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U(N) INVARIANCE IN EXTENDED SUPERGRAVITY

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U(N) invariance of previously established extended supergravity theories (O(1), O(2), O(3)) is shown to be also compatible with the O(4) theory at the first non-trivial order in the Newton constant. If U(4) invariance remains true to all orders the non-polynomial feature of this theory is enormously simplified. As an example we determine, to all orders, the non-polynomial correction to the spin-1 kinetic term using some of the constraints of supersymmetry.

Supergravity theories with a global internal symmetry group O(N) have been constructed [1–5] up to $N = 3$. It has been observed [6] that in these theories the real symmetry O(N) can be extended to a unitary group U(N) provided that chiral transformations on the fermions and duality transformations on the vector fields are introduced. These duality transformations generalize to non-Abelian groups the duality invariance of the Maxwell-Einstein system [7]. This result is expected because, according to Haag et al. [8] the spinor supersymmetry charges belong to the fundamental representation of U(N). It has also been shown that duality transformations are an essential ingredient for the finiteness of one-loop diagrams in extended supergravity [9]. Supergravity theories with $N \geqslant 4$ seem more difficult to construct due to the presence of scalar fields which apparently introduce a non-polynomiality in the interactions. This is already suggested by the work of Das [10] for the O(4) theory where signals of this non-polynomiality are manifest.

It is the aim of this paper to develop techniques which seem useful to handle this non-polynomiality. Supersymmetry is equivalent to a system of algebraic and differential equations for unknown (non-polynomial) functions of the scalar fields which are present both in the Lagrangian and transformation laws. Writing the complete system and solving it would be a straightforward but cumbersome procedure. However, an enormous simplification takes place under the conjecture that the theory is U(4) invariant. This is strongly motivated by the fact that the O(4) theory is actually U(4) symmetric at the first non-trivial order of the interaction and that the supersymmetry equations we have considered are consistent with U(4) invariance. As a result of this investigation we compute the exact modification of the spin-1 kinetic term due to the gravitational interaction with the scalar fields. The complete Lagrangian of the O(4) theory contains a Vierbein $V_{a\mu}$, a quartet of spin 3/2 fields ψ^i_μ and spin 1/2 fields χ^i, a sextet of vector fields A^{ij}_μ and a doublet of spin 0 fields A and B of opposite parity. All these fields belong to the same irreducible multiplet as the graviton.

Because of gauge invariance and local supersymmetry possible non-polynomiality can only occur because of the scalar fields. We consider only the part of the Lagrangian that contains the vector fields. This has the following general form

$$\begin{aligned} V^{-1}\mathcal{L} = &-\tfrac{1}{8}\,[g_1F^2 - g_2FF^* - g_3F\widetilde{F}^* - g_4F\widetilde{F}] \\ &+\tfrac{1}{4}\,[f_1FP + f_2FP^* + f_3F\widetilde{P}^* + f_4F\widetilde{P}\,] \\ &+\tfrac{1}{4}\,[h_1FQ + h_2FQ^* + h_3F\widetilde{Q}^* + h_4F\widetilde{Q}] \end{aligned} \qquad (1)$$

in which

$$F^{*ij}_{\mu\nu} = \tfrac{1}{2}\epsilon^{ijkl}F^{kl}_{\mu\nu}; \qquad \widetilde{F}^{ij}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma},$$

$$FF = F^{\mu\nu ij}F^{ij}_{\mu\nu}; \qquad F\widetilde{F} = F^{\mu\nu ij}\widetilde{F}^{ij}_{\mu\nu}; \ \ldots,$$

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and:

$$P^{ij}_{\mu\nu} = \bar{\psi}^i_\mu \psi^j_\nu - \bar{\psi}^i_\nu \psi^j_\mu - \mathrm{i}\epsilon_{\mu\nu}{}^{\rho\sigma} \bar{\psi}^i_\rho \gamma_5 \psi_\sigma{}^j,$$

$$Q^{ij}_{\mu\nu} = \frac{\mathrm{i}}{2} \epsilon^{ijkl} \bar{\psi}^k_\lambda \sigma_{\mu\nu} \gamma^\lambda \chi^l, \tag{2}$$

all the notation and conventions are the same as in ref. [10]. All coefficients g_i, h_i and f_i are a priori arbitrary functions of KA and KB, where K is the gravitational constant.

The relevant part of the supersymmetry transformation laws of the fields are

$$\delta_S V_{a\mu} = -\mathrm{i}K\bar{\epsilon}^i \gamma_a \psi^i_\mu,$$

$$\delta_S A = \frac{1}{\sqrt{2}} \bar{\epsilon}^i \chi^i a; \qquad \delta_S B = \frac{\mathrm{i}}{\sqrt{2}} \bar{\epsilon}^i \gamma_5 \chi^i b,$$

$$\delta_S \chi^i = -[p_1 F^* + p_2 F + p_3 \widetilde{F} + p_4 \widetilde{F}^*]^{ij}_{\alpha\beta} \sigma^{\alpha\beta} \epsilon^j, \tag{3}$$

$$\delta_S \psi^i_\lambda = \mathrm{i}[q_1 F + q_2 F^* + q_3 \widetilde{F}^* + q_4 \widetilde{F}]^{ij}_{\alpha\beta} \sigma^{\alpha\beta} \gamma_\lambda \epsilon^j.$$

The p_i, q_i, a and b are functions of A and B to be determined[‡1].

In order to impose U(4) symmetry we require invariance only under transformations involving (real) symmetric matrices as O(4) invariance is already fulfilled. As noticed in [6], U(N) invariance is a property of the equations of motion but not of the action.

Generalization of the transformation laws of the U(3) theories suggests the following transformation laws for the fields involved

$$\delta_U V_{a\mu} = 0, \qquad \delta_U \psi^i_\mu = \mathrm{i}\gamma_5 \Lambda^{ik} \psi^k_\mu,$$

$$\delta_U \chi^i = -\mathrm{i}\gamma_5 (\Lambda^{ik} - \delta^{ik} \operatorname{Tr} \Lambda) \chi^k, \tag{4}$$

$$\delta_U A = B \operatorname{Tr} \Lambda, \qquad \delta_U B = -A \operatorname{Tr} \Lambda,$$

$$\delta_U F^{ij}_{\mu\nu} = \Lambda^{ik} \widetilde{G}^{jk}_{\mu\nu} - \Lambda^{jk} \widetilde{G}^{ik}_{\mu\nu}. \tag{5}$$

in which

$$\begin{aligned} G = {} & g_1 F - g_2 F^* - g_3 \widetilde{F}^* - g_4 \widetilde{F} \\ & - f_1 P - f_2 P^* - f_3 \widetilde{P}^* - f_4 \widetilde{P} \\ & - h_1 Q - h_2 Q^* - h_3 \widetilde{Q}^* - h_4 \widetilde{Q}. \end{aligned} \tag{6}$$

Let us comment on how the transformation laws (4) and (5) have been obtained. The field equations for the vector fields are precisely

$$D_\mu G^{\mu\nu ij} = 0. \tag{7}$$

The consistency of the variation of $\delta F^{ij}_{\mu\nu}$ with the Bianchi identity $D_\mu(\widetilde{F}^{\mu\nu ij}) = 0$ is automatically satisfied[‡2]. The transformation laws (4) are determined in such a way that the kinetic terms of all the fields but the vector fields are U(4) invariant[‡3]. These transformations provide also simple variations for the bilinear fermionic tensors $P^{ij}_{\mu\nu}$ and $Q^{ij}_{\mu\nu}$ occurring in the Noether couplings. A non-trivial point is that the Noether couplings (order K) are left invariant by these transformations.

The requirement of SU(4) covariance of the equations of motion for the vector fields imply the following algebraic constraints

$$g_1 g_4 + g_2 g_3 = 0, \; g_1^2 + g_3^2 - g_2^2 - g_4^2 = \text{const.} \tag{8}$$

These equations are obtained by demanding that terms which contain F or F^* in δG cancel while terms in $\widetilde{F}$ and $\widetilde{F}^*$ have constant coefficients.

Next, requiring the coefficients of the P and Q terms to cancel in the variation gives two identical systems of 4 linear homogeneous algebraic equations for the f_i and q_i. The vanishing of the determinant of this system is a requirement for the existence of non-zero, SU(4) invariant Noether couplings. Computing this determinant yields:

$$\Delta = (1 - g_1^2 - g_3^2 + g_2^2 + g_4^2) + 4(g_1 g_4 + g_2 g_3)^2. \tag{9}$$

Its vanishing is indeed automatic once eqs. (8) are fulfilled, with the normalization condition:

$$g_1^2 + g_3^2 - g_2^2 - g_4^2 = 1. \tag{10}$$

In fact because of conditions (8) and (10), not only Δ vanishes but also all the 3 × 3 determinants as is easily seen by finding two linearly independent solutions from which we obtain the general solution as a linear

‡1 It was noticed in the erratum to ref. [10] that the transformation laws of A and B are indeed modified at order K^2.

‡2 One could in principle add to δF terms proportional to $\widetilde{G}^*$. This however would violate the SU(4) invariance of the stress energy tensor $T_{\mu\nu}$.

‡3 Although the kinetic terms of the A and B fields are modified at order K^2, the modification is only a function of $A^2 + B^2$ and thus, the kinetic term of the scalar fields is U(4) invariant.

combination with arbitrary parameters α_f and β_f which can a priori be functions of A and B.

$$f_1 = \alpha_f(1 + g_1) + \beta_f g_4, \qquad f_2 = -\alpha_f g_2 + \beta_f g_3,$$
$$f_3 = -\alpha_f g_3 - \beta_f g_2, \qquad f_4 = -\alpha_f g_4 + \beta_f(1 + g_1). \tag{11}$$

The h_i are related to the g_i by the same relations, with α_f, β_f replaced by two other parameters α_h, β_h.

In a similar way, we can relate the p_i and q_i which appear in the transformation laws of χ and ψ to the g_i by requiring that the spinor charges transform as a $\underline{4}$ under SU(4).

This again leads to the same system of homogeneous linear equations and the solution is again of the form (11) with α_f, β_f being replaced by α_p, β_p or α_q, β_q. So SU(4) invariance has reduced the problem from 22 down to 12 arbitrary functions.

Now we use U(1) invariance to further reduce the number of unknown functions. Since now Tr $\Lambda \neq 0$, the A and B fields transform and we get systems of differential equations. As $K^2(A^2 + B^2) \equiv \rho^2$ is unaffected we only get information about the dependence of the functions on the variable $\theta = \mathrm{arctg}(A/B)$.

We shall only write two of the differential equations satisfied by the g's. They are:

$$\frac{\partial g_1}{\partial \theta} = 2g_1 g_4, \qquad \frac{\partial g_4}{\partial \theta} = 1 + g_4^2 - g_1^2. \tag{12}$$

The two other equations are compatible with SU(4) invariance. The best strategy, for solving (12), is to wait until we also have the constraints of supersymmetry, however at this stage we can already obtain the g_i's up to arbitrary functions of ρ, for instance

$$g_1 = \frac{[1 - H^2(\rho)]^{1/2}}{1 + H(\rho)\cos 2(\theta - \theta_0(\rho))}. \tag{13}$$

Studying the system obeyed by the 8 functions α, β one finds the remarkable feature that they are independent of θ. In the same way U(1) invariance of the supersymmetric transformation laws of A and B imply that $a = b = a(\rho^2)$.

A result of U(4) invariance is that the equation of motion of the vector fields transform into the Bianchi equation as required:

$$\delta G^{ij}_{\mu\nu} = \Lambda^{ik}\tilde{F}^{jk}_{\mu\nu} - \Lambda^{jk}\tilde{F}^{ik}_{\mu\nu}. \tag{14}$$

We have also verified that the equations of motion of the graviton and of the scalar fields are U(4) invariant with the previously derived transformation laws and constraints on the functions occurring in the action.

Finally, we use parity to further restrict our choice of functions. As one easily sees all (1, 2) functions are scalars and all (3, 4) pseudoscalars. As parity is conserved, and we now know that the α, β coefficients, being functions of $A^2 + B^2$ only are scalars, we conclude to the vanishing of all β coefficients which are parity violating. Similarly, one concludes that $\theta_0 = 0$.

So, using U(4) and parity leaves us with only 6 unknown functions $H, \alpha_f, \alpha_q, \alpha_p, \alpha_h, a$ of one variable $\rho = K(A^2 + B^2)^{1/2}$.

Comparing our results at that stage with those obtained in ref. [10] we observe that they are compatible with them. In particular, one verifies the relations:

$$f_i/\alpha_f = h_i/\alpha_h = p_i/\alpha_p = q_i/\alpha_q. \tag{15}$$

It becomes very tempting to use additional information which would explicitly give us those unknown functions. As this can only come from supersymmetry we establish equations satisfied by the unknown functions at all order in K. This can be done most easily by requiring that terms of the type $F^2\chi\epsilon$ and $F^2\psi\epsilon$ cancel in the variation of the action, at all orders in the A, B fields.

Many terms cancel automatically because of relation (15). In addition, one obtains a non-trivial system of equations. From the $F^2\chi\epsilon$ terms, we get differential equations:

$$\frac{\partial g_1}{\partial B} = -\frac{\partial g_4}{\partial A}; \quad \frac{\partial g_4}{\partial B} = \frac{\partial g_1}{\partial A}; \quad \frac{\partial g_2}{\partial B} = \frac{\partial g_3}{\partial A}; \quad \frac{\partial g_2}{\partial A} = -\frac{\partial g_3}{\partial B} \tag{16}$$

$$a\frac{\partial g_1}{\partial A} = 2\sqrt{2}(h_1 q_2 - h_4 q_3)$$

$$a\frac{\partial g_2}{\partial A} = \sqrt{2}(-h_1 q_1 + h_3 q_3 - h_2 q_2 + h_4 q_4),$$

$$a\frac{\partial g_3}{\partial A} = -\sqrt{2}(h_1 q_4 + h_3 q_2 + h_2 q_3 + h_4 q_1), \tag{17}$$

$$a\frac{\partial g_4}{\partial A} = -2\sqrt{2}(h_1 q_3 + h_4 q_2).$$

From the $F^2\psi\epsilon$ terms we get algebraic constraints:

$$4\sum_i q_i f_i + \sum_i p_i h_i = 0 \tag{18}$$

$$-Kg_1 + \sum_i f_i q_i - \frac{1}{4}\sum_i p_i h_i = 0$$

$$Kg_2 + f_1 q_2 + f_2 q_1 + f_4 q_3 + f_3 q_4 - \tfrac{1}{4}(p_1 h_2 + p_2 h_1 + p_3 h_4 + p_4 h_3) = 0. \qquad (19)$$

Let us recall that this system is obtained by using supersymmetry alone, and not U(4). A remarkable feature of it is the appearance of the Cauchy-Riemann conditions for the couples (g_1, g_4); (g_2, g_3). Thus, from supersymmetry alone one knows that:

$$g_1 - ig_4 = k_1(z), \qquad g_3 - ig_2 = k_2(z) \qquad (20)$$

where $z = K(B + iA)$ and k_1, k_2 are analytic functions to be determined. The conditions (8) and (10) simply become:

$$k_1^2 + k_2^2 = 1. \qquad (21)$$

One can verify that this system is not compatible with $g_1 = 1, g_2 = g_3 = g_4 = 0$ except when all h_i, q_i vanish. Thus supersymmetry is seen to imply an inherent non-polynomiality of the theory.

This system of equations cannot by itself determine the 20 functions appearing in it. However we can solve it to a great extent by looking for U(4) invariant solutions.

Putting together eqs. (12) obtained from U(4) invariance and the analyticity condition obtained from supersymmetry we obtain the following differential equation for the function $k_1(z)$;

$$z\frac{dk_1(z)}{dz} = k_1^2(z) - 1. \qquad (22)$$

Whose solution is given by:

$$k_1(z) = \frac{1 - z^2}{1 + z^2}. \qquad (23)$$

In (23) a scaling constant has been chosen so as to reproduce the result of [10] at the lowest non-trivial order in K. From eq. (21) k_2 is given by:

$$k_2(z) = \frac{2z}{1 + z^2}. \qquad (24)$$

Thus, the g_i are fully determined already at this stage. One checks that they agree with those of ref. [10] up to the relevant orders. A rather interesting feature is that a limiting value exists for the A and B fields, namely $B = 0$, $KA = \pm 1$ for which the vector boson stress energy tensor becomes infinite.

A non-trivial check of U(4) compatibility with supersymmetry comes out at the inspection of eqs. (17–19). Since we know the g_i explicitly, eqs. (17) give 4 possibly different expressions for the product $\alpha_h \alpha_q$ which have all to be identical and to depend only on ρ^2. This is indeed what happens and one obtains:

$$\alpha_q \alpha_h = \frac{K}{2\sqrt{2}} a(\rho^2). \qquad (25)$$

Similarly, eqs. (19) give two identical expressions for $\alpha_f \alpha_q - \frac{1}{4}\alpha_p \alpha_h$ which indeed turn out to depend only upon $A^2 + B^2$:

$$\alpha_f \alpha_q - \tfrac{1}{4}\alpha_p \alpha_h = \frac{K}{4}(1 - K^2(A^2 + B^2)). \qquad (26)$$

Finally, the constraints (18) give a last relation between $\alpha_q \alpha_f$ and $\alpha_p \alpha_h$:

$$4\alpha_q \alpha_f + \alpha_p \alpha_h = 0. \qquad (27)$$

So we obtain a system of 3 equations for 5 unknown ones, leaving us just 2 unknown functions of $A^2 + B^2$. Our results are then that using a subsection of the action and of the transformation laws containing 22 arbitrary functions, a few of the supersymmetry constraints and U(4) invariance, one is left with only two unknown functions of one parameter. Clearly, the task of determining all the unknown functions in the whole action is rather formidable, but we hope to have shown that the use of U(N) invariance imposed beforehand is a consistent and valuable shortcut to those calculations.

Geometric techniques have been found to obtain the O(1) and O(2) theories [11]. In the absence of such techniques for O(N) theories with $N > 2$, where non-gauge fields (spin 1/2 and 0) appear, this seems at present the most powerful method to handle the non-polynomiality due to the scalar fields.

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