To be submitted to
Phys. Rev. Letters

ISTITUTO NAZIONALE DI FISICA NUCLEARE
Laboratori Nazionali di Frascati

LNF-77/54(P)
7 Dicembre 1977

R. Benzi and G. Martinelli: VALIDITY OF THE SCALING LAW FOR THE SPECIFIC HEAT IN THREE DIMENSIONAL ISING MODEL.

Servizio Documentazione
dei Laboratori Nazionali di Frascati
Cas. Postale 13 - Frascati (Roma)

INFN - Laboratori Nazionali di Frascati
Servizio Documentazione

LNF-77/54(P)
7 Dicembre 1977

R. Benzi[(x)] and G. Martinelli: VALIDITY OF THE SCALING LAW FOR THE SPECIFIC HEAT IN THREE DIMENSIONAL ISING MODEL.

It is well known[1] that for the three dimensional Ising model the value of the critical exponents α and ν, respectively for the specific heat and for the correlation length, obtained by means of the high temperature expansion seem to be not in agreement with the scaling law:

$$\alpha = 2 - 3\nu. \qquad (1)$$

The disagreement has led[1,2] to suggesting that simple scaling is not valid and eq. (1) should be modified to:

$$\alpha + \delta = 2 - 3\nu, \qquad (2)$$

where $\delta \simeq -0.03$. However the exponents are extracted from the high temperature expansion assuming a pure scaling law of the kind:

$$C(T) \sim (T - T_c)^{-\alpha}\left[1 + O(T - T_c)\right]. \qquad (3)$$

On the other hand the renormalization group[3] theory tells us that instead of (3) we must test the scaling law:

(x) Istituto di Fisica dell'Università di Roma.

$$C(T) \sim (T - T_c)^{-\alpha}\left[1 + O(T - T_c)^{\omega\nu} + O(T - T_c)\right] , \qquad (4)$$

where $\omega\nu$ is of order 1/2.

The aim of this letter is to find a direct test of scaling law (1). We find that (1) is valid but we are not able to extimate $\omega\nu$. The reason for that will become clear later.

Let us sketch the argument that leads to scaling laws in Quantum Field Theory (QFT). Our starting point is the definition of α and ν. When $T \sim T_c$ (critical temperature) the specific heat C and the correlation length ξ have a T-dependence[3]:

$$C \equiv \langle \Phi_B^2 \Phi_B^2 \rangle \sim (T - T_c)^{-\alpha} ,$$
$$\xi \equiv \frac{1}{m_R} \sim (T - T_c)^{-\nu}, \qquad (5)$$

where we have expressed C and ξ in terms of the corresponding field quantities that is the two point correlation function (energy density) of Φ^2 for C and the inverse of the renormalized mass m_R for ξ. The index B stands for bare. Because $\alpha \sim \frac{1}{8}$ it is better, numerically, to work with $\partial C/\partial T$ which is a more singular quantity. Then we express $\partial C/\partial T$ in terms of the renormalized mass and of the coupling constant g through some relation known in QFT[3]:

$$\frac{\partial C}{\partial T} = \langle \Phi_B^2 \Phi_B^2 \Phi_B^2 \rangle = Z_{\phi^2}^3 \langle \Phi_R^2 \Phi_R^2 \Phi_R^2 \rangle =$$
$$= (Z_{\phi^2})^3\, m_R^{3(D-2)-2D} F(g) = \left(\frac{\partial m_R^2}{\partial T}\right)^3 m_R^{D-6} F(g) ,$$

where D is the dimension of the space, in our case D = 3.

Then we consider the quantity Q defined as:

$$Q \equiv \frac{\partial C/\partial T}{m_R^{D-6}(\partial m_R^2/\partial T)^3} = F(g) . \qquad (6)$$

From (5) it easy to see that if scaling law (1) is valid then $F(g)$ is not singular at $T = T_c$, and we have:

$$Q(T) \sim Q(0) + A(T - T_c)^{\omega\nu} + B(T - T_c) + O(T - T_c)^2 . \qquad (7)$$

On the other hand if the scaling law (2) is valid we get, for Q, a singular behaviour caracterized by the exponent δ, namely:

$$Q(T) \sim (T - T_c)^{\delta} , \qquad (8)$$

where we recall that δ is supposed to be -0.03.

The prescription is than: compute an high temperature expansion of Q defined by eq. (6) and by means of this expansion compute the behaviour of Q near the critical temperature. If Q is singular (eq. (8)) then the scaling law (2) is valid; if Q is regular (eq. (7)) the scaling law (1) is valid. The advantage is that using this method we test directly the scaling law without having to compute in an independent way the critical temperature, α and ν.

From the high temperature series[4, 5] it is possible to get the high temperature expansion of C and $R \equiv m_R^{-2}$. In terms of C and R, Q becomes:

$$Q = \frac{\partial C/\partial T \, R^{4,5}}{(\partial R/\partial T)^3} = \frac{\sum_1 A_n \beta^n}{\sum_1 B_n \beta^n} . \qquad (\beta = 1/T)$$

Now define Δ as the exponent of the leading term of Q. By standard technique[6] it is easy to compute Δ by means of the equations:

$$\Delta = \lim_{n \to \infty} \Delta_n ,$$

where

$$\Delta_n = n \left[1 - \frac{A_n}{B_n} \frac{B_{n-1}}{A_{n-1}} \right] . \qquad (9)$$

In Table I we report the value of Δ_n obtained using eq. (9). To estimate Δ it is possible to use standard extrapolations based on Neville tables. Table II is just the Neville table of Δ_n up to quadratic extrapolation. One can get further information using extrapolation of the form[7] :

$$\Delta'_n = \frac{1}{2}\left[(n+\varepsilon)\Delta_n - (n+\varepsilon-2)\Delta_{n-2}\right] ,$$

where $\varepsilon = 1/2$. In Table III we report the values of Δ'_n.

From Tables I, II and III there is no evidence for Δ being a negative number of order 0.03. Indeed the estimate suggest that Δ is very small and positive and this would correspond to a scaling law relation in the direction opposite to the claimed one. We neglect this last possibility and we think there are good evidence for Δ being 0.

Let us try to compute the correction to the scaling law (2). Let us suppose that $\Delta_n \underset{n\to\infty}{\to} n^{-\lambda}$. Standard arguments imply that :

$$\lambda = \lim_{n\to\infty} F_n ,$$

where

$$F_n = -n\left[\frac{\Delta_n}{\Delta_{n-1}} - 1\right] .$$

The F_n are shown in Table IV and their limit is consistent with $\lambda = 1$. Therefore we can see only the coefficient of the regular correction to the scaling law, $O(T - T_c)$, and then $\omega\nu$ cannot be extimated. Indeed it is quite possible that this last term is present but with a quite small coefficient which may be characteristic of spin 1/2 Ising model, the coefficient of the terms being not an universal quantity.

The authors are gratefull to G. Parisi for having explained to us the question and for illuminating discussion.

REFERENCES.

[1]M. Ferer, M.A. Moore and M. Wortis, Phys. Rev. Lett. 22, 1382 (1968).

[2]M.E. Fisher, Rept. Progr. Phys. 30, 615 (1967).

[3]G. Parisi, Lectures given at the Cargese Summer School, 1973.

[4]M.A. Moore, D. Jasnow and M. Wortis, Phys. Rev. Lett. 22, 940 (1969).

[5]M.F. Sykes, J.L. Martin and D.L. Hunter, Proc. Phys. Soc. (London) 91, 671 (1967).

[6]D. Jasnow and M. Wortis, Phys. Rev. 176, 739 (1968).

[7]M.E. Fisher, The Theory of Equilibrium Critical Phenomena.

TABLE I

$\Delta_2 = -1.2525$	$\Delta_7 = -0.3483$
$\Delta_3 = -0.7813$	$\Delta_8 = -0.3058$
$\Delta_4 = -0.5908$	$\Delta_9 = -0.2722$
$\Delta_5 = -0.4793$	$\Delta_{10} = -0.2450$
$\Delta_6 = -0.4037$	$\Delta_{11} = -0.2226$

TABLE II

n	linear extrapolations	quadratic extrapolations
5	- 0.033268	
6	- 0.025569	- 0.0101719
7	- 0.016105	+ 0.0075555
8	- 0.0086124	+ 0.013865
9	- 0.00344699	+ 0.014632
10	- 0.00021840	+ 0.012695
11	+ 0.0015601	+ 0.0095634

TABLE III

n	Δ'_n
5	0.049291
6	0.017362
7	0.0119089
8	0.0120978
9	0.012985
10	0.013367
11	0.01307587

TABLE IV

n	F_n
5	0.9435
6	0.9468
7	0.9604
8	0.9760
9	0.9891
10	1.0000
11	1.0065