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A. Turrin: PULSE-SHAPE DEPENDENCE IN POPULATION INVERSION WITH FREQUENCY CHIRPING. -

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ABSTRACT

The response of a two-level atom to an optical pulse which has a sech envelope and a tanh frequency sweep is calculated. This model is an extension of the one considered by Allen and Eberly. The model is shown to give respectable agreement, in the adiabatic and near-adiabatic limits, with the Landau-Zener formula, that gives the transition probability in the case where a constant-intensity linearly chirped radiation is assumed. This indicates that, as far as significant population inversion is concerned, the transition probability is not affected considerably by the shape of the (slowly varying) optical pulse.

Allen and Eberly in their 1975 book/1/ (AE), have provided an analytical solution of the optical Bloch equations/2, 3/ in the case where the detuning $\dot{\Phi}$ and the field envelope $\mathcal{E}$ of a laser pulse have the forms

$$\dot{\Phi} = \delta \tanh(\alpha t) , \tag{1a}$$

$$\mu \mathcal{E} = (\alpha^2 + \delta^2)^{1/2} \operatorname{sech}(\alpha t) , \tag{1b}$$

where μ is the dipole transition matrix element in units of $\hbar$ and α and δ are arbitrary (constant) parameters.

They find for the resulting atomic inversion w

$$w = \tanh(\alpha t) , \tag{1c}$$

i.e., complete inversion at $t = +\infty$.

On the other hand, for a constant-intensity linearly chirped pulse, the response of a two-level atom initially in the ground state

$$\underset{(t=-\infty)}{A = 0}, \qquad \underset{(t=-\infty)}{B = 1}, \tag{2}$$

is described by the well-known Landau and Zener (LZ) formula[4-9]

$$P_{LZ} = \underset{(t=+\infty)}{AA^*} = 1 - \exp\left[-(\pi/2)(\Omega^2/\lambda)\right] \tag{3}$$

which gives the probability of transition to the excited state in the long-time limit. Here $\Omega = \mu\mathcal{E}$ (= constant) and λ is the (constant) time rate of the chirped pulse.

Note that Eq. (3) predicts complete inversion in the adiabatic limit $\Omega^2/\lambda = \infty$ only.

The purpose of the present letter is to provide a natural mathematical framework for a detailed comparison of the two models. As a preliminary remark we observe that there is not a significant difference between the AE result expressed by Eq. (1c) and the "corresponding" LZ result. To see how this comes about, one may consider $\dot{\Phi}$ as a linear function with a slope given by the value assumed by $\ddot{\Phi}$ just at the resonance crossing, i.e. $\ddot{\Phi} = \ddot{\Phi}(0) = \delta\alpha$. Then one can put $\lambda = \delta\alpha$ and (from Eq. (1b) $\Omega^2 = (\mu\mathcal{E}(0))^2 = \alpha^2 + \delta^2$ in Eq. (3), so that one obtains

$$P_{LZ} = 1 - \exp\left[-(\pi/2)(\alpha/\delta + \delta/\alpha)\right] > 1 - \exp(-\pi) \; (\cong 0.957) \tag{3a}$$

for any α/δ, and this corresponds to an almost complete inversion too.

As an extension of the AE model, let the detuning be described by Eq. (1a) and $\mu\mathscr{E}$, more generally, by

$$\mu\mathscr{E} = \varepsilon \,\mathrm{sech}\,(\alpha t) , \tag{4}$$

where ε denotes an arbitrary (constant) parameter.

In the rotating wave approximation, the time-dependent Schrödinger equation for the two-level atom interacting with the optical field leads to the pair of equations/4, 6, 7, 10/

$$i\dot{A} = (\mu\mathscr{E}/2) B \exp(i\Phi) , \tag{5a}$$

$$i\dot{B} = (\mu\mathscr{E}/2) A \exp(-i\Phi) . \tag{5b}$$

These coupled differential equations can readily be separated by differentiation. Thus, we have the equation for A :

$$\ddot{A} - (i\delta - \alpha)\tanh(\alpha t)\dot{A} + (\varepsilon/2)^2 \,\mathrm{sech}^2(\alpha t) A = 0 . \tag{6}$$

It can be transformed into a hypergeometric differential equation/11/

$$z(1-z) A'' + \left[c - (a+b+1)z\right] A' - ab A = 0 \tag{7}$$

by the transformation

$$z = (1/2)(1 + \tanh(\alpha t)) \tag{8}$$

($' \equiv d/dz$), with the three constants

$$2a = i(\delta/\alpha) \pm \left[(\varepsilon/\alpha)^2 - (\delta/\alpha)^2\right]^{1/2} , \tag{9a}$$

$$2b = i(\delta/\alpha) \mp \left[(\varepsilon/\alpha)^2 - (\delta/\alpha)^2\right]^{1/2} , \tag{9b}$$

$$2c = i(\delta/\alpha) + 1 . \tag{9c}$$

As time goes from $-\infty$ to $+\infty$, z spans the interval $0 \leftrightarrow 1$.

The general solution of Eq. (7) can be written in the form

$$A = CF(a, b; c; z) + Dz^{1-c} F(a-c+1, b-c+1; 2-c; z) , \tag{10}$$

where C and D are integration constants and the F's are hypergeometric functions (Eqs. 15. 5. 3 and 15. 5. 4 of Ref. /11/).

In order that the boundary conditions expressed by Eqs. (2) be satisfied we must set $C = 0$. To determine D, it is necessary to reconsider Eq. (5a) which, in terms of the new independent variable z, transforms into the equation

$$2\alpha z(1-z)A' = -(i/2)\,\varepsilon\,\mathrm{sech}(\alpha t)\,B\,\exp(i\Phi)\ . \qquad (11)$$

This gives, in the limit $t = -\infty$,

$$A'A'^{*} = (\varepsilon/(2\alpha))^{2}/z\ . \qquad (12)$$

Differentiating Eq. (10) (with $C = 0$), we have

$$A' = D(1-c)z^{-c}F(a-c+1,\, b-c+1;\, 1-c;\, z) \qquad (13)$$

(use of Eq. 15. 2. 4 of Ref. /11/ has been made); combining Eq. (12) and (13) we obtain, in the limit $t = -\infty$,

$$DD^{*} = (\varepsilon/\alpha)^{2}/\left[1+(\delta/\alpha)^{2}\right]\ . \qquad (14)$$

Now, in the limit $t = +\infty$, we find

$$\underset{(t=+\infty)}{AA^{*}} = DD^{*}\left|\Gamma(2-c)\,\Gamma(c-a-b)/(\Gamma(1-a)\,\Gamma(1-b))\right|^{2}\ , \qquad (15)$$

where the Γ's are gamma functions.

In deriving this expression for the transition probability we have made use of Eq. 15. 1. 20 of Ref. /11/.

Finally, we develop the squared modulus in Eq. (15) and use Eq. (14). We get for the desired transition probability

$$P = \underset{(t=+\infty)}{AA^{*}} = 1-\cos^{2}(\left[x^{2}-y^{2}\right]^{1/2})\,\mathrm{sech}^{2}y \qquad \text{for } x \geq y\ , \qquad (16a)$$

$$P = \underset{(t=+\infty)}{AA^{*}} = 1-\cosh^{2}(\left[y^{2}-x^{2}\right]^{1/2})\,\mathrm{sech}^{2}y \qquad \text{for } x \leq y\ , \qquad (16b)$$

where $x = (\pi/2)(\varepsilon/\alpha)$ and $y = (\pi/2)(\delta/\alpha)$.

To facilitate comparison with the LZ formula, we express (3) in terms of x and y as follows:

$$P_{LZ} = 1 - \exp(-x^2/y) . \tag{17}$$

By looking first at Eq. (16b), we obtain that it reduces precisely to the LZ formula (17) at $\delta \to \infty$, $\alpha \to 0$, with $\alpha\delta$ finite.

Another remark which can be made about Eq. (16b) is that it becomes close to unity for $y \gg 1$ and $x \approx y$. But in this case one obtains $x^2/y \approx y$ ($\gg 1$) and gets P_{LZ} close to unity too.

Finally let us discuss Eq. (16a). Maxima and minima of P occur at

$$x^2 = y^2 + \left[(k + 1/2)\pi\right]^2 \qquad \text{(maxima)}, \tag{18a}$$

and

$$x^2 \simeq y^2 + (k\pi)^2 \qquad \text{(minima)} , \tag{18b}$$

($k = 0; 1; 2; \ldots$) so that

$$P_{max} = 1 , \tag{19a}$$

and

$$P_{min} \cong 1 - \text{sech}^2 y . \tag{19b}$$

The corresponding values for P_{LZ} become

$$P_{LZ} > 1 - \exp\left[-(2k + 1)\pi\right] \gtrsim 0.957 , \tag{20a}$$

and

$$P_{LZ} \begin{cases} > 1 - \exp(-2k\pi) & \text{for } k \neq 0 , \qquad (20b) \\ = 1 - \exp(-y) & \text{for } k = 0 \text{ (i.e., } x = y) . \qquad (20c) \end{cases}$$

By Eqs. (20a) and (20b) one sees that P_{LZ} remains very close to unity in any case.

On Eqs. (19b) and (20c) we merely note that both predict almost complete inversion for $y \gg 1$.

In conclusion, we have shown that both the AE and LZ models predict almost complete inversion in the near-adiabatic limit (we are mainly interested in). This suggests that the transition probability does not depend upon the details of any other form one may devise for the pulse envelope, provided that it is described by a well-behaving function of time.

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