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ABSTRACT.

A calculation is presented of the asymptotic final states of a spin 1/2 particle, initially in the ground state, in the presence of a perturbing magnetic field when the precession angular frequency sweeps exponentially toward an asymptotic value which is close to a spin resonance line. The resulting very simple formula reduces (for the ANL-ZGS) to the Froissart and Stora formula⁽⁴⁾ in the case where the depolarizing resonance is passed through, or, if not, to the adiabatic following approximation formula. Very large γ -intervals between adjacent resonances are available for the flat-topped ZGS.

1. - INTRODUCTION

Acceleration of polarized protons to high energies is now standard practice⁽¹⁾ at the Argonne ZGS. It is achieved by rapidly changing at proper times the vertical betatron wavenumber Q so that the intrinsic depolarization resonances (which occur when $\gamma G = k \pm Q$) are jumped during acceleration. Imperfection resonances (characterized by $\gamma G = k$) do not look harmful during normal ZGS operation (here, γ is the ratio of total energy of particle to its rest energy; $G = g/2-1$; g is the gyromagnetic factor; $G = 1.79$ for protons; k is any integer number; $Q = .8$).

2.

In addition, in the past year a variety of beautiful machine studies⁽²⁾ have been undertaken at the ZGS to investigate the beam depolarization due to:

- i) suitable perturbations deliberately introduced in the field of the Accelerator;
- ii) the effect of imperfection resonances in the case where the beam is allowed to circulate on a long flat-top placed far from any intrinsic resonance.

Experiments ii) were originally motivated by the idea⁽²⁾ that one might be able to store high-energy polarized proton beams, and therefore these ii) experiments are the ones which we are interested in, in the present paper. The preliminary encouraging experimental results are summarized in Fig. 14 of Ref. ⁽²⁾, that shows the depolarization rates at energies close to the $\gamma G = k = 6$ imperfection resonance.

To interpret these results and to calculate the γ -space available in working between two adjacent resonances, we derive in the present paper a closed-form solution to the spin motion equations of a spin 1/2 particle subject to a more and more slowly increasing precession angular frequency (modeled as an exponential function) which intersects (or does not) the resonance line. Then by examining the asymptotic forms of the solution at early and late times, we derive a simple formula for the final state as a function of few Machine parameters which are considered as basic.

2. - THE EXPONENTIAL MODEL

As a simple model that exhibits the changeover from the accelerating period to the flat-top period, let

$$\gamma(t) - \gamma_{\text{res}} = \Delta\gamma - \gamma_0 \exp(-\lambda t), \quad (1)$$

$$\Delta\gamma = \gamma_{\infty} - \gamma_{\text{res}}$$

where $-\infty < t < +\infty$, γ_{res} is the resonance "energy" and γ_{∞} (constant) denotes the asymptotic γ value. $\Delta\gamma$ may be positive or negative. The positive constant γ_0 should not appear in the final expression for the transition probability, because it is dependent on the origin chosen for the time axis.

In order to see how the exponential model works, let us consider with more detail the case where $\Delta\gamma > 0$. In such a case we may choose the origin of the time axis just at the resonance crossing point by letting $\gamma(0) = \gamma_{\text{res}}$, so that Eq. (1) becomes

$$\gamma(t) - \gamma_{\text{res}} = \Delta\gamma \{1 - \exp(-\lambda t)\} \quad (1a)$$

$$(\Delta\gamma > 0)$$

and consequently

$$\dot{\gamma}_{\text{res}} = \dot{\gamma}(0) = \lambda \Delta\gamma \quad (\Delta\gamma > 0). \quad (1b)$$

Thus, from Eq. (1b) it follows that the exponential model works well if

$$\Delta\gamma < \dot{\gamma}_n / \lambda \quad (\text{case } \Delta\gamma > 0) \quad (2)$$

where $\dot{\gamma}_n$ (=constant) is the normal $\dot{\gamma}$ in the Synchrotron.

If we take $\dot{\gamma}_n = 12/\text{sec}$ and $^{(3)} 1/\lambda \approx 1 \cdot 10^{-2}$ sec for the ZGS we obtain $\Delta\gamma < .12$, giving ample γ -space to study the spin resonance phenomena in the neighbourhood of the resonance tails.

The effect of phase oscillations in passing through an imperfection resonance will be neglected; in other words, we will assume that a series of "few" rapid traversals of a weak resonance occurs. Later it will be shown that this assumption is consistent with the results, at least for $\dot{\gamma} > .07/\text{sec}$.

4.

3. - AN APPROXIMATE APPROACH

In this paragraph we will give an approximate expression for the asymptotic value (i. e. at $t \rightarrow +\infty$) of the vertical component S_z of the polarization vector \bar{S} , starting with the initial conditions $S_x(t \rightarrow -\infty) = 0$, $S_y(t \rightarrow -\infty) = 0$, $S_z(t \rightarrow -\infty) = 1$.

To assess the accuracy of the above-mentioned approximate formula, in the subsequent paragraphs 4 and 5 a rigorous treatment of the problem will be developed. Since the approximate and exact expressions which come out are nearly identical, one may easily follow the formulation given in the present paragraph, without making the (more convincing) calculation developed in paragraphs 4 and 5. The advantage of the subsequent approximate approach is that it is of great help in visualizing the evolution of the polarization vector \bar{S} .

Two situations are of practical interest:

- i) Flat-top placed slightly above a resonance line ($\Delta\gamma > 0$). In this case $\gamma - \gamma_{res}$ may be modeled as a linear function with a slope given by the value assumed by $\dot{\gamma}$ just at the resonance crossing, i. e. $\dot{\gamma} = \dot{\gamma}_{res} = \dot{\gamma}(0) = \lambda \Delta\gamma$. Thus, $S_z(t \rightarrow +\infty)$ may be expressed by the Froissart and Stora formula⁽⁴⁾ and one gets Eq. (22) of the present paper, where ω (= constant) is strength of the perturbation, ω_c is the angular velocity of the particle and $\delta\gamma/\gamma_1$ is the Machine's energy resolution.
- ii) Flat-top placed slightly below a resonance line ($\Delta\gamma < 0$). In this case $\gamma - \gamma_{res}$ varies slowly, \bar{S} may be thought to be adiabatically following the effective driving field, so that the adiabatic following approximation may be adopted to describe the polarization vector evolution. Thus we have at $t \rightarrow +\infty$ Eq. (23) of the present paper.

4. - THE SPIN MOTION EQUATIONS

In the rotating perturbing field approximation and in a reference frame which is attached to the particle and which rotates about the main field direction with angular frequency equal to the rotating field angular frequency, the Schrödinger time-dependent equation leads to the pair of coupled equations⁽⁴⁾

$$\dot{f} = -i (\omega/2) g \exp(-i\chi) \quad (3a)$$

$$\dot{g} = -i (\omega/2) f \exp(i\chi), \quad (3b)$$

where $g(t)$ and $f(t)$ are the occupation numbers of the states $(+1/2)$ and $(-1/2)$ and

$$\chi = \omega_c G \int (\gamma - \gamma_{res}) dt. \quad (4)$$

With the normalization condition $ff^* + gg^* = 1$, the evolution of the vertical component of the polarization vector S_z is given by the equation

$$S_z = 1 - 2 gg^* \quad (5)$$

(the star denotes complex conjugation).

Using Eq. (1) and introducing the dimensionless independent variable $\tau = (\omega/2)t$ and the dimensionless parameters $a = 2\lambda/\omega$, $\alpha = 2\omega_c G \gamma_0/\omega$ and $b = \omega_c \Delta\gamma G/\omega$, Eqs. (3) transform into

$$df/d\tau = -i g \exp(-i\chi) \quad (3c)$$

$$dg/d\tau = -i f \exp(i\chi), \quad (3d)$$

where

$$\chi = - \int \{ a \exp(-a\tau) - 2b \} d\tau \quad (4a)$$

6.

Combining Eqs. (3c) and (3d), we obtain, e. g., for g ,

$$d^2 g / d\tau^2 + i \{ \alpha \exp(-a\tau) - 2b \} dg / d\tau + g = 0. \quad (6)$$

On introduction of the function F by the substitution

$$g = F \exp(+i\chi/2) \quad (7)$$

we get the following equation for F :

$$d^2 F / d\tau^2 + \{ 1 + i\alpha \exp(-a\tau)/2 + (\alpha \exp(-a\tau) - 2b)^2 / 4 \} F = 0. \quad (8)$$

Finally, with the substitutions

$$u = i \alpha \exp(-a\tau) / a \quad (9)$$

for the independent variable u ,

$$p_0 = (1 + b^2)^{1/2} / a, \quad p_+ = p_0 + b/a$$

for the parameters and

$$F = \exp(-u/2) u^{-ip_0} w \quad (10)$$

for the new unknown function w , we arrive at the equation

$$uw'' + \{ (1 - 2ip_0) - u \} w' + ip_+ w = 0, \quad (11)$$

where the prime denotes differentiation with respect to u .

Eq. (11) is the well known confluent hypergeometric equation⁽⁵⁾.

Thus, the conclusion can be drawn that the coupled differential Eqs. (3)

for the occupation numbers f and g can be solved exactly when the exponential model given by Eq. (1) is adopted.

5.- ANALYSIS

Eq. (11) has two linearly independent solutions, which we take as y_1 and y_7 (i. e. Eqs. 13. 1. 12 and 13. 1. 18) of Reference ⁽⁵⁾. Notations adopted in Ref. ⁽⁵⁾ will be used throughout.

Thus, we write for g

$$\exp\{(u-i\chi)/2\} g = u^{-ip_0} [CM(-ip_+, 1-2ip_0, u) + D \exp(u)U(1-ip_-, 1-2ip_0, -u)], \quad (12)$$

where $p_- = p_0 - b/a$, and C and D are integration constants.

We are interested in C and D in the limits $t \rightarrow -\infty$, $t \rightarrow +\infty$, which correspond to $u \rightarrow +i\infty$ and $u \rightarrow +i0$, respectively.

Using the appropriate forms for M and U (Eqs. 13. 5. 1 and 13. 5. 2 of Ref. ⁽⁵⁾) we have, for $u \rightarrow +i\infty$,

$$M(-ip_+, 1-2ip_0, u) = \exp(p_+ \pi) u^{ip_+} \Gamma(1-2ip_0) / \Gamma(1-ip_-) \quad (13a)$$

and

$$U(1-ip_-, 1-2ip_0, -u) = (-u)^{ip_-} / (-u), \quad (13b)$$

where the Γ 's are gamma functions.

Now, if the particle is initially ($t \rightarrow -\infty$) in the spin state $(+1/2)$, we have $g=0$ ($f^*=1$), and consequently $C = 0$. Thus, the limiting form for g is expressed by

$$g(u \rightarrow +i\infty) \simeq -D \exp\{(u+i\chi)/2\} u^{-ip_0} (-u)^{ip_-} / u. \quad (14)$$

8.

To determine D, it is necessary to reconsider Eq. (3d), which, in terms of the new independent variable u, transforms into the equation

$$a u g' = i f \exp(i \chi). \quad (3c)$$

This gives, in the limit $u \rightarrow +i\infty$,

$$a^2 u u^* |g'|^2 = 1. \quad (15)$$

Boundary condition expressed by Eq. (15) will determine DD^* , as follows:

Differentiating Eq. (14) we have

$$g'(u \rightarrow +i\infty) \simeq (1/2)(i\chi' + 1) g(u \rightarrow +i\infty). \quad (16)$$

According to Eqs. (9) and (4a) one gets $(1/2)(i\chi' + 1) = 1 - ib/(au)$, so that

$$g'(u \rightarrow +i\infty) \simeq g(u \rightarrow +i\infty). \quad (17)$$

Putting boundary condition (15) into Eq. (17) (namely Eq. (14)) one finds

$$DD^* = \exp \left\{ -(p_0 + p_-)\pi \right\} / a^2. \quad (18)$$

Now, in the limit $t \rightarrow +\infty$ (i. e. $u \rightarrow +i0$), we write

$$g(u \rightarrow +i0) = D \exp \left\{ (i\chi + u)/2 \right\} u^{-ip_0} U(1 - ip_-, 1 - 2ip_0, -u). \quad (19)$$

The appropriate asymptotic form for U as $u \rightarrow +i0$ can be expressed by taking Eq. 13.1.3 of Reference ⁽⁵⁾, and replacing in it the M. functions by unity. One gets in the limit $u \rightarrow +i0$,

$$U(1 - ip_-, 1 - 2ip_0, -u) = -i\pi / \sinh(2p_0\pi). \quad (20)$$

$$\cdot \left[1 / \left[\Gamma(1 + ip_+) \Gamma(1 - 2ip_0) \right] - (-u)^{2ip_0} / \left[\Gamma(1 - ip_-) \Gamma(1 + 2ip_0) \right] \right].$$

Finally, we develop the squared modulus $|g(u \rightarrow +i0)|^2$ and use Eq. (18). We get for the desired transition probability

$$|g(t \rightarrow +\infty)|^2 = \left[\frac{\sinh(p_+ \pi)}{ap_+ \exp(p_- \pi)} + \frac{\sinh(p_- \pi) \exp(p_+ \pi)}{ap_-} \right] \cdot \left[2 ap_0 \sinh(2 p_0 \pi) \right]^{-1} \quad (21)$$

In writing Eq. (21) we have dropped a rapidly oscillating term which comes in, since its time average value is zero. Such an oscillatory asymptotic behaviour is (qualitatively) similar to that which one can find into the Rabi solution (case where $\gamma(t) - \gamma_{\text{res}} = \Delta\gamma = \text{constant}$). Therefore, it should be not surprising that our result has some relation to Rabi's oscillation.

Some algebraic rearrangement leads for $1 - S_z = 2|g(t \rightarrow +\infty)|^2$ the more useful form

$$1 - S_z = 1 + \left[1 + \exp(-4\pi p_0) - 2 \exp(-2 p_- \pi) \right] \cdot b (1+b^2)^{-1/2} \left[1 - \exp(-4\pi p_0) \right]^{-1}, \quad (21a)$$

where

$$p_0 = (1+b^2)^{1/2} / a, \quad p_- = p_0 - b/a,$$

$$b = \omega_c (\gamma G) (\Delta\gamma / \gamma) / \omega,$$

and

$$a = 2\lambda / \omega.$$

6. - FINAL EXPRESSION AND NUMERICAL ESTIMATES FOR TRANSITION PROBABILITIES IN THE FLAT-TOPPED ZGS

Let us investigate firstly the case where the flat-top field is close to the resonance $\gamma G = k = 6$. An inspection of the experimental data collected

in both Fig. (10) and Fig. (12) of quoted Reference⁽²⁾ readily reveals that $\pi\omega^2/(2\omega_c) \approx .016$ rad/sec for the unperturbed ZGS (i. e. without pulsing poleface windings of the Machine). This shows that for sufficiently large values of \dot{B} ($> \sim 100$ Gauss/sec) polarization is not affected considerably by synchrotron oscillations.

If we take $\omega_c = 10^7$ rad/sec and $1/\lambda \approx 1 \cdot 10^{-2}$ sec, it follows $\exp(-4 p_0 \pi) < 2 \cdot 10^{-9}$. Moreover, assuming for the Machine's energy resolution $|\delta\gamma/\gamma| \approx 2 \cdot 10^{-4}$, we get $|b| > 38$ for any $|\Delta\gamma/\gamma| > |\delta\gamma/\gamma|$, and $p_- = 1/(2ab)$. Thus Eq. (21a) can be replaced by the Froissart and Stora⁽⁴⁾ solution

$$1 - S_z = 2 \cdot 2 \exp \left[- \pi \omega^2 / (2 \omega_c (\gamma G) \lambda (|\Delta\gamma/\gamma|)) \right] \quad (22)$$

in the case where $|\Delta\gamma/\gamma| > |\delta\gamma/\gamma|$.

Similarly, for any $|\Delta\gamma/\gamma| < |\delta\gamma/\gamma|$, we have $2\pi p_- \gg 1$, and the adiabatic following approximation solution

$$1 - S_z = 1 - \left[1 + (\omega / (\omega_c \gamma G |\Delta\gamma/\gamma|))^2 \right]^{-1/2} \quad (23)$$

holds in the case where $|\Delta\gamma/\gamma| < |\delta\gamma/\gamma|$.

Eqs. (22) and (23) give

$$1 - S_z = \begin{cases} .35 \cdot 10^{-3} & \text{just below} \\ .25 & \text{just above} \end{cases} \quad \text{the } \gamma G = k=6 \text{ resonance.}$$

Finally, let us consider the case where the flat-top is placed slightly below the $\gamma G = k - Q$ intrinsic resonance, with $k=8$. From Eq. (64) and Table I of the quoted Reference⁽¹⁾ we get $\omega = 7 \cdot 10^3$ rad/sec, and consequently $4 p_0 \pi \gg 1$, $2\pi p_- \gg 1$ and $|b| \approx 1 \cdot 10^4$, $|\Delta\gamma/\gamma| > 1 \cdot 10^4$, $|\delta\gamma/\gamma| \approx 2$.

From Eq. (23) we have numerically

$$1 - S_z = \begin{cases} .1 \text{ just below the } \gamma G = 8 - Q \text{ resonance line} \\ .5 \cdot 10^{-2} \text{ for } \Delta\gamma/\gamma = - 1. \cdot 10^{-3}. \end{cases}$$

7. - CONCLUSION

The effect of depolarizing resonances for protons circulating on an exponentially growing flat-top field in the ZGS has been calculated using both an elementary and a rigorous procedure. Resonant depolarization can be evaluated with good accuracy by making use of Eqs. (22) and (23) of the present paper. It turns out that we can avoid significant beam-depolarization even when we work with a flat-top which is placed slightly below ($|\Delta\gamma/\gamma| \approx \text{few } 10^{-3}$) an intrinsic resonance tail.

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