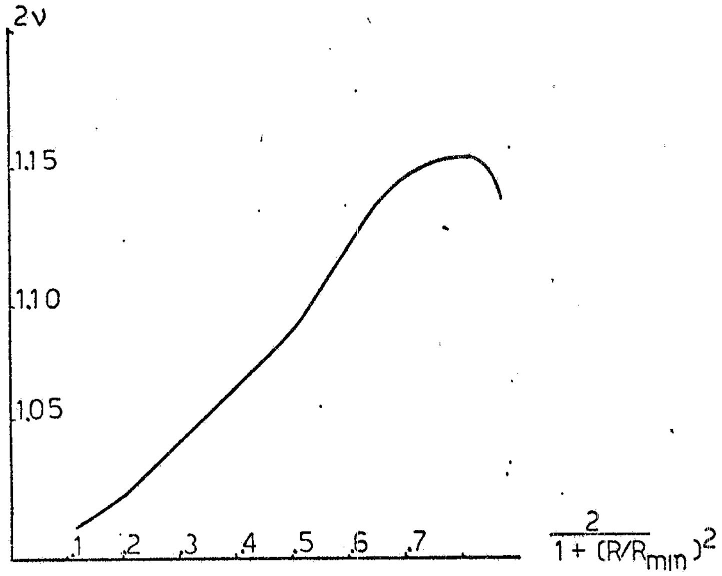


FIG. 13

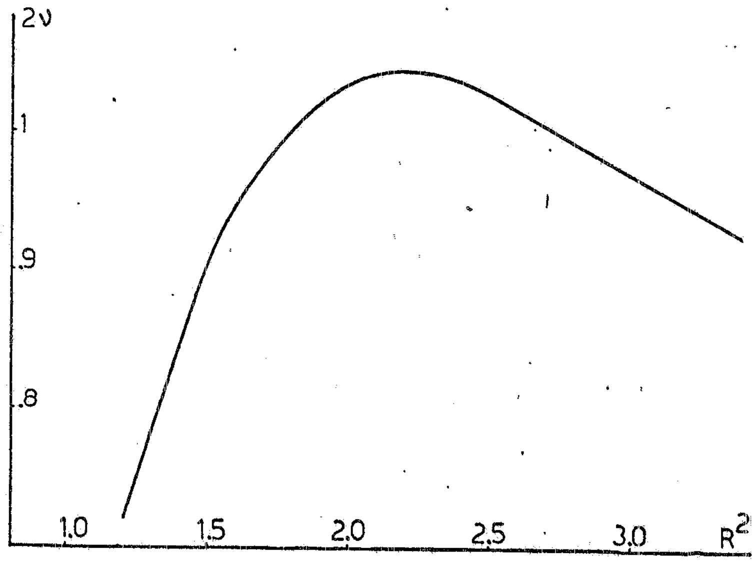


FIG. 14

model that is $\gamma = 1$, $2\nu = 1$. Note that we should obtain, when $\lambda \gg 1$, the same results as in the Ising model because in the limit $\lambda \longrightarrow \infty$ (taking $\frac{m}{2} + \frac{\lambda}{4!} = 1$) the $\lambda\Phi^4$ theory becomes the Ising model. The $\lambda\Phi^4$ theory differs from the Ising model for the value of the semi-invariants H_{2n}. For the Ising model:

$$H_{2n} = \frac{d^{2n}}{dJ^{2n}} \ln \cos h J \Big|_{J=0}$$

for the $\lambda\Phi^4$ theory they are

$$H_{2n} = \frac{d^{2n}}{dJ^{2n}} \ln \left[\int dx \exp - \left(\frac{m^2 x^2}{2} + \frac{\lambda x^4}{4!} - Jx \right) \right] \Big|_{J=0}$$

In Table I we give the critical exponents γ and ν for D=2, 3 and λ =0. 1, 1, 10, 100. For λ =100 we obtain values of ν and γ very near the values of the Ising model. In the case of D=3 γ and ν depend very weakly on λ as expected. When D=2 for small λ we obtain values near the gaussian ones which is quite obvious. Howevere we want to point out that for D=2 we have encountered the same troubles

TABELLA I

Coupling constant	D = 2	D = 3
0.1	$\gamma = 1.47$	$\gamma = 1.11$
	$2\nu = 1.47$	$2\nu = 1.16$
1	$\gamma = 1.53$	$\gamma = 1.18$
	$2\nu = 1.55$	$2\nu = 1.17$
10	$\gamma = 1.65$	$\gamma = 1.20$
	$2\nu = 1.90$	$2\nu = 1.18$
100	$\gamma = 1.70$	$\gamma = 1.20$
	$2\nu = 2.04$	$2\nu = 1.20$

as for the Ising model that is the series of γ in Padè analysis are not stable.

For D=4 we have found values of γ and 2ν very near 1 with deviation only of one, two per cent.

These results show that our model reproduces with a good approximation the results for the Ising model for the $\lambda\Phi^4$ theory found on the lattice. Our hope is that these kind of techniques may be applied to other theories such as gauge theories(8).

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