Submitted to
Rivista del Nuovo Cimento

COMITATO NAZIONALE PER L'ENERGIA NUCLEARE
Laboratori Nazionali di Frascati

LNF-75/46(P)
24 Ottobre 1975

S. Ferrara: SUPERSYMMETRY (FERMI-BOSE SYMMETRY). A NEW INVARIANCE OF QUANTUM FIELD THEORY.

Servizio Documentazione
dei Laboratori Nazionali di Frascati del CNEN
Casella Postale 70 - Frascati (Roma)

Laboratori Nazionali di Frascati del CNEN
Servizio Documentazione

LNF-75/46(P)
24 Ottobre 1975

S. Ferrara[(x)]: SUPERSYMMETRY (FERMI-BOSE SYMMETRY). A NEW INVARIANCE OF QUANTUM FIELD THEORY.

1. - INTRODUCTION. -

One of the dreams of theoretical physicists has been for a long time, the search for non trivial generalizations of space-time symmetries mainly in order to incorporate, not in an ad hoc way, properties that fundamental interactions of particles exhibit in Nature beyond relativistic invariance.

Two years ago Wess and Zumino(98, 91, 89, 90), generalizing from dual model super-gauge symmetries, succeded to construct an algebraic structure in physical four-dimensional space-time, now called supersymmetry, which is quite remarkable in at least two respects:

a) the irreducible representations of this symmetry combine fermions with bosons;
b) it is a truely relativistic spin-containing symmetry and the structures of known previously stated no-go theorems(60, 11) are circumvented.

The algebraic structure underlying supersymmetry is not a Lie algebra and this is the main reason why previous attempts to construct non trivial generalizations of relativistic symmetries in the framework of conventional Lie algebras failed.

Surprisingly enough, later investigations have shown that almost all known renormalizable interactions of local Quantum Field Theory can be arranged in such a way to manifest this symmetry.

Moreover, this symmetry has been shown not only to be preserved by the quantum corrections but also to provide the less divergent model field theories known up to now.

It is the aim of this article to give an almost complete review of the main applications of supersymmetry invariance in the Physics of particles and fields.

(x) - CERN, Geneva (Switzerland); Address after December 1975: Laboratoire de Physique Theorique, CNRS, Ecole Normale Superieure, Paris (France);
Permanent address: Laboratori Nazionali di Frascati (Italy).

2. - THE SUPERSYMMETRY ALGEBRA. -

The supersymmetry algebra, as naturally arises generalizing from dual models(1, 42, 57, 70), is obtained adding to the conformal algebra of space-time two Majorana spinors Q_α, S_α and a chiral charge Π.

The two spinor charges, to be called restricted and special supersymmetry transformation generators, admit the following commutation relations with the conformal and chiral charges:

$$\begin{aligned}
&[Q_\alpha, D] = \frac{i}{2} Q_\alpha && [S_\alpha, D] = -\frac{i}{2} S_\alpha \\
&[Q_\alpha, M_{\mu\nu}] = i(\sigma_{\mu\nu}Q)_\alpha && [S_\alpha, M_{\mu\nu}] = i(\sigma_{\mu\nu}S)_\alpha \\
&[Q_\alpha, P_\mu] = 0 && [S_\alpha, K_\mu] = 0 \qquad (2\text{-}1)\\
&[Q_\alpha, K_\mu] = -i(\gamma_\mu S)_\alpha && [S_\alpha, P_\mu] = i(\gamma_\mu Q)_\alpha \\
&[Q_\alpha, \Pi] = -\frac{3}{4} i(\gamma_5 Q)_\alpha && [S_\alpha, \Pi] = \frac{3}{4} i(\gamma_5 S)_\alpha,
\end{aligned}$$

and the following anticommutation relations among themselves

$$\begin{aligned}
&\{Q_\alpha, \overline{Q}_\beta\} = -2P_\mu\gamma^\mu_{\alpha\beta} \qquad \{S_\alpha, \overline{S}_\beta\} = 2K_\mu\gamma^\mu_{\alpha\beta} \qquad (2\text{-}2)\\
&\{Q_\alpha, \overline{S}_\beta\} = 2(\sigma^{\mu\nu}M_{\mu\nu} - D + 2\gamma_5\Pi)_{\alpha\beta}.
\end{aligned}$$

It is for instance in this general form that the supersymmetry algebra in four-dimensions was originally discovered by Wess and Zumino(91).

We note that we are using a Majorana representation for the γ matrices i.e. they are real and satisfy the properties

$$\gamma_i^2 = -\gamma_o^2 = 1 \qquad \sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu] \qquad (2\text{-}3)$$

and

$$\overline{\Psi} = \Psi\gamma^o = \gamma_o\Psi \qquad \text{for Majorana spinor.} \qquad (2\text{-}4)$$

If we introduce parameters α's which are totally anticommuting spinors (they can be regarded as odd elements of a Grassmann Algebra)

$$\alpha_i\alpha_j = -\alpha_j\alpha_i \qquad i, j = 1, \ldots, 4 \qquad (2\text{-}5)$$

then the anticommutation relations (2-2) become commutation relations among the infinitesimal action $\delta = \overline{\alpha}Q$ of an element of the algebra. Moreover we observe that a general supersymmetry transformation is an eight (anticommuting) parameter transformation because we must associate a Majorana spinor α^o to the charge Q_α and an independent Majorana spinor α' to the charge S_α, they can embedded in a linear x dependent spinor(91)

$$\alpha(x) = \alpha^o + \gamma_\mu x^\mu \alpha' \qquad (2\text{-}6)$$

which is in fact the most general solution of the differential equation

$$(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu - \frac{1}{2} g_{\mu\nu}\gamma\cdot\partial)\,\alpha(x) = 0 \qquad (2\text{-}7)$$

which ensures that the product of two such transformations

$$\xi_\mu(x) = 2i\bar{\alpha}_1(x)\gamma_\mu\alpha_2(x) \;, \tag{2-8}$$

is indeed a conformal transformation of the space-time point x, i.e. it satisfies the differential equation

$$\partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) - \frac{1}{2} g_{\mu\nu} \partial \cdot \xi(x) = 0 \,. \tag{2-9}$$

We know that the general solution of (2-9) is

$$\xi_\mu(x) = c_\mu + \omega_{\mu\nu} x^\nu + \varepsilon x_\mu + a_\mu x^2 - 2 x_\mu a \cdot x \qquad (\omega_{\mu\nu} = - \omega_{\nu\mu}) \,, \tag{2-10}$$

this means that the parameters involved in (2-10) can be uniquely expressed, because of (2-8), in terms of the anticommuting parameters α_1^o, α_1^1, α_2^o, α_2^1 as the following bilinear expressions

$$\begin{aligned} c_\mu &= 2i\,\bar{\alpha}_1^o \gamma_\mu \alpha_2^o \;, \\ a_\mu &= 2i\,\bar{\alpha}_1^1 \gamma_\mu \alpha_2^1 \;, \\ \varepsilon &= 2i(\bar{\alpha}_1^o \alpha_2^1 - \bar{\alpha}_2^o \alpha_1^1), \\ \omega_{\mu\nu} &= i(\bar{\alpha}_1^o [\gamma_\mu, \gamma_\nu] \alpha_2^1 - \bar{\alpha}_2^o [\gamma_\mu, \gamma_\nu] \alpha_1^1) \,, \end{aligned} \tag{2-11}$$

the remaining independent bilinear combination of the α's being the parameter of the chiral transformation

$$\eta = 4i(\bar{\alpha}_1^1 \gamma_5 \alpha_2^o - \bar{\alpha}_2^1 \gamma_5 \alpha_1^o) \,. \tag{2-12}$$

We remark that the above algebraic system given by eqs. (2-1), (2-2) can be written in a much more elegant form if we embed tha two spinor charges in a eight-dimensional spinor χ_a which transforms according to the Dirac representation of the spinor group SU(2,2) (locally isomorphic to O(4,2)) and the corresponding 15 conformal generators in a 6x6 skew-symmetric matrix J_{AB} which corresponds to the generators of the O(4,2) (conformal) algebra.

Then we get[31]

$$\begin{aligned} [\chi_a, J_{AB}] &= i\gamma_{ABa}{}^b \chi_b \\ [\chi_a, \Pi] &= -\frac{3}{4} i \gamma_{7a}{}^b \chi_b \\ \{\chi_a, \bar{\chi}_b\} &= 2(\gamma_{ab}^{AB} J_{AB} + 2\gamma_{7ab}\Pi) \;, \end{aligned} \tag{2-13}$$

where $\gamma_7 = \frac{1}{6!} \varepsilon^{ABCDEF} \gamma_A \gamma_B \gamma_C \gamma_D \gamma_E \gamma_F$ is the chiral transformation matrix.

We observe that the previous algebraic system, which is an example of Graded Lie Algebra[12, 13, 39, 53, 54, 58, 59, 66, 85, 43], could be useful, with regard to possible applications to Physics, only for massless systems. This is of course due to the fact that such a closed algebra contains the conformal charges. However there is a non trivial subalgebra[92, 73, 33, 71, 72] of the previous structure generated by the Poincare generators $M_{\mu\nu}$, P_ρ and the spinor charge Q_α

$$\{Q_\alpha, \bar{Q}_\beta\} = -2P_\mu \gamma^\mu_{\alpha\beta} \;, \tag{2-14}$$

$$\left[Q_\alpha, P_\mu\right] = 0 \qquad \left[Q_\alpha, M_{\mu\nu}\right] = i(\sigma_{\mu\nu}Q)_\alpha \tag{2-15}$$

whose elements commute with the square mass operator $P^\rho P_\rho$ and which therefore can very well be a symmetry of a massive system.

We will confine ourselves, in the sequel, to the smaller graded algebra defined by eqs. (2-14), (2-15) in order to avoid the usual problems connected with massless particles.

Finally we note that the product of two transformations generated by the Q_α give, because of eq. (2-14)

$$\left[\delta_1, \delta_2\right] = \left[\bar{\alpha}_1 Q, \bar{\alpha}_2 Q\right] = -2\,\bar{\alpha}_1 \gamma^\mu \alpha_2 P_\mu \tag{2-16}$$

a four-dimensional translation with parameter $a_\mu = -2\bar{\alpha}_1 \gamma_\mu \alpha_2$ which is a nilpotent quantity i. e. $a_\mu^P = 0$ for $P \leq 8$ due to the anticommuting properties of the α's. As a consequence any translated quantity $f(x+a)$ will depend only on a finite number of derivatives at the point x.

We shall see in the sequel that this is the fundamental property which allows to construct representations of this relativistic symmetry with a finite number of particle fields.

3. - NON TRIVIAL MIXING OF INTERNAL SYMMETRIES WITH SUPERSYMMETRY. -

The supersymmetry algebra given by eqs. (2-14), (2-15) can be generalized in a non trivial way, assuming that the spinor charge belong to some representation of an internal symmetry group G. This has been achieved by Salam, Strathdee(76, 77, 78) and Wess, Zumino(100).

Imagine the previous algebra written in two component Weyl formalism

$$\begin{aligned} &\left\{Q_\alpha, Q_\beta\right\} = 0 \qquad &&\left\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\right\} = 0 \\ &\left\{Q_\alpha, \bar{Q}_{\dot\beta}\right\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \qquad &&\left[Q_\alpha, P_\mu\right] = 0 . \end{aligned} \tag{3-1}$$

Then if the Q^L_α transform now according to some representation of G and the $\bar{Q}^L_{\dot\beta}$ according to the complex conjugate representation, the algebra (3-1) will become

$$\begin{aligned} &\left\{Q^L_\alpha, Q^M_\beta\right\} = \left\{\bar{Q}^L_{\dot\alpha}, \bar{Q}^M_{\dot\beta}\right\} = 0 \\ &\left\{Q^L_\alpha, \bar{Q}^M_{\dot\beta}\right\} = 2\,\delta^{LM}\sigma^\mu_{\alpha\dot\beta} P_\mu \\ &\left[Q^L_\alpha, P_\mu\right] = 0 \qquad \left[Q^L_\alpha, B_l\right] = i S_l^{LM} Q^M_\alpha \\ &\left[B_l, B_m\right] = i f^k_{lm} B_k , \end{aligned} \tag{3-2}$$

where S_l^{LM} are the hermitian matrices of the representation to which the Q^L_α belong and B_l are the generators of the (compact) internal symmetry group.

Of course we have written only the relevant commutators and anticommutators, the others being obvious.

Actually the algebraic system given by (3-2) can be further generalized in the sense that the first anticommutator does not need to vanish. Namely it can have the form

$$\left\{Q^L_\alpha, Q^M_\beta\right\} = \varepsilon_{\alpha\beta} Z^{LM} , \tag{3-3}$$

where Z^{LM} are operators which commute with any element of the graded Lie algebra

$$\left[Z^{LM}, G\right] = 0 \qquad \text{for} \qquad G = Z^{LM}, Q_\alpha, P_\mu, M_{\mu\nu}, B_1 . \tag{3-4}$$

It has been shown by Haag and collaborators(47) that the above graded Lie algebra is indeed the most general (non trivial) symmetry of the S matrix, for massive systems. The only further possible generalization consists in adding an internal symmetry G' which commutes with all elements of the graded Lie algebra. Note that the only freedom one has, is the group G and the representation for the spinor charges.

Surprisingly enough Haag and collaborators(47) have further shown that, if the internal symmetry is combined with the enlarged algebra (2-1), which holds for massless particles (note that this is true modulo infrared problems !), then the group G itself is fixed to be U(N) and the spinor charges are in the fundamental (N dimensional) representation of the group. The only remaining freedom is, in this case, just the number N of spinor charges one has and a complete fusion among space-time symmetries and internal symmetries is obtained. We observe that only in this case all boson charges, namely the internal symmetry generators and the conformal charges, can be written as well defined bilinear expressions of the Fermi charges, so this is probably a possible reason why in the zero mass case the algebra is so stringent.

Unfortunatly the more general algebra we are talking about is not useful in order to construct field theory models because it always leads to non-renormalizable interactions. Also from a phenomenological point of view, the algebra considered in this section, turns out to be not very interesting because the spectrum of states is rather unphysical (apart the problem of mass degeneracy), for any choise of the group G and of its representations. We will discuss these aspects in the next section.

4. - PARTICLE SUPERMULTIPLETS. -

Following Salam and Strathdee(76, 77) the representations of the supersymmetry algebra can be studied in a very simple way using the Wigner method of induced representations(76).

Consider for instance the previous algebraic system of commutation relations

$$\left[Q^L_\alpha, P_\mu\right] = 0 \qquad \left[Q^L_\alpha, M_{\mu\nu}\right] = i(\sigma_{\mu\nu}Q^L)_\alpha$$
$$\left\{Q^L_\alpha, \bar{Q}^M_\beta\right\} = \delta^{LM}\gamma^\mu_{\alpha\beta} P_\mu \tag{4-1}$$

where Q^L_α belong to some representation of an internal symmetry compact group G. The construction of unitary representations of (4-1) begins with the observation that the Q's charges leave invariant the manifold of states with given momentum P_μ. On this manifold the anticommutators of the Q's become a fixed set of numbers and in fact these commutators generate a Clifford algebra. We consider first the simplest case where there is no internal symmetry at all, and we restrict first to the case of non-vanishing mass. The four-momentum, being time-like, can always be assumed of the form $P_\mu = (M, \underline{0})$. The little algebra of the supersymmetry algebra is therefore generated by the Q's and by the angular momentum $\vec{J}$, the O(3) group being the little group of a time-like vector. In terms of Weyl spinors the anticommutation relations in eq. (4-1) become

$$\left\{Q_\alpha, Q_\beta\right\} = \left\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\right\} = 0 \qquad \left\{Q_\alpha, \bar{Q}_{\dot\beta}\right\} = \delta_{\alpha\dot\beta} , \tag{4-2}$$

having chosen a suitable normalization.

The Q's satisfy the algebra of creation and distruction operators and can be used in a familiar way to build a 4-dimensional Fock space with positive metric. Indeed one can start with a particle state $|J, J_3\rangle$ to be considered as the Clifford vacuum i.e.

$$\bar{Q}_{\dot{\alpha}}|J, J_3\rangle = 0 \tag{4-3}$$

and build up the states

$$|J, J_3\, n_1 n_2\rangle = Q_1^{n_1} Q_2^{n_2} |J, J_3\rangle \tag{4-4}$$

where n_1, n_2 take values (0, 0), (0, 1), (1, 0), (1, 1). These states span a $4(2J+1)$ dimensional irreducible representation of the little algebra. The spin-parity content of such representation is $(J-1/2)^{\eta}$, $J^{i\eta}$, $J^{-i\eta}$, $(J+1/2)^{-\eta}$ where $\eta = \pm i, \pm 1$ for J integer or halfinteger. Such irreducible representation is therefore, when reduced to the Poincare group, the direct sum of four inequivalent representations

$$|M, J-1/2, \eta\rangle \oplus |M, J, i\eta\rangle \oplus |M, J, -i\eta\rangle \oplus |M, J, -\eta\rangle \tag{4-5}$$

which are mixed together by the spinor charges Q's.

We consider now some examples. The simplest representations are those with J = = 0, 1/2, 1, 3/2. Their particle contents is respectively: a scalar, a pseudoscalar and a spin 1/2; a pseudoscalar and a spin 1/2; a pseudoscalar (scalar), a vector (pseudovector) and two spin 1/2; a vector, a pseudovector, a spin 1/2 and a spin 3/2; a pseudovector (vector), two spin 3/2 and a tensor (pseudotensor).

In the case of zero mass it can be show that there are only four physical states spanned by the Q's ($J \neq 0$) of helicities $\pm\lambda$, $\pm(\lambda+1/2)$. The second and the fourth of the previously described multiplets contain messless particles of helicities respectively (1/2, 1) and (3/2, 2), and can be therefore regarded as the multiplets which contain the photon and the graviton in a supersymmetric world. The reason why, for zero mass, there are only two helicity states, is very simple: if P_μ is light-like, then it can be assumed of the form $P_\mu = (1, 0, 0, 1)$. The little algebra becomes

$$\{Q_1, \bar{Q}_1\} = 1 \qquad \{Q_2, \bar{Q}_2\} = \{Q_1, \bar{Q}_2\} = \{\bar{Q}_1, Q_2\} = 0 \tag{4-6}$$

the physical states are $|\lambda\rangle$ and $Q_1|\lambda\rangle$, while $Q_2|\lambda\rangle$, $Q_1Q_2|\lambda\rangle$ are zero-norm states.

We now consider particle supermultiplets classified according to the supersymmetry algebra non trivially mixed with internal simmetries. As before, the rest frame anticommutation relations are

$$\begin{aligned} \{Q^L_\alpha, Q^M_\beta\} &= \{\bar{Q}^L_{\dot{\alpha}}, \bar{Q}^M_{\dot{\beta}}\} = 0 \\ \{\bar{Q}^L_\alpha, \bar{Q}^M_{\dot{\beta}}\} &= \delta^{LM}\delta_{\alpha\dot{\beta}}\,. \end{aligned} \tag{4-7}$$

If the Q^L_α belong to a real n dimensional representation of the internal symmetry group G, then they will give 2^{2n} independent new states when applied to a given state which acts as a ground state. In the case where the representation is complex thay will erect 2^{4n} independent states. The dimension of the representation of the little algebra will be 2^{2n} times the dimension of the ground state. We are going now to consider some examples. Take the Q^L_α to be in the fundamental rappresentation of SU(2), then n=2 and the algebra will give 16 new states when applied to a Clifford vacuum $|J, J_3; I, I_3\rangle$ defined by the equation

$$\bar{Q}^L_{\dot{\alpha}}|J, J_3; I, I_3\rangle = 0\,. \tag{4-8}$$

Consider the lowest representation, namely $|J, J_3; I, I_3\rangle = |0, 0\rangle$. The $(J, I)^P$ content of the states of the little algebra will be

$$(0, 0)^+ \oplus (1/2, 1/2)^i \oplus (1, 0)^- \oplus (0, 1)^- \oplus (1/2, 1/2)^{-i} \oplus (0, 0)^+\,. \tag{4-9}$$

The structure of any irreducible representation will be given by vector multiplication of such representation with an $O(3)\otimes SU(2)$ multiplet (J, I) which acts as a ground state. It will be noticed that the rest frame states can be classified into $SU(4)$ multiplets, namely as

$$16 = 1 \oplus 4 \oplus 6 \oplus \bar{4} \oplus 1 . \tag{4-10}$$

This is because the rest frame algebra contains the $SU(4)$ algebra as a subalgebra. Its generators are given by

$$\frac{1}{2}\left\{ Q^L_\alpha, \bar{Q}^M_\beta \right\} . \tag{4-11}$$

This is true in general, in the sense that, if we start with n spinor charges $Q^L_\alpha(L=1,\dots,n)$ the 2^{2n} (2^{4n}) states can be uniquely decomposed into a direct sum of irreducible representations of the $SU(2n)$ $(SU(4n))$ algebra, the spin running from 0 up to $n/2\,(n)$.

As an other example, consider the Q^L_α transforming according to the vector representation of $O(3)$. In this case $2^{2n} = 64$ and the states can be classified according to $SU(6)$ representations

$$64 = 1 \oplus 6 \oplus 15 \oplus 20 \oplus \overline{15} \oplus \bar{6} \oplus 1 . \tag{4-12}$$

The spin-isospin-parity contents $(J, I)^P$ of these representations is

$$\begin{aligned} &(0,0)^+ \oplus (1/2,1)^i \oplus (0,0)^- \oplus (1,1)^- \oplus (0,2)^- \oplus (3/2,0)^{-i} \oplus (1/2,1)^{-i} \\ &\oplus (1/2,2)^{-i} \oplus (0,0)^+ \oplus (1,1)^+ \oplus (0,2)^+ \oplus (1/2,1)^i \oplus (0,0)^- . \end{aligned} \tag{4-13}$$

As a final example we consider the case where the charges are in the quark representation, i.e. they are $SU(3)$ triplets. In this case the lowest representation is of dimension $2^{12} = 4096$ and these states can be classified according to $SU(12)$ as

$$\begin{aligned} 4096 = {} & 1 \oplus 12 \oplus 66 \oplus 220 \oplus 495 \oplus 792 \oplus 924 \oplus \\ & \overline{792} \oplus \overline{495} \oplus \overline{220} \oplus \overline{66} \oplus \overline{12} \oplus 1 . \end{aligned} \tag{4-14}$$

The maximum spin is $J = 3$ (singlet). Of course all the unitary representations of the supersymmetry algebra are obtained by vector multiplication of such lowest representations (where the ground state is a scalar singlet) with an arbitrary representation of the internal symmetry group and with an arbitrary spin carried by the ground state. The dimension of a given representation will be

$$D = 2^{2n} \times (2J+1) \times d . \tag{4-15}$$

Being J the spin and d the dimension of the representation which defines the ground state.

5. - SUPERFIELDS AND LAGRANGIAN MODELS. -

The first non trivial representation of the supersymmetry algebra given in the literature was obtained by Wess and Zumino[91] in the following form

$$\begin{aligned} \delta A(x) &= \bar{\alpha}\,\psi(x) \\ \delta B(x) &= \bar{\alpha}\,\gamma_5\psi(x) \\ \delta\psi(x) &= \partial_\mu(A(x) - \gamma_5 B(x))\,\gamma^\mu\alpha + (F(x) + \gamma_5 G(x))\alpha \end{aligned} \tag{5-1}$$

$$\delta F(x) = i\bar{\alpha}\gamma^{\mu}\partial_{\mu}\psi(x) \tag{5-1}$$

$$\delta G(x) = i\bar{\alpha}\gamma_5\gamma^{\mu}\partial_{\mu}\psi(x) .$$

Where A and F are scalars and B and G are pseudoscalars, being ψ a Majorana spinor. Note that, on the mass-shell, the following relations hold

$$(\Box - m^2)A = (\Box - m^2)B = 0$$
$$(i\gamma\cdot\partial + m)\psi = 0 . \tag{5-2}$$

Which means that the representation is reducible because

$$\delta(mA(x) + F(x)) = \delta(mB(x) + G(x)) = 0 \tag{5-3}$$

and therefore one can take

$$F(x) = -mA(x) \qquad\qquad G(x) = -mB(x) . \tag{5-4}$$

This shows that the multiplet of fields given in (5-1) describes the off-shell particle field multiplet which corresponds to the J=0 representation according to the nomenclature given in the previous section.

In order to construct field representations of supersymmetry, following Salam and Strathdee(73), one considers the action of a group element $e^{-\bar{\alpha}Q}$ over the group manifold

$$e^{-ix\cdot P}e^{-\bar{\theta}Q} = e^{-ix\cdot P - \bar{\theta}Q} . \tag{5-5}$$

From the anticommutation relations of the Q's one gets

$$e^{-\bar{\alpha}Q}e^{-ix\cdot P - \bar{\theta}Q} = e^{-i(x+i\bar{\alpha}\gamma\theta)\cdot P - (\bar{\theta}+\bar{\alpha})Q}$$

or in infinitesimal form, for the "superfield" $\phi(x,\theta) = e^{-ixP-\bar{\theta}Q}\phi(0,0)$

$$\delta\phi(x,\theta) = (-\frac{\partial}{\partial\bar{\theta}} + i\gamma^{\mu}\theta\partial_{\mu})\phi(x,\theta) \tag{5-6}$$

analogously

$$\delta\phi(x,\theta) = i\partial_{\mu}\phi(x,\theta)$$
$$\delta\phi(x,\theta) = -i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu} + \bar{\theta}\sigma_{\mu\nu}\frac{\partial}{\partial\bar{\theta}})\phi(x,\theta) \tag{5-7}$$

respectively for translations and Lorentz transformations.

Following Ferrara, Wess, Zumino(33) it is convenient to use Weyl spinors θ_{α} $\bar{\theta}_{\dot{\alpha}}$ related to the Dirac spinor θ_{α} by the relations

$$\theta_{\alpha} = \frac{1}{2}(1 - i\gamma_5)\theta \qquad\qquad \bar{\theta}_{\dot{\alpha}} = \frac{1}{2}(1 + i\gamma_5)\theta . \tag{5-8}$$

Them it follows that the superfield can also be written as

$$\phi(x,\theta,\bar{\theta}) = e^{-ixP + i\theta Q + i\bar{\theta}\bar{Q}}\phi(0,0,0) . \tag{5-9}$$

The main point is that, due to the anticommuting properties of the θ's, $\phi(x,\theta,\bar\theta)$ is a finite power series, namely

$$\begin{aligned}\phi(x,\theta,\bar\theta) &= C + \theta\chi + \bar\theta\bar\chi + \theta\theta M + \bar\theta\bar\theta\bar M \\ &+ \theta\sigma_\mu\bar\theta v^\mu + \theta\theta\bar\theta\bar\lambda + \bar\theta\bar\theta\theta\lambda + \theta\theta\bar\theta\bar\theta D\end{aligned} \tag{5-10}$$

where, without loss of generality, we can assume ϕ to be real.

Because of the transformation law (5-6) one easily realizes that the individual components transform into

$$\begin{array}{lll} C \rightarrow \chi & \chi \rightarrow \partial A,\ M,\ M^*,\ v_\mu & M,\ M^* \rightarrow \partial\chi\,,\ \lambda \\ \lambda \rightarrow \partial M,\ \partial M^*,\ \partial v_\mu,\ D\ . & & D \rightarrow \partial\lambda \end{array} \tag{5-11}$$

under supersymmetry transformations. In terms of component fields one can write the infinitesimal transformation as follows

$$\delta\begin{pmatrix} C \\ \chi \\ M \\ N \\ v_\mu \\ \lambda \\ D\end{pmatrix} = \begin{pmatrix} 0 & i\bar\alpha\gamma_5 & 0 & 0 & 0 & 0 & 0 \\ -\gamma_5\gamma\cdot\partial\alpha & 0 & \alpha & \gamma_5\alpha & \gamma^\mu\alpha & 0 & 0 \\ 0 & i\bar\alpha\gamma\cdot\partial & 0 & 0 & 0 & i\bar\alpha & 0 \\ 0 & i\bar\alpha\gamma_5\gamma\cdot\partial & 0 & 0 & 0 & i\bar\alpha\gamma_5 & 0 \\ 0 & i\bar\alpha\partial_\mu & 0 & 0 & 0 & i\bar\alpha\gamma_\mu & 0 \\ 0 & 0 & \gamma\cdot\partial\alpha & \gamma_5\gamma\cdot\partial\alpha & -2\sigma_{\mu\nu}\partial^\nu\alpha & 0 & \gamma_5\alpha \\ 0 & 0 & 0 & 0 & 0 & i\ \gamma\cdot & 0 \end{pmatrix}\begin{pmatrix} C \\ \chi \\ M \\ N \\ v_\mu \\ \lambda \\ D\end{pmatrix} \tag{5-12}$$

So one sees that, as expected, the superfield notation is nothing but an economic way to deal with matrix multiplication[62].

In order to further develop the tensor calculus of supersymmetry we observe that the group-manifold can be parametrized in three different but equivalent ways[33],

$$\begin{aligned} & e^{-ixP + i\theta Q + i\bar\theta\bar Q} \\ & e^{-ixP + i\theta Q}\, e^{i\bar\theta\bar Q} \\ & e^{-ixP + i\bar\theta\bar Q}\, e^{i\theta Q}\ . \end{aligned} \tag{5-13}$$

The three expressions in (5-13) correspond to three different (equivalent) realizations of supersymmetry representations i.e. to three different definitions of superfields which are related by the following identities

$$\phi(x,\theta,\bar\theta) = \phi_1(x+i\theta\sigma_\mu\bar\theta,\theta,\bar\theta) = \phi_2(x-i\theta\sigma_\mu\bar\theta,\theta,\bar\theta)\ . \tag{5-14}$$

We note that, once a superfield has been defined as type I or II, it becomes intrinsically complex so that the reality condition cannot be longer imposed.

This fact can be better understood from the fact that $i\theta\sigma_\mu\bar\theta$ is a pure imaginary shift on the first argument.

The group action on ϕ_1 and ϕ_2 is respectively

$$\delta_\alpha\phi_1 = (\alpha\frac{\partial}{\partial\theta} + \bar\alpha\frac{\partial}{\partial\bar\theta} + 2i\theta\sigma_\mu\bar\alpha\partial^\mu)\phi_1\ , \tag{5-15}$$

$$\delta_\alpha \phi_2 = (\alpha \frac{\partial}{\partial\theta} + \bar{\alpha}\frac{\partial}{\partial\bar{\theta}} - 2i\alpha\sigma_\mu \bar{\theta}\partial^\mu)\phi_2 \ . \tag{5-15}$$

From (5-15) it follows immediatly that $\partial/\partial\bar{\theta}$, $\partial/\partial\theta$ are invariant differentiations on type I or type II superfields. Therefore for any given superfield ϕ (for example in real basis) there are two invariant differentiations(33, 78)

$$\begin{aligned} D_\alpha \phi &= (\frac{\partial}{\partial\theta} + i\alpha\sigma_\mu \bar{\theta}\partial^\mu)\phi \\ D_{\dot{\alpha}}\phi &= (-\frac{\partial}{\partial\bar{\theta}} - i\theta\sigma_\mu \bar{\alpha}\partial^\mu)\phi \ . \end{aligned} \tag{5-16}$$

We shall call D_α, $\bar{D}_{\dot{\alpha}} = (\bar{D}_\alpha)$ the covariant derivatives. They obey the following relations

$$\begin{aligned} &\{D_\alpha, D_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \qquad \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \\ &D_\alpha D_\beta D_\gamma = \bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}\bar{D}_{\dot{\gamma}} = 0 \ . \end{aligned} \tag{5-17}$$

The last line being obvious.

Because of the existence of the covariant derivatives a (complex) superfield can be reduced further inposing on it the contraints

$$D_\alpha \phi = 0 \qquad \text{or} \qquad \bar{D}_{\dot{\alpha}}\phi = 0 \ . \tag{5-18}$$

A superfield satisfying (5-18) is called a chiral field, according to the nomenclature of Salam and Strathdee(78) (it was originally called a scalar superfield).

A general (real) superfield will be called a vector superfield. Strictly speaking there are superfileds which are nor scalar nor vector i.e. superfields which satisfy the relation

$$DD\phi = 0 \qquad \text{or} \qquad \bar{D}\bar{D}\phi = 0 \ . \tag{5-19}$$

They are called linear superfields(33). However, as they have not been used in any interesting application, we will omit them from our discussion.

Chiral fields are called left-handed or right-handed if they satisfy the equation

$$\bar{D}_{\dot{\alpha}}\phi_L = 0 \qquad \text{or its complex conjugate} \qquad D_\alpha \phi_R = 0 \tag{5-20}$$

The reason of this definition is that, when translated in I or II basis, they do not longer depend on the right-handed (left-handed) part of the Majorana spinor θ. We note that the general form of ϕ_{IL} is

$$\phi_{IL} = A + \theta\psi + \theta\theta F \tag{5-21}$$

being A, ψ, F complex fields. Their transformation law is, in term of real fields

$$\delta \begin{pmatrix} A \\ B \\ \psi \\ F \\ G \end{pmatrix} = \begin{pmatrix} 0 & 0 & i\bar{\alpha} & 0 & 0 \\ 0 & 0 & i\bar{\alpha}\gamma_5 & 0 & 0 \\ \gamma\cdot\partial\alpha & -\gamma_5\gamma\cdot\partial\alpha & 0 & \alpha & \gamma_5\alpha \\ 0 & 0 & i\bar{\alpha}\gamma\cdot\partial & 0 & 0 \\ 0 & 0 & i\bar{\alpha}\gamma_5\gamma\cdot\partial & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ \psi \\ F \\ G \end{pmatrix} \tag{5-22}$$

We have confined, so far, our discussion to superfields without an additional (external) Lorentz index $\{\alpha\}$. Of course such an index is completely irrelevant for supersymmetry transformations so it can be added without difficulty to the previous superfields without modifications of the established results.

Multiplication among different superfields is also well defined. It will be an essential ingredient in order to build up supersymmetric model field theories. For example one easily verifies the following relations

$$\phi_V \psi_V = (\phi\psi)_V \qquad \phi_L \psi_L = (\phi\psi)_L$$
$$\phi_R \psi_R = (\phi\psi)_R \qquad \phi_L \psi_R \pm \phi_R \psi_L = (\phi\psi)_{\pm V}$$

where $\phi_R = \overline{\phi}_L$, $\psi_L = \overline{\psi}_R$ and the subscript V stands for vector superfield.

Moreover

$$DD\phi_V(x, \theta, \overline{\theta}) = (DD\phi)_R(x, \theta, \overline{\theta}) .$$

If ϕ_L has components A, ψ, F according to (5-21) then $\phi'_L = \overline{D}\,\overline{D}\,\phi_L$ has components respectively given by:

$$F,\ G,\ \gamma\cdot\partial\psi,\ \Box A,\ \Box B .$$

In terms of component fields the covariant derivatives are nothing but finite-dimensional matrices[71,72]. For example the previous operation is given by the following matrix in the component space

$$\begin{pmatrix} F \\ G \\ \gamma\cdot\partial\psi \\ \Box A \\ \Box B \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \gamma\cdot\partial & 0 & 0 \\ \Box & 0 & 0 & 0 & 0 \\ 0 & \Box & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ \psi \\ F \\ G \end{pmatrix} \qquad (5\text{-}23)$$

In order to derive lagrangian field theories which are invariant under supersymmetry transformations, we remark that, because of (5-6) the last component field of a given multiplet always transforms as a total derivative,

$$\delta\,\phi_{LAST}(x) = i\gamma^{\mu}\partial_{\mu}\chi(x) . \qquad (5\text{-}24)$$

Being $\chi(x)$ some Fermi field. Therefore it follow that

$$\int d^4x\, \phi_{LAST}(x) \qquad (5\text{-}25)$$

is invariant.

The first supersymmetric model field theort which has been studied is the self interaction of a chiral multiplet, namely the interaction of the A, B, ψ, F, G component fields[92]. In order to derive the corresponding langrangian we will use extensively the superfield techniques. In particular we will show that the kinetic term, the mass term and the interaction term are separately invariant under supersymmetry.

Kinetic term: starting withthe chiral multiplet S ($\overline{D}_{\dot{\alpha}} S = 0$) the kinetic energy is the D component of the following real (vector) superfield $S\overline{S}$

$$S\overline{S}_D = -\frac{1}{2}(\partial_{\mu}A)^2 - \frac{1}{2}(\partial_{\mu}B)^2 - \frac{i}{2}\overline{\psi}\,\gamma\cdot\partial\psi + \frac{1}{2}F^2 + \frac{1}{2}G^2 . \qquad (5\text{-}26)$$

Which is equal, up to a four-divergence, to the F component of the chiral bilinear $S\bar{D}\bar{D}\bar{S}$.
Mass term : the mass term corresponds to the F component of the chiral multiplet S^2

$$m S^2_F = m(FA + GB - \frac{i}{2}\bar{\psi}\psi) . \tag{5-27}$$

Interaction term : the interaction corresponds to the F component of the chiral multiplet S^3

$$gS^3_F = g(FA^2 - FB^2 + 2GAB - i\bar{\psi}\psi A + i\bar{\psi}\gamma_5\psi B) . \tag{5-28}$$

Therefore :

$$L(x) = L_k + L_m + L_I . \tag{5-29}$$

Using the equations of motion for F and G :

$$\begin{aligned} F + mA + g(A^2 - B^2) &= 0 \\ G + mB + 2gAB &= 0 . \end{aligned} \tag{5-30}$$

The lagrangian (5-29) takes the more familiar form

$$\begin{aligned} L(x) = &-\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{i}{2}\bar{\psi}\gamma\cdot\partial\psi + \frac{1}{2}m^2A^2 + \frac{1}{2}m^2B^2 - \\ &- \frac{i}{2}m\bar{\psi}\psi - gmA(A^2 + B^2) - \frac{g^2}{2}(A^2 + B^2)^2 - ig\bar{\psi}(A - \gamma_5 B)\psi . \end{aligned} \tag{5-31}$$

i.e. a combination of cubic, quartic and Yukawa interactions with a well defined relation between masses and coupling constants.

The Noether supersymmatry current is given by the following expression[92, 20]

$$J_\mu(x) = \gamma\cdot\partial(A - \gamma_5 B)\gamma_\mu\psi - (F + \gamma_5 G)\psi , \tag{5-32}$$

and is conserved as a consequence of the equations of motion.

It has been shown by Iliopoulos and Zumino[51] that this model is renormalizable preserving the symmetry. This means that the infinities can be absorbed in supersymmetric counterterms of the form given by eq. (5-29).

Moreover, remarkably enough, it turns out that there is only one divergence, a logarithmic infinity due to a wave function renormalization of the multiplet. We observe, en passant, that the fields F and G are auxiliary fields and do not correspond to any asymptotic state.

The renormalized mass and coupling constant are given by the following relation[51,88]

$$m_r = Z m_B \qquad g_r = Z^{3/2} g_B \tag{5-33}$$

where Z is the wave function renormalization.

It has been shown by Ferrara, Iliopoulus and Zumino[32] that the renormalized one particle irreducible n point functions $\Gamma_{\phi_r \ldots \phi_n}$, where the external lines stand for any field of the chiral multiplet, satisfy the following Callan-Symanzik equations, at each order of perturbation theory

$$\left[m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g)\right]\Gamma_{\phi_1\ldots\phi_n}(P_i; m, g) = \frac{m}{2g}\delta(g)\,\Gamma_{A,\phi_1\ldots\phi_n}(0, P_i; m, g) \tag{5-34}$$

with $\beta(g)$, $\gamma(g)$ and $\delta(g)$ expressed in terms of a single function

$$\beta(g) = \frac{3}{2} g \frac{f}{1+f} , \qquad \gamma(g) = \frac{1}{2} \frac{f}{1+f} , \qquad \delta(g) = \frac{1}{1+f} \tag{5-35}$$

being f(g) defined as

$$f(g) = m_B \frac{\partial \log Z}{\partial m_B} = \frac{g^2}{4\pi^2} + \cdots \tag{5-36}$$

note that (5-35) imply the relation(32)

$$\beta(g) = 3 g \gamma(g) . \tag{5-37}$$

Which shows that there is not fixed point which can be reached in a continuous way from the origin.

In order to understand the previous result in a simple way we can use supersymmetric perturbation theory(8, 9, 14, 40, 41, 48, 50, 79, 8).

The previous lagrangian (and the corresponding action) can be written as a density over a superspace whose point are labelled by x, θ_α, $\bar{\theta}_{\dot{\alpha}}$. Spinor integration is a well defined operation. For instance one makes use of the following fundamental property(5, 6)

$$\int d\theta_i \theta_j = \delta_{ij} . \tag{5-38}$$

Using (5-38) the action of the previous system can be written as

$$\int d^4x \left[\int d^2\theta \frac{1}{2} S \bar{D} \bar{D} \bar{S} + mS^2 + \frac{1}{3!} gS^3 + h.c.\right] + \int d^4x \left[\int d^2\theta\, JS + h.c.\right] \tag{5-39}$$

The free propagator is obtained for $g = 0$

$$S(1) = \frac{1}{16(m^2 - \Box_1)} (\bar{D}_1 \bar{D}_1 \bar{J}(1) - mJ(1)) \tag{5-40}$$

Using the functional differentiation rule

$$\frac{\delta S(1)}{\delta S(2)} = \delta_S(1,2) = \delta^4(x_1 - x_2 + i\bar{\theta}_1 \gamma \theta_2) \delta^2(\theta_1 - \theta_2)$$

$$\frac{\delta \bar{S}(1)}{\delta \bar{S}(2)} = \delta_{\bar{S}}(1,2) = \delta^4(x_1 - x_2 + i\bar{\theta}_1 \gamma \theta_2)\, \delta^2(\bar{\theta}_1 - \bar{\theta}_2) \tag{5-41}$$

$$\frac{\delta S(1)}{\delta \bar{S}(2)} = \frac{\delta \bar{S}(1)}{\delta S(2)} = 0 , \qquad \delta^2(\theta_1 - \theta_2) = \frac{1}{4}(\theta_1 - \theta_2)(\theta_1 - \theta_2)$$

one obtain

$$\langle S(1)\bar{S}(2)\rangle = \frac{\delta S(1)}{\delta i \bar{J}(2)} = \frac{-i}{16(m^2+k^2)}\, e^{\bar{\theta}_1 \gamma \theta_2 k - \theta_{12} \sigma \bar{\theta}_{12} k}$$

$$\langle S(1) S(2)\rangle = \frac{\delta S(1)}{\delta i J(2)} = \frac{im}{16(m^2+k^2)}\, e^{\bar{\theta}_1 \gamma \theta_2 k}\, \theta_{12} \theta_{12} .$$

Consider now the one loop diagrams for the self-energy[14]. They are

$$\Sigma_{++}(1,2) = \Delta_{++}(1,2)\,\Delta_{++}(2,1)$$
$$\Sigma_{+-}(1,2) = \Delta_{+-}(1,2)\,\Delta_{-+}(2,1)$$

which contribute to Σ_{++}, Σ_{+-}. Because $\Delta^2_{++} = 0$, $\Sigma_{++} = 0$. For Σ_{+-} one gets

$$\Sigma_{+-}(q, \theta_1, \theta_2) \cong \int d^4k \, \frac{1}{k^2+m^2} \, \frac{1}{(k+q)^2+m^2} \, \times$$
$$\times \; e^{\bar\theta_1\gamma\theta_2 k - \theta_{12}\sigma\bar\theta_{12}k} \times e^{-\bar\theta_1\gamma\theta_2(k+q) + \theta_{12}\sigma\bar\theta_{12}(k+q)} =$$
$$= e^{-\bar\theta_1\gamma\theta_2 q + \theta_{12}\sigma\bar\theta_{12}q} \int d^4k \, \frac{1}{k^2+m^2} \, \frac{1}{(k+q)^2+m^2}$$

i.e. a logarithmic divergence. This divergence can be absorbed in a counter term of the type $S\bar{S}$ i.e. in a wave function renormalization. Because $\Sigma_{++} = 0$ no mass counter term is needed.

We consider now the vertex correction[14]. The only graphs one can write are:

$$V_{+++}(1,2,3) = \Delta_{++}(1,2)\,\Delta_{++}(2,3)\,\Delta_{++}(3,1)$$
$$V_{++-}(1,2,3) = \Delta_{++}(1,2)\,\Delta_{+-}(2,3)\,\Delta_{-+}(3,1)$$

which contribute to V_{+++}, V_{++-}. However, because $\theta_{12}\theta_{12}\theta_{13}\theta_{13}\theta_{23}\theta_{23} = 0$, $V_{+++} = 0$. For the V_{++-} vertex we get

$$\int d^4k \, \frac{1}{k^2+m^2} \, \frac{1}{(k+q)^2+m^2} \, \frac{1}{(k+t)^2+m^2} \, e^{\bar\theta_1\gamma\theta_2 k} \, \theta_{12}\theta_{12} \, \times$$
$$\times \; e^{\bar\theta_2\gamma\theta_3 k - \theta_{23}\sigma\bar\theta_{23}k} \, e^{\bar\theta_3\gamma\theta_1 k + \theta_{31}\sigma\bar\theta_{31}k} =$$
$$= \int d^4k \, \frac{1}{k^2+m^2} \, \frac{1}{(k+q)^2+m^2} \, \frac{1}{(k+t)^2+m^2} \, \theta_{12}\theta_{12}$$

i.e. a finite result. Therefore there is no independent coupling constant renormalization.

In the same way one can show that all higher n-point functions are finite. Moreover Capper and Leibbrandt[9] have established the superficial degree of divergence of any graph

$$d = 4L - 2I + 2\mu \qquad \text{where} \quad \mu = \min n_+ - 1,\; n_- - 1$$

being L, I, n_+, n_- respectively the number of loops, propagators and vertices of any graph. Using the following topological relations

$$I + E_+ = 3n_+, \qquad I + E_- = 3n_-, \qquad L = I - (n-1), \qquad n = n_+ + n_-$$

one finally gets

$$d = 2 - 2E_- - 4N,$$

where $E_\pm$ are number of external +, - lines and $N = \frac{E_+ - E_-}{3}$ which has been assumed to be positive (if, on the other hand $E_- > E_+$ then $d = 2 - 2E_+ - 4N$ where $N = \frac{E_- - E_+}{3}$. The power counting formula shows that the only divergent graphs are those with $N = 0$, $E_- = E_+ = 1$. i.e. the self-energy graphs Σ_{+-}.

Finally we would like to mention that the super S^4 theory, as shown first by Lang and Wess[55], leads to a non-renormalizable lagrangian. This fact can also be understood, in a very simple way, using supergraphs and the previous power-counting formulae.

6. - SPONTANEOUS SYMMATRY BREAKING AND GOLDSTONE FERMIONS. -

The notion of supersymmetry is very elegant, however, as we have seen, it leads to a complete degeneracy of masses inside a supermultiplet. As Nature does not show such multiplets it is very compeling to look for a spontaneous symmetry breaking of this symmetry.

It has been shown by Salam, Strathdee and Wess, Zumino[74, 51] that when the vacuum is not a supersymmetric singlet, then a Goldstone phenomenon emerges with the immediate implication that the system must contain a massless spin 1/2 fermion because of the following general argument:

the infinitesimal change of a Dirac field ψ_α under a supersymmetry transformation is

$$\delta\psi_\alpha(x) = \left[\psi_\alpha(x), \int d^3x' \bar{a}_\gamma J_{o\gamma}(x')\right] \tag{6-1}$$

where $J_{\mu\gamma}(x)$ is the vector spinor current. The vacuum expectation value of (6-1) yelds to the expression

$$\partial_\mu \langle T^* \bar{a} J_\mu(x) \psi_\alpha(0)\rangle = -\delta_4(x) \langle \delta\psi_\alpha(0)\rangle \tag{6-2}$$

because of the non-invariance of the vacuum

$$\langle \delta\psi_\alpha(0)\rangle = a \langle \phi \rangle \tag{6-3}$$

where ϕ is some scalar field related to ψ_α through supersymmetry transformations.

Eq. (6-2) gives

$$\partial_\mu \langle T^* J_{\mu\alpha}(x) \psi_\beta(0)\rangle = \langle\phi\rangle \delta_4(x) \delta_{\alpha\beta} \tag{6-4}$$

the Fourier transform yelds to

$$\int d^4x \; e^{ikx} \langle T^* J_{\mu\alpha}(x) \; \psi_\beta(0)\rangle = M(k^2) k_\mu \delta_{\alpha\beta} + \cdots \tag{6-5}$$

which because of (6-4) gives finally

$$k^2 M(k^2) = \langle\phi\rangle \tag{6-6}$$

i.e. a simple pole in $M(k^2)$ at $k^2=0$ with residue $\langle\phi\rangle$. This indicate that an intermediate state which is a massless spin 1/2 particle contributes to $M(k^2)$.

The notion of Goldstone fermion is very exciting since it might give a fundamental explanation of the neutrino[4, 24].

Salam and Strathdee[82] have further shown that in any spontaneously broken theory, where the multiplet described in the previous section are involved (i. e. chiral multiplets and vector multiplets), the expression of the Goldstone field in terms of the basic fields is

$$\chi_\alpha(x) = \sum_j \langle F_j - \gamma_5 G_j \rangle \psi_j(x) - \frac{1}{2} \sum_{ik} \langle \gamma_5 D_{jk} \rangle \lambda_{jk}(x) \tag{6-7}$$

implying that a necessary and sufficient condition for spontaneous breakdown is that at least one of the auxiliary fields F, G and D admit a non-vanishing expectation value. The subscript of these fields refer to a given internal symmetry index. The D, λ fields are therefore assumed to be matrices because the vector multiplet can only appear as a gauge multiplet in a given renormalizable lagrangian. In the next section we will encounter an explicit lagrangian model where the Goldstone phenomenon occurs.

7. - GAUGE INVARIANCE AND SUPERSYMMETRY. -

The model we have described in the previous sections consists of a scalar, pseudoscalar and a spin 1/2 particle in interaction among each others. However Lagrangian field theories which combine supersymmetry with gauge invariance have also been constructed(93, 34, 75, 19) and shown to be renormalizable in a way consistent with supersymmetry and gauge invariance (21, 22, 37, 48, 68, 83, 84, 18).

The gauge superfield is a real vector multiplet $V(x, \theta, \bar\theta)$. To define gauge transformations in a way which is consistent with supersymmetry we must enlarge[93] the gauge function to an entire chiral multiplet $\Lambda(x, \theta, \bar\theta)$ $(\bar D_{\dot\alpha} \Lambda = 0)$ so that

$$\delta V(x, \theta, \bar\theta) = i(\Lambda - \bar\Lambda) \tag{7-1}$$

under a gauge transformation. In terms of component fields (7-1) reads

$$\begin{aligned} \delta\chi &= \psi & \qquad \delta\lambda &= 0 \\ \delta C &= B & \qquad \delta v_\mu &= \partial_\mu A \\ \delta M &= F & \qquad \delta D &= 0 \\ \delta N &= G & & \end{aligned} \tag{7-2}$$

So one realizes that C, χ, M, N are not physical degrees of freedom (they can be in fact gauged away) while λ, D are gauge invariant quantities.

The supersymmetric extension of the electromagnetic strength corresponds to the following chiral field

$$W_\alpha = \bar D \bar D D_\alpha V \tag{7-3}$$

which contains the component fields λ, D, $F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$, $\gamma\cdot\partial\lambda$ and in fact $\delta W_\alpha = 0$ under (7-1).

The free supersymmetric lagrangian[91] is given by the F component of the chiral multiplet $W^\alpha W_\alpha$

$$W^\alpha W_{\alpha\,F} = -\frac{1}{4} F^2_{\mu\nu} - \frac{i}{2} \bar\lambda \gamma\cdot\partial\lambda + \frac{1}{2} D^2 . \tag{7-4}$$

The gauge superfield can be put in interaction with a complex chiral multiplet $S = S_1 + iS_2$ then providing the supersymmetric extension of QED. If one defines $S_\pm = S_1 \pm iS_2$ $(\bar D_\alpha S_\pm = 0)$ the gauge transformation on $S_\pm$ is

$$\delta S_{\pm} = \mp i g \Lambda S_{\pm} \tag{7-5}$$

and the corresponding supersymmetric minimal coupling is[93]

$$\frac{1}{2}(S_+\bar{S}_+ e^{gV} + S_-\bar{S}_- e^{-gV}) \tag{7-6}$$

of course also the mass term

$$2\,m S_+ S_- \tag{7-7}$$

is both gauge invariant and supersymmetric.

The supersymmetric lagrangian is just given by the following expression

$$(W^{\alpha} W_{\alpha})_F + m(S_+S_-)_F + \frac{1}{4}(S_+\bar{S}_+ e^{gV} + S_-\bar{S}_- e^{-gV})_D \;. \tag{7-8}$$

The quite unconventional form of the interaction term is merely due to the presence of the unphysical field $C(x)$ (of zero dimension) which is however essential in order to write the covariant derivantive in a supersymmetric way. The important point is that, using the gauge freedom, one can fix $C = \chi = M = N = 0$ (Wess, Zumino gauges)[93]. In these gauges the Langrangian given by (7-8) reduces to the familiar form:

$$\begin{aligned} L = & -\frac{1}{2}(\partial_\mu A_1)^2 - \frac{1}{2}(\partial_\mu A_2)^2 - \frac{1}{2}(\partial_\mu B_1)^2 - \frac{1}{2}(\partial_\mu B_2)^2 - \frac{i}{2}(\bar{\psi}_1\gamma\cdot\partial\psi_1 - \\ & -\frac{i}{2}\bar{\psi}_2\gamma\cdot\partial\psi_2 - \frac{1}{2}m^2(A_1^2+A_2^2+B_1^2+B_2^2) - \frac{i}{2}m(\bar{\psi}_1\psi_1+\bar{\psi}_2\psi_2) - \frac{1}{4}F_{\mu\nu}^2 - \\ & -\frac{i}{2}\bar{\lambda}\gamma\cdot\partial\lambda - g v_\mu\left[(A_1\overleftrightarrow{\partial}_\mu A_2)+(B_1\overleftrightarrow{\partial}_\mu B_2) - i\bar{\psi}_1\gamma_\mu\psi_2\right] - ig\bar{\lambda}\left[(A_1+\gamma_5 B_1)\psi_2 - \right. \\ & \left. -(A_2+\gamma_5 B_2)\psi_1\right] - \frac{1}{2}g^2 v_\mu^2(A_1^2+A_2^2+B_1^2+B_2^2) - \frac{1}{2}g^2(A_1B_2 - A_2B_1)^2 \end{aligned} \tag{7-9}$$

where we used the equations of motion of the auxiliary fields

$$F_i + mA_i = 0, \qquad G_i + mB_i = 0, \qquad D + g(A_1B_2 - A_2B_1) = 0. \tag{7-10}$$

The abelian gauge model we have described so far, provides the simplest example of spontaneous breakdown of supersymmetry if we add to the previous lagrangian the term[30]

$$\frac{\xi}{g} V_D = \frac{\xi}{g} D \tag{7-11}$$

which is supersymmetric and gauge invariant. Upon elimination of the D component field this term adds to the previous Lagrangian the quantity: $-\xi(A_1B_2 - A_2B_1)$. The mass matrix can be diagonalized introducing the new fields

$$a_1 = \frac{1}{\sqrt{2}}(A_1 - B_2), \qquad a_2 = \frac{1}{\sqrt{2}}(B_1 + A_2),$$

$$b_1 = \frac{1}{\sqrt{2}}(A_1 + B_2), \qquad b_2 = \frac{1}{\sqrt{2}}(A_2 - B_1),$$

and the resulting potential is

$$\frac{1}{2}(m^2 - \xi)(a_1^2 + a_2^2) + \frac{1}{2}(m^2 + \xi)(b_1^2 + b_2^2) + \frac{g^2}{8}(a_1^2 + a_2^2 - b_1^2 - b_2^2)^2. \qquad (7\text{-}12)$$

At this point one sees already that supersymmetry is spontaneously broken, since the masses of the fields in the same multiplet are no longer equal.

For $|\xi| < m^2$ the fields a_i, b_i have a vanishing expectation value, while

$$\langle D \rangle = -\frac{\xi}{g}, \qquad \delta\lambda = -\frac{\xi}{g}\gamma_5 \alpha + \cdots$$

and therefore λ is a Goldstone spinor. For $|\xi| > m^2$ one of the quadratic terms has a negative coefficient. Now gauge invariance is also spontaneously broken and the vector field v_μ acquires a mass $m_v^2 = 2(\xi - m^2)$ through the usual Higgs mechanism. It turns out that the Goldstone fermion is now a linear combination of λ, $\gamma_5\psi_1$ and ψ_2.

Salam, Strathdee(75), Ferrara and Zumino(34) succeeded to construct non abelian supersymmetric gauge theories.

In this case the gauge transformation for the gauge superfield looks more complicated. For a finite local gauge transformation one has

$$e^{gV} \rightarrow e^{-ig\Lambda^+} e^{gV} e^{ig\Lambda} \qquad (7\text{-}13)$$

where matrix multiplication is understood. Note, in fact, that $V = \lambda^a V^a$, $\Lambda = \lambda^a \Lambda^a$ being λ^a the group generators in the vector representation. The finite gauge transformation (7-13) corresponds to a complicated (non linear) infinitesimal transformation for the gauge field

$$\delta V_a = \Lambda^b C_{ba}(V) + \bar{\Lambda}^b \bar{C}_{ba}(V) \qquad (7\text{-}14)$$

where $C_{ba}(V) = iC^1_{ba}(V) + C^2_{ba}(V)$ is a infinite power series in V_a. However in the Wess, Zumino gauges

$$V_a V_b V_c = 0$$

so that

$$C^1_{ba} = \delta_{ba}, \qquad C^2_{ba} = -f_{bac} V^c \qquad (7\text{-}15)$$

and one recovers the usual Yang-Mills transformations for the fields

$$v^a_\mu, \quad \lambda^a, \quad D^a.$$

The pure supersymmetric Yang-Mills Lagrangian is provided by the F component of the following chiral multiplet

$$\mathcal{L}_{Y.M.} = T_r(W^\alpha W_\alpha)_F \qquad (7\text{-}16)$$

with

$$W_\alpha = \bar{D}\bar{D}(e^{-gV} D_\alpha e^{gV}). \qquad (7\text{-}17)$$

The invariance of (7-16) can be checked using the local transformation of (7-17). In terms of component fields (7-16) reduces to (Wess-Zumino gauge)

$$\mathrm{Tr}\left(-\frac{1}{4}G^2_{\mu\nu} - \frac{i}{2}\bar{\lambda}\gamma\cdot\mathcal{D}\lambda\right) \qquad (7\text{-}18)$$

where

$$G_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu + ig\left[v_\mu, v_\nu\right]$$

$$\mathcal{D}_\mu \lambda = \partial_\mu \lambda + ig\left[v_\mu, \lambda\right]$$

and $D^a = 0$ because of the eqs. of motion.

The lagrangian (7-18) implies the quite striking result that a Yang-Mills interaction with a fermion in the adjoint representation is automatically supersymmetric. The corresponding conserved (gauge invariant) spinor current is[100, 36]

$$J_\mu = -\frac{1}{2}\,\mathrm{Tr}\,(G^{\rho\sigma}\gamma_\rho\gamma_\sigma\gamma_\mu\lambda)\,. \tag{7-19}$$

The Yang-Mills supersymmetric system can be coupled to a matter supermultiplet (A_i, B_i, ψ_i) belonging to some representation of the group. The form of the Yang-Mills coupling of the matter is completely analogous to the abelian case. In particular, if also the matter belongs to the adjoint representation, the resulting lagrangian is[34]

$$\mathcal{L}_{\mathrm{MATTER}} = \mathrm{Tr}(-\frac{1}{2}(\mathcal{D}_\mu A)^2 - \frac{1}{2}(\mathcal{D}_\mu B)^2 - \frac{i}{2}\bar{\psi}\gamma\cdot\mathcal{D}\psi - \frac{1}{2}m^2A^2 - \frac{1}{2}m^2B^2 - \\ - \frac{i}{2}m\bar{\psi}\psi + \frac{g^2}{2}\left[A, B\right]^2 + g\bar{\lambda}\left[A + \gamma_5 B, \psi\right])\,. \tag{7-20}$$

Remarkably enough we observe[34] that the supersymmetric Yang-Mills theory with just one matter multiplet in the adjoint representation manifests an additional symmetry, for $m = 0$. In fact, because of the particular form of the lagrangian, one can combine the two Majorana spinors λ and ψ into a complex Dirac spinor

$$\chi = \frac{1}{\sqrt{2}}\,(\lambda + i\psi)\,. \tag{7-21}$$

This leads to the langrangian

$$\mathrm{Tr}(-\frac{1}{4}G^2_{\mu\nu} - \frac{1}{2}(\mathcal{D}_\mu A)^2 - \frac{1}{2}(\mathcal{D}_\mu B)^2 - \frac{i}{2}\bar{\chi}\gamma\cdot\overleftrightarrow{\mathcal{D}}\chi - \\ - ig\bar{\chi}\left[A + \gamma_5 B, \chi\right] + \frac{g^2}{2}\left[A, B\right]^2)\,. \tag{7-22}$$

This langrangian, in addition of being invariant under Yang-Mills and supersymmetry transformations, is also invariant under a "baryon number" phase transformation $\chi \to e^{i\varphi}\chi$, $\chi^* \to e^{-i\varphi}\chi^*$ (the other fields remaining unchanged) and therefore is invariant under a global U(I) group[80, 81]. The langrangian under consideration, being supersymmetric, admits a conserved spinor current. We leave, as an exercise, to verify that this current is given by the following expression[100]

$$J_\mu = \mathrm{Tr}\,(-\frac{1}{2}G^{\rho\sigma}\gamma_\rho\gamma_\sigma\gamma_\mu\chi + ig\left[A, B\right]\gamma_5\gamma_\mu\chi - i\gamma\cdot\mathcal{D}\,(A - \gamma_5 B)\gamma_\mu\chi)\,.$$

Renormalizability of supersymmetric gauge theories has been proved using different techniques[21, 37, 49, 68, 83, 84]. In these theories, like in the model without gauge fields, the same phenomenon occurs, namely the absence of mass renormalization for the matter multiplet[37, 18].

The main problem one has to deal with, in order to prove renormalizability, is the fact that one has to abandon the Wess-Zumino gauges, where the lagrangian looks renormalizable by power counting, and to consider a supersymmetric gauge where the lagrangian is intrinsically non polynomial.

It is straightforward to show that the supersymmetric generalization of the usual Fermi type gauge breaking term $1/\alpha\,(\partial_\mu v^\mu)^2$ is given by[21, 37]

$$\frac{1}{\alpha}\,(D D V \bar{D} \bar{D} V)_D \qquad (7\text{-}23)$$

moreover one has to introduce also Faddeev-Popov ghosts which are anticommuting chiral fields. The new feature of supersymmetric gauge theories is therefore the presence of spin 1/2 fields which obey Bose statistics. These new ghosts are decoupled in the Wess-Zumino non covariant gauges but they are needed for general gauges.

The regularization procedure is also a problem in supersymmetric gauge theories[37]. In fact dimensional regularization[87] cannot be longer used because of the fact that supersymmetry transformations depend on the space-time dimension[16]. However it turns out, from the works of Ferrara, Piguet, Schweda[37, 67] that the higher covariant derivative method and the BPHZ renormalization scheme can be generalized without difficulties to supersymmetric theories.

Concerning the problem of spontaneous supersymmetry breaking[28], we would like to mention that the mechanism used in the abelian situation[30] (Fayet, Iliopoulos), cannot be longer used in the non abelian case. In fact, in this case, the D component of the vector multiplet is no longer a scalar (it belongs to the adjoint representation) and would break explicitely gauge invariance. The method can only be applied to a non semisimple group like U(2) with SU(2) as a residual symmetry, but there is no obvious generalization to semisimple groups. The same difficulty would not arise if one requires a spontaneous breakdown of the gauge group[61, 63, 64, 65, 78], the supersymmetry remaining unbroken. Remarkably, as shown by O'Rafeartaigh and collaborators[7], this is sufficient in order to build up a theory which is both asymptotically free and infrared convergent, i.e., this is sufficient to equip all but an abelian subset of the Yang-Mills fields with masses.

8. - ASYMPTOTIC FREEDOM IN NON ABELIAN GAUGE THEORIES. -

A question of great physical interest, in a non abelian gauge theory of particles, is its ultraviolet behaviour and in particular when it can be asymptotically free[44, 69, 87].

We observe that, because of the results discussed in the previous section, the supersymmetric Yang-Mills theory, if we do not add some extra-couplings like $g' d_{ijk} S^i S^j S^k$ turns out to be controled by one coupling constant only, the gauge coupling constant, so there is one Callan-Symanzik function $\beta(g)$ common to all fields[34, 86].

If one defines the following group invariants

$$\begin{aligned} T(R)\,\delta^{ab} &= \mathrm{Tr}\,(R^a R^b) \qquad\qquad C_2(R)\,I = R^a R^a \\ C_2(G)\,\delta^{ab} &= f^{acd} f^{b}{}_{cd} \end{aligned} \qquad (8\text{-}1)$$

where R^a are the generators of the representation to which the matter multiplet (A_i, B_i, ψ_i) belongs, and $C_2(G)$ is the quadratic Casimir in the adjoint representation, i.e. in the representation of the gauge multiplet $(\lambda_{jk}, v_{\mu jk})$, then the Callan-Symanzik function is given, up to second loop, by the following expression[52]

$$\beta(g) = \frac{A}{16\pi^2}\, g^3 + \frac{B}{(16\pi^2)^2}\, g^5 \qquad (8\text{-}2)$$

with

$$A = n\,T(R) - 3\,C_2(G) \tag{8-3}$$

$$B = 4n\,C_2(R)\,T(R) + 2C_2(G)\,T(R) - 6C_2^2(G) \tag{8-4}$$

being n the number of matter multiplets.

The condition for asymptotic freedom is[34, 52]

$$n\,T(R) - 3C_2(G) < 0\,. \tag{8-5}$$

Note that for $A = 0$ the theory is never asymptotically free because[52]

$$B(A=0) = 12\,C_2(R)\,C_2(G) > 0\,.$$

There is a wide class of solution of (8-5). For example, if the matter field belongs to the adjoint representation and $G = SU(N)$, then (8-3), (8-4) simplify into

$$A = N(n-3)\,, \qquad B = 6N^2(n-1)\,, \tag{8-6}$$

therefore for $n=0$ $A<0$, $B<0$ (pure Yang-Mills), for $n=1$ $A<0$, $B=0$ for $n=2$ $A<0$, $B>0$ and we have a perturbative zero at

$$\frac{g^2}{16\pi^2} = \frac{1}{6N}\,.$$

For $n \geq 3$ the theory is infrared free.

A problem[96], recently solved by O'Rafeartaigh and collaborators[7], is that of having infrared convergence in an asymptotically free supersymmetric theory. Namely to equip all the gauge mesons with mass but an abelian subset. They have shown that this can be achieved through a spontaneous breakdown of the gauge group (supersymmetry remaining unbroken). This mechanism can be triggered if one adds to the original Yang-Mills langrangian

$$\begin{aligned}\mathcal{L} = {} & \mathrm{Tr}\left(-\frac{1}{4}G^2_{\mu\nu} - \frac{i}{2}\bar{\lambda}\gamma\cdot\mathcal{D}\lambda\right) - \frac{i}{2}\bar{\psi}\gamma\cdot\mathcal{D}\psi - \frac{1}{2}\mathcal{D}_\mu A\,\mathcal{D}^\mu A^+ - \\ & -\frac{1}{2}\mathcal{D}_\mu B\,\mathcal{D}^\mu B^+ - \frac{1}{2}m^2(A^2+B^2) - \frac{i}{2}\,m\,\bar{\psi}\psi + ig\Big[\bar{\lambda}(A^+R\psi) + \\ & + \bar{\lambda}\gamma_5(B^+R\psi) + \mathrm{h.c.}\Big] + \frac{g^2}{2}\,(A^+RB + \mathrm{h.c.})^2\end{aligned} \tag{8-7}$$

a supersymmetric matter-matter interaction of the form

$$g'\,d_{ijk}\,S^i S^j S^k \tag{8-8}$$

which is of course locally Yang-Mills invariant. Note, however, that a term like (8-8) cannot be generated from (8-7) because it is odd in S_i while (8-7) is even.

If the matter field itself belongs to the adjoint representation then, irrespectively of the new term (8-8), the condition for having asymptotic freedom is unchanged i.e. $n < 3$. However the important point is that, the resulting potential one obtains by adding (8-8) to (8-7), has not a SU(N) symmetric minimum and in fact it originates a spontaneous breakdown of the gauge group.

In turns out that the simplest way to achieve infrared convergence is that of having two matter multiplets in the adjoint representation of SU(3). This model is rather unphysical, in particular it still contains massless scalar particles, however it shows that, in supersymmetric theories, the problem of infrared convergence, is less severe than in conventional theo ries.

9. - SUPERSYMMETRIC EXTENSIONS OF THE U(2) WEINBERG-SALAM AND U(1) HIGGS MODELS. -

Attempts have been made in order to construct realistic models for unified weak and electromagnetic interactions with the additional properties of supersymmetry invariance. The main difficulty one encounters is that, if one wantsto build up a realistic model with the neutrino as a Goldstone fermion, there is place only for one kind of neutrino in a supersymmetric theory, just because the number of Goldstone particles cannot exceed the number of spinor charges in the theory. On the other hand, if one wantsto introduce two types of neutrinos, then one necessarily must double the charges and therefore provide the supersymmetry algebra of a non trivial mixing with an internal symmetry group.

Unfortunatly it has been shown that renormalizable lagrangians with this reacher symmetry do not exist[25, 26, 38]. A model calculation showing this fact has been carried out by Capper and Leibbrandt[10].

The most realistic model, restricted to the electron sector, has been given by Fayet[27]. It can be regarded as the supersymmetric extension of the $SU(2) \otimes U(1)$ Weinberg-Salam model of weak and electromagnetic interactions.

The original lagrangian contains a doublet of complex chiral multiplets and a singlet interacting with a triplet and a singlet of vector multiplets, i.e. the gauge supermultiplets of $SU(2) \otimes U(1)$.

Spontaneous symmetry breaking occurs in two steps. First there is a spontaneous breakdown of SU(2) with the gauge group U(1) and supersymmetry remaining unbroken. A super-Higgs mechanism[78] is operating in such a way that three real chiral multiplets are eliminated and three (real) vector multiplets acquire a mass. The resulting particle multiplets are therefore :

a) The massless vector multiplet (γ, ν) which contains the photon and the neutrino ;

b) The massive (neutral) vector multiplet (Z, E_0, z) which contains the Z neutral vector boson, a neutral lepton and a scalar with the same masses ;

c) The charged massive vector multiplet $(W_\pm, E_\pm, e_\pm, z_\pm)$ which contains the $W_\pm$ vector mesons, the electron and a new lepton and a charged scalar z all with the same mass ;

d) A complex (neutral) chiral multiplet (e_0, ω, ϕ) which contains a lepton and two scalars.

A second step, spontaneous breaking of supersymmetry occurs (the gauge group U(1) remaining unbroken). The ν particle becomes a Goldstone fermion. All masses inside the multiplets remain the same but for the electron multiplet where the masses of the leptons E_-, e_- are splitted on each side of the W_- mass according to the formula

$$m_{E_-} = m_{W_-} + \xi \ , \qquad\qquad m_{e_-} = m_{W_-} - \xi$$

where ξ is a parameter which measures the symmetry breaking. Z, $W_\pm$ are the ordinary vector mesons of weak interactions. ν, γ are the electron neutrino and the photon. e_- is the electron and E_- an heavy lepton. e_0, E_0 are two heavy neutrinos.

It can be shown that the present model solves the difficulty related to the vanishing of the anomalous magnetic moment in supersymmetric gauge theories.

In fact, it has been shown by Fayet that the graphs, which in a supersymmetric theory exactly cancel the QED graph for the g-2 are higly suppressed in this spontaneously broken version of the gauge theory, as previously suggested by Ferrara and Remiddi[35].

Fayet[29] succeeded also to construct in a very elegant way the supersymmetric version of the U(1) Higgs model. Such a model describes the interaction of a massless vector multiplet with a chiral multiplet.

The lagrangian, in the Wess-Zumino gauge, is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{i}{2}\bar{\lambda}\gamma\cdot\partial\lambda - i\bar{\psi}_L\gamma\cdot\mathcal{D}\psi_L - \mathcal{D}_\mu\varphi^*\mathcal{D}^\mu\varphi + + ie\sqrt{2}\,(\bar{\psi}_L\lambda\varphi + \varphi^*\bar{\lambda}\psi_L) - \frac{1}{2}(\xi + e\varphi^*\varphi)^2 . \tag{9-1}$$

If $\xi < 0$ supersymmetry remains intact but gauge invariance is broken in the vacuum. If

$$\phi = -\frac{i}{\sqrt{2}}(A - iB) \qquad \text{and} \qquad E = \psi_L + \lambda_R$$

we can choose $\langle A\rangle = 0$ $\langle B\rangle = \delta > 0$ where δ satisfies the relation $\xi + \frac{1}{2}e\delta^2 = 0$.

As a consequence of a super Higgs mechanism the chiral multiplet disappears and the vector multiplet (B, E, v_μ) becomes massive with mass given by $m = e\delta$.

The new lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - i\bar{E}\partial\!\!\!/ E - \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{1}{2}(m+eB)^2 v_\mu^2 - - \frac{1}{2}(mB + \frac{e}{2}B^2)^2 - i(m+eB)\bar{E}E + e\bar{E}_L\gamma_\mu E_L v^\mu . \tag{9-2}$$

Remarkably enough the lagrangian in (9-2) is nothing but the most general self-interaction of a massive supermultiplet which does not contain quartic terms in the Fermi fields.

This lagrangian has in fact the most general form

$$L = V\bar{D}DD\bar{D}V + m^2V^2 + (\xi V + \frac{\delta^2}{4}e^{2eV}) \tag{9-3}$$

where, if $\xi < 0$, spontaneous symmetry breaking does not occur. If one makes the following identifications

$$B = \delta e^{eC} , \qquad\qquad E = \lambda_R + e\delta e^{eC}\chi_L .$$

Then (9-3) is just the Higgs lagrangian. In other words, the supersymmetric Higgs model is nothing but the self-interaction of a massive vector multiplet.

10. - SUPERCURRENT AND THE GRAVITON MULTIPLET. -

It has been shown up to now that almost all renormalizable theories, including gauge theories, can be made supersymmetric and in addition it has been realized that their supersymmetric version has less divergent quantum corrections.

On the other hand it is well known that the classical Einstein theory of gravitation does not have a consistent quantum counterpart.

It is therefore of great theoretical interest to investigate the properties of a supersymmetric version of gravitation[99] and in particular if it leads to a renormalizable theory.

As a very preliminary step to this problem, Ferrara and Zumino[36, 23] have studied in details the transformation properties of the various conserved quantities which are present in a symmetric matter system.

Let us consider first a massless supersymmetric system, i.e. a system described by a lagrangian field theory which is (classically) invariant under the 24 parameter Graded Lie Algebra introduced in section 2.

It has been shown by the previous authors that the spinor current can always be redefined in such a way that the improved spinor current

$$\chi_\mu = J_{\mu IMP}. \tag{10-1}$$

which satisfies $\gamma \cdot \chi = 0$, is a member of a real supermultiplet having the following general structure

$$\begin{aligned} \sigma^{\alpha\dot{\alpha}}_\mu V_{\alpha\dot{\alpha}}(x,\theta) &= C_\mu(x) + i\bar{\theta}\gamma_5\chi_\mu(x) + i\bar{\theta}\gamma_5\gamma^\lambda\theta(v_{\lambda\mu}(x)) - \\ &- \frac{1}{2}\varepsilon_{\lambda\mu}^{\ \ \nu\rho}(\partial_\nu C_\rho(x) - \partial_\rho C_\nu(x)) - \frac{1}{4}\bar{\theta}\theta\bar{\theta}\gamma_5\gamma\cdot\partial\chi_\mu(x) + \\ &+ \frac{1}{4}\bar{\theta}\gamma_5\theta\bar{\theta}\gamma\cdot\partial\chi_\mu(x) + \frac{1}{8}(\bar{\theta}\theta)^2\Box C_\mu(x) \end{aligned} \tag{10-2}$$

where

$$C_\mu = \frac{1}{2}J^5_\mu, \qquad v_{\lambda\mu} = -\frac{3}{2}\theta_{\lambda\mu}, \qquad \chi_\mu = -\frac{3}{4}J_{\mu IMP} \tag{10-3}$$

are the conserved axial current, the traceless energy momentum tensor and the conserved spinor current. The form (10-2), together with the conservation equations and the conditions

$$\gamma\cdot\chi = 0, \qquad \theta^\rho_\rho = 0$$

is actually equivalent to the following covariant conservation equations

$$D^\alpha V_{\alpha\dot{\alpha}} = 0, \qquad \bar{D}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} = 0. \tag{10-4}$$

The supermultiplet $V_{\alpha\dot{\alpha}}$ actually contains all the local currents of a supersymmetric system. The corresponding elements (charges) of the Graded Lie Algebra are obtained as moments of the local currents contained in the multiplet.

If one performs an infinitesimal supersymmetry transformation $\alpha(x)$ as defined in section 2, then the multiplet $V_{\alpha\dot{\alpha}}$ undergoes the following transformation

$$\begin{aligned} \delta C_\mu &= i\bar{\alpha}\gamma_5\chi_\mu \\ \delta\chi_\mu &= \frac{1}{2}\gamma^\lambda(v_{\lambda\mu}+v_{\mu\lambda})\alpha - \frac{1}{2}\gamma^\lambda\varepsilon_{\lambda\mu}^{\ \ \nu\rho}\partial_\mu C_\rho\alpha - \partial_\lambda C_\mu\gamma_5\gamma^\lambda\alpha - \\ &- C_\mu\gamma_5\gamma^\lambda\partial_\lambda\alpha - 2C_\lambda\gamma_5\gamma^\lambda\partial_\mu\alpha \\ \delta v_{\lambda\mu SYM} &= \frac{i}{2}\bar{\alpha}(\partial_\lambda\chi_\mu + \partial_\mu\chi_\lambda) - \frac{i}{2}\bar{\alpha}(\gamma_\lambda\gamma\cdot\partial\chi_\mu + \gamma_\mu\gamma\cdot\partial\chi_\lambda) + \\ &+ 5i(\partial_\lambda\bar{\alpha}\chi_\mu + \partial_\mu\bar{\alpha}\chi_\lambda). \end{aligned} \tag{10-5}$$

It is therefore clear that the above supermultiplet should play in a supersymmetric theory a role analogous to the stress-tensor in a usual theory.

In the case of a massive system the equations (10-4) become partial conservation equations. They are

$$D^{\alpha} V_{\alpha\dot{\alpha}} = \bar{D}_{\dot{\alpha}} S \qquad (10\text{-}6)$$

S being a chiral multiplet ($D_{\alpha} S = 0$).

These equations imply that the supermultiplet $V_{\alpha\dot{\alpha}}$ contains now a scalar and a pseudoscalar field beyond an axial vector and the conserved spinor current and stress tensor.

If one implements the equations (10-4) with the mass-shell condition

$$(\Box - m^2)\, V_{\alpha\dot{\alpha}} = 0 \qquad (10\text{-}7)$$

then the multiplet $V_{\alpha\dot{\alpha}}$ can be used to describe a multiplet of particles of spin 1, 3/2, 3/2 and 2. This multiplet corresponds to the $J = 3/2$ multiplet according to the classification given in section 4. For $m = 0$ the eqs. (10-4) become gauge conditions and the remaining degree of freedom are two particles of helicities 3/2 and 2.

This may very well be the graviton supermultiplet which should play a fundamental role in a supersymmetric theory of gravitation. Finally we note that the massless spin 3/2 particle in the supersymmetric gravitational theory has the exact counterpart in the massless spin 1/2 particle of the photon multiplet.

11. - ACTION PRINCIPLE IN SUPERSPACE. -

Arnowitt, Nath and Zumino[2, 3] developed an action principle in curved superspace which was a natural extension of the usual geometry of Minkowsky space to a Riemann geometry.

They considered a point on a superspace labelled by coordinates $z^A = (x^{\mu}, \theta^{\alpha})$ (where x^{μ} is the usual space-time coordinate and θ^{α} is a Majorana spinor) and generalized the supersymmetry transformations to a local gauge group of arbitrary coordinate transformations in superspace

$$z^{A'} = z^{A'}(z) \,. \qquad (11\text{-}1)$$

These transformations leave the line-element

$$ds^2 = dz^A\, g_{AB}(z)\, dz^B \qquad (11\text{-}2)$$

invariant.

The $g_{AB} = \{ g_{\mu\nu}, g_{\mu\alpha}, g_{\alpha\beta} \}$ are gauge supermultiplets and can be interpreted as the metric tensor of superspace.

The Riemann geometry of this space can be constructed and leads to a curvature tensor

$$R^{D}_{\ ABC} = -\Gamma^{D}_{\ AC,B} + (-1)^{bc}\, \Gamma^{D}_{\ AB,C} + (-1)^{b(c+d+e)}\, \Gamma^{E}_{\ AB}\, \Gamma^{D}_{\ EC} - (-1)^{c(d+e)}\, \Gamma^{E}_{\ AC}\, \Gamma^{D}_{\ EB} \qquad (11\text{-}3)$$

and Christoffel affinity

$$\Gamma^{A}_{\ BC} = (-1)^{bc}\, \frac{1}{2} \left[(-1)^{bd}\, g_{AD,B} + (-1)^{ad}\, \eta_{ab}\, g_{BD,A} - g_{AB,D} \right] g^{DC} \qquad (11\text{-}4)$$

$$\eta_{ab} = (-1)^{a+b+ab} \,.$$

An action principle can be formulated in superspace starting from the following action, which is invariant under general coordinate transformations in superspace

$$A = \int d^8z \sqrt{-g}\ R \tag{11-5}$$

where R is the scalar curvature

$$R = (-1)^a g^{AB} R_{AB} \tag{11-6}$$

and R is the contracted Riemann tensor.

The action principle leads to the supersymmetric Einstein equations

$$R_{AB} = 0 , \tag{11-7}$$

which could eventually be generalized adding to the original action an allowed additional invariant

$$\int d^8z \ \sqrt{-g}$$

which would give a cosmological term in (11-7)

$$R_{AB} = \lambda g_{AB} . \tag{11-8}$$

The actual weakness of this approach is due to the fact that usual supersymmetry transformations in Minkowsky space, which one would naturally associate to a flat limit of the above curved space, are not solution of (11-7) nor of (11-8). This is a consequence of the fact that $g_{AB}(z)$ is not constant in flat space and its actual form does not satisfy the previous equations.

Woo[97] has in fact shown that the usual supersymmetric equations of flat space cannot be identified with the corresponding covariant equations in curved space in the flat limit. In order to make such an identification a limiting procedure has to be taken into account.

Such limiting procedure probably corresponds to a contraction of the local gauge group of general coordinate transformations in superspace into a (geometrically different) local group whose flat limit would be identified with the flat supersymmetry.

The construction of this contracted geometry and the corresponding consistent formulation of a supersymmetric Einstein theory of gravitation is still an open problem.

REFERENCES. -

(1) - Y. Aharnov, A. Casher and L. Susskind, Phys. Letters 35B, 512 (1971).
(2) - R. Arnowitt and P. Nath, Phys. Letters 56B, 177 (1975); Northeastern Univ. preprint Nub-2259 (1975); Nub-2261 (1975).
(3) - R. Arnowitt, P. Nath and B. Zumino, Phys. Letters 56B, 81 (1975).
(4) - W. A. Bardeen, to be published.
(5) - F. A. Berezin, The method of 2nd quantization (Academic Press, 1966).
(6) - F. A. Berezin and G. I. Katz, Mathemat. Sbornik (USSR) 82, 343 (1970); English Translat. vol. II.
(7) - S. Browne, L. O'Rafeartaigh and T. Sherry, Dublin preprint Dias-TP-75-16 (1975).
(8) - D. M. Capper, Trieste preprint IC/74/66 (1974).
(9) - D. M. Capper and G. Leibbrandt, Nuclear Phys. B85, 492 (1975).
(10) - D. M. Capper and G. Leibbrandt, Nuclear Phys. B85, 503 (1975).
(11) - S. Coleman and J. Mandula, Phys. Rev. 159, 1251 (1967).
(12) - L. Corwin, Y. Ne'eman and S. Sternberg, Tel Aviv report TAUP 448-74 (1974).
(13) - L. Corwin, Y. Ne'eman and S. Sternberg, Rev. Modern Phys. 47, 573 (1975).
(14) - R. Delbourgo, Nuovo Cimento 25A, 646 (1975).
(15) - R. Delbourgo, Lett. Nuovo Cimento 12, 293 (1975).
(16) - R. Delbourgo and V. B. Prasad, London preprint ICTP/74/6 (1974).
(17) - R. Delbourgo and P. Jarvis, London preprint ICTP/74/9 (1975).
(18) - R. Delbourgo, London preprint ICTP/74/19 (1975).
(19) - R. Delbourgo, A. Salam and J. Strathdee, Phys. Letters 51B, 475 (1974).
(20) - B. De Wit, Phys. Rev. D11, 898 (1975).
(21) - B. De Wit, Stony Brook preprint ITP-SB-75-14 (1975).
(22) - B. De Wit, Stony Brook preprint ITP-SB-75-18 (1975).
(23) - B. De Wit and D. Z. Freedman, Stony Brook preprint ITP-SB-75-24 (1975).
(24) - B. De Wit and D. Z. Freedman, Stony Brook preprint ITP-SB-75-19 (1975).
(25) - P. H. Dondi, Karlsruhe preprint (1975).
(26) - P. H. Dondi and M. Sohnius, Nuclear Phys. B81, 317 (1974).
(27) - P. Fayet, Nuclear Phys. B90, 104 (1975).
(28) - P. Fayet, Ecole Normale Superieure preprint PTENS 75-1 (1975).
(29) - P. Fayet, Ecole Normale Superieure preprint PTENS 75-7 (1975).
(30) - P. Fayet and J. Iliopoulos, Phys. Letters 51B, 461 (1974).
(31) - S. Ferrara, Nuclear Phys. B77, 73 (1974).
(32) - S. Ferrara, J. Iliopoulos and B. Zumino, Nuclear Phys. B77, 413 (1974).
(33) - S. Ferrara, J. Wess and B. Zumino, Phys. Letters 51B, 239 (1974).
(34) - S. Ferrara and B. Zumino, Nuclear Phys. B79, 413 (1974).
(35) - S. Ferrara and E. Remidii, Phys. Letters 55B, 347 (1974).
(36) - S. Ferrara and B. Zumino, Nuclear Phys. B87, 207 (1975).
(37) - S. Ferrara and O. Piguet, Nuclear Phys. B93, 261 (1975).
(38) - R. J. Firth and J. D. Jenking, Durham preprint (1974).
(39) - Z. Frölicher and A. Nijenhuis, Proc. Natl. Acad. Sci. USA 43, 239 (1975).
(40) - K. Fujikawa and W. Lang, Nuclear Phys. B88, 61 (1975).
(41) - K. Fujikawa and W. Lang, Nuclear Phys. B88, 77 (1975).
(42) - O. L. Gervais and B. Sakita, Nuclear Phys. B34, 633 (1971).
(43) - P. Goddard, Nuclear Phys. B88, 429 (1975).
(44) - D. J. Gross and F. Wilczek, Phys. Rev. Letters 30, 1346 (1973).
(45) - D. Grosser, Lett. Nuovo Cimento 12, 649 (1975).
(46) - D. Grosser, Nuclear Phys. B92, 120 (1975).
(47) - R. Haag, J. T. Lopuszanski and M. Sohnius, Nuclear Phys. B88, 257 (1975).
(48) - J. Honerkamp, F. Krause, M. Scheunert and M. Schlindwein, Freiburg preprint (1974).
(49) - J. Honerkamp, F. Krause, M. Scheunert and M. Schlindwein, Freiburg preprint (1975).
(50) - M. Huq, Belfield preprint, Dublin (1974).
(51) - J. Iliopoulos and B. Zumino, Nuclear Phys. B76, 310 (1974).
(52) - T. R. D. Jones, Nuclear Phys. B87, 127 (1975).
(53) - K. Kodaira, L. Niremberg and D. C. Spencer, Ann. Path. 75, 536 (1962).
(54) - K. Kodaira and D. C. Spencer, Ann. Math. 67, 328 (1958).

(55) - W. Lang and J. Wess, Nuclear Phys. B81, 249 (1974).
(56) - L. Mezincescu and V. Ogievetsky, Dubna preprint E2-8277 (1974).
(57) - A. Neveu and J. Schwarz, Nuclear Phys. B31, 86 (1971).
(58) - A. Nijenhuis, Proc. K. Ned. Acad. Wet. A58, 3 (1955).
(59) - A. Nijenhuis and W. Richardson Jr., Bull. Am. Math. Soc. 70, 406 (1966); 72, 1 (1966).
(60) - L. O'Rafeartaigh, Phys. Rev. Letters 14, 575 (1965); Phys. Rev. 139B, 1052 (1965).
(61) - L. O'Rafeartaigh, Phys. Letters 56B, 41 (1975).
(62) - L. O'Rafeartaigh, Nuclear Phys. B89, 418 (1975).
(63) - L. O'Rafeartaigh, Dublin preprint SB Dias-TP-74-46 (1974).
(64) - L. O'Rafeartaigh, Dublin preprint SB Dias-TP-75-9 (1975).
(65) - L. O'Rafeartaigh, Lectures notes, Comm. of the Dublin Inst. of Adv. Studies, Series A (Th. Phys.) 22 (1975).
(66) - A. Pais and V. Rittenberg, Rockfeller Univ. report COO-3232B-74 (1974).
(67) - O. Piguet and M. Schweda, Nuclear Phys. B92, 334 (1975).
(68) - O. Piguet and A. Rouet, Münich preprint MPI-PAE/PTH 18/75 (1975).
(69) - H. D. Politzer, Phys. Rev. Letters 30, 1346 (1973).
(70) - P. Ramond, Phys. Rev. D3, 2415 (1971).
(71) - W. Rühl and B. C. Yunn, Kaiserlautern preprint (August 1974).
(72) - W. Rühl and B. C. Yunn, Kaiserlautern preprint (September 1974).
(73) - A. Salam and J. Strathdee, Nuclear Phys. B76, 477 (1974).
(74) - A. Salam and J. Strathdee, Phys. Letters 49B, 465 (1974).
(75) - A. Salam and J. Strathdee, Phys. Letters 51B, 353 (1974).
(76) - A. Salam and J. Strathdee, Nuclear Phys. B80, 499 (1974).
(77) - A. Salam and J. Strathdee, Nuclear Phys. B84, 127 (1975).
(78) - A. Salam and J. Strathdee, Phys. Rev. D11, 521 (1975).
(79) - A. Salam and J. Strathdee, Nuclear Phys. B86, 142 (1975).
(80) - A. Salam and J. Strathdee, Nuclear Phys. B87, 85 (1975).
(81) - A. Salam and J. Strathdee, Nuclear Phys. B93, 23 (1975).
(82) - A. Salam and J. Strathdee, Trieste preprint IC/75/38 (1975).
(83) - A. A. Slavnov, Dubna preprint E2-8308 (1974).
(84) - A. A. Slavnov, Dubna preprint E2-8449 (1975).
(85) - D. C. Spencer, Ann. Math. 76, 306 (1962).
(86) - M. Suzuki, Nuclear Phys. B83, 269 (1974).
(87) - G. t'Hooft, Nuclear Phys. B61, 455 (1973); B62, 444 (1973).
(88) - H. S. Tsao, Phys. Letters 53B, 381 (1974).
(89) - D. V. Volkov and V. P. Akulov, Phys. Letters 46B, 109 (1973).
(90) - D. V. Volkov and V. A. Soroka, JETP Letters 18, 529 (1973).
(91) - J. Wess and B. Zumino, Nuclear Phys. B70, 39 (1974).
(92) - J. Wess and B. Zumino, Phys. Letters 49B, 52 (1974).
(93) - J. Wess and B. Zumino, Nuclear Phys. B78, 1 (1974).
(94) - J. Wess, Bonn Lecture Notes (1974).
(95) - P. West, Nuclear Phys. B91, 289 (1975).
(96) - G. Woo, MIT preprint (November 1974).
(97) - G. Woo, MIT preprint (March 1975).
(98) - B. Zumino, in Renormalization and invariance in quantum field theory (Ed. E. R. Caianello) (Plenum Press, 1973), p. 367.
(99) - B. Zumino, Nuclear Phys. B89, 535 (1975).
(100) - B. Zumino, Proc. 17th Intern. Conf. on High Energy Physics, London (1974).