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E. Etim, M. Greco and Y. Srivastava: DUALITY FOR VECTOR AND AXIAL-VECTOR CURRENT PROPAGATORS.

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ABSTRACT. -

Duality for vector and axial-vector current spectral functions is formulated in terms of superconvergence of their corresponding θ-current-current vertices. We study in detail the saturation of these sum rules in a model with infinite particle spectra. We obtain: (i) for the asymptotic value of $R \equiv \sigma(e^+e^- \to \text{hadrons}) / \sigma(e^+e^- \to \mu^+\mu^-) \simeq 4.2$ in the SU(4) scheme, (ii) Weinberg's first sum rule and the KSFR relation, and (iii) $f_k/f_\pi \simeq 1.2$.

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The idea of duality in e^+e^- annihilation between the low and the high energy scaling behaviour of the total cross section into hadrons has been studied recently[1], in the framework of the canonical trace anomaly of the energy momentum tensor[2-3]. In complete analogy with strong interactions an entire set of sum rules have been derived, relating the asymptotic value of R to the low energy behaviour of $\sigma(e^+e^- \to \text{hadrons})$. In particular the zeroth moment sum rule

$$(1) \qquad \int \left[\operatorname{Im} \pi(s) - \frac{\alpha R}{3} \right] ds = 0,$$

with $\operatorname{Im} \pi(s) = s\, \sigma_{had}(s)/4\pi\alpha$, has been discussed in connection with experimental data, and shows almost exact saturation with $R \simeq 2.5$, also locally, up to $q^2 \simeq 10\ \text{GeV}^2$, the threshold for production of the newly discovered particles. This result is in agreement with the extended (or generalized) vector dominance model which gives $R = 8\pi^2/f_\rho \sim 2.5$. Beyond this range, addition of the set of new vector mesons (charmed and/or colored) modifies the value of R.

The purpose of the present note is two-fold. Firstly we formulate the idea of duality in terms of a superconvergence sum rule satisfied by $\Delta(q^2)$, where $\Delta(q^2)$ is the Green's function associated with the vertex θ-j_μ-j_ν, θ the trace of the energy momentum tensor and j_μ the electromagnetic current. In this connection we study further properties of our model, in particular the validity of the canonical trace identities derived by Chanowitz and Ellis[2] and Crewther[3].

We then extend our considerations to trace identities with axial and strange-vector currents, by saturating those currents with an infinite series of appropriate $J^P = 0^-, 0^+, 1^-$ and 1^+ mesons lying on parallel Regge trajectories. Using only asymptotic chiral and SU(3) symmetry and scaling, we obtain the first Weinberg sum rule[4]

(2) $$\frac{m_\varrho^2}{f_\varrho^2} = \frac{m_A^2}{f_A^2} + f_\pi^2 ,$$

as well as the SKFR relation[5]

(3) $$\frac{m_\varrho^2}{f_\varrho^2} = 2 f_\pi^2 .$$

The strange channel provides one verifiable relation

$$\frac{f_k}{f_\pi} \simeq 1.2 ,$$

where f_k and f_π are the kaon and pion decay constants.

Defining the Green's functions

(4) $$\Delta_{\mu\nu}(q) = \int d^4x\, d^4y\, e^{iqy} \langle 0/T(\theta(x)\, J_\mu(y)\, J_\nu(0))/0\rangle ,$$

(5) $$\pi_{\mu\nu}(q) = i\int d^4x\, e^{iqx} \langle 0/T(J_\mu(x)\, J_\nu(0))/0\rangle ,$$

we have the following anomalous trace identity

(6) $$\Delta(q^2) = -2q^2 \frac{\partial}{\partial q^2} \pi(q^2) - \frac{e^2 R}{6\pi^2} ,$$

with

(7) $$\Delta_{\mu\nu}(q) = (q_\mu q_\nu - g_{\mu\nu} q^2)\, \Delta(q^2)$$

(8) $$\pi_{\mu\nu}(q) = (q_\mu q_\nu - g_{\mu\nu} q^2)\, \pi(q^2) .$$

Writing a dispersion relation for $\Delta(q^2)$

$$\Delta(q^2) = \Delta(0) + \frac{q^2}{\pi}\int_{s_0}^{\infty} ds \frac{\mathrm{Im}\,\Delta(s)}{s(s-q^2)},$$

and using eq. (6) one obtains

$$(9)\quad \Delta(q^2) = \Delta(0) - \frac{2}{\pi}\frac{q^2}{s_0-q^2}\mathrm{Im}\,\pi(\infty) - \frac{2q^2}{\pi}\int_{s_0}^{\infty}\frac{ds}{(s-q^2)^2}\mathrm{Im}\left[\pi(s)-\pi(\infty)\right].$$

Let us assume that θ is sufficiently "soft" so that $\Delta(q^2) \to 0$ as $q^2 \to \infty$. Eqs. (6) and (9) then give

$$(10)\qquad \Delta(0) = -\frac{2}{\pi}\,\mathrm{Im}\,\pi(\infty) = -\frac{e^2 R}{6\pi^2}.$$

Moreover if $q^2\Delta(q^2) \to \frac{2}{\pi}c$ as $q^2 \to \infty$ we obtain from eq. (9)

$$(11)\qquad c = \frac{\alpha R}{3}s_0 - \int_{s_0}^{\infty} ds\left[\mathrm{Im}\,\pi(s) - \frac{\alpha R}{3}\right].$$

Since $s_0 = 4\mu^2 \simeq 0$ the duality sum rule (eq. (1)) implies that $\Delta(q^2)$ is superconvergent, i.e. $c = 0$. This simple example illustrates (i) that duality for $\pi(s)$ is a reflection of superconvergence of $\Delta(s)$, and (ii) that deviations from superconvergence are caused by threshold effects. We assume, consistently with phenomenological evidence, that eq. (1) is always satisfied, and hence $c = \alpha R s_0/3$. This is also verified in the model discussed below. Eq. (11) is modified if different asymptotic limits for $\Delta(s)$ are assumed, e.g. those given by asymptotic free field theory models.

Now we turn to an explicit model[1] (EVMD) in which the anomalous trace identities are not imposed but occur naturally. Furthermore the model allows for precise answers to the questions of superconvergence, duality, etc.

Taking into account both "old" ($V \equiv \rho, \omega, \varphi, \rho' \ldots$) and new ($V' \equiv J/\psi(3.1)$, $\psi(3.7)$, $\psi(4.2)$...) vector mesons, and specifying the mass and the coupling constants spectra as $m_{in}^2 = m_{io}^2(1+a_i n)$ and $f_{in}^2 = f_{io}^2(1+b_i n)$, ($i = \varrho, \omega, \varphi, \psi$) the field current and field-trace identities are

$$j_\mu(x) = \sum_i \sum_{n=0}^{\infty} e \frac{m_{in}^2}{f_{in}^2} V_\mu^{in} \tag{12}$$

$$\theta(y) = \sum_i \sum_{n=0}^{\infty} m_{in}^2 : V_\mu^{in} V_\mu^{in} : . \tag{13}$$

Using eqs. (4) and (5) we then obtain

$$\begin{aligned} \pi(q^2) &= q^2 \sum_i \sum_{n=0}^{\infty} \left(\frac{e}{f_{in}}\right)^2 \frac{1}{m_{in}^2 - q^2} = \\ &= -\sum_i \frac{e^2}{b_i f_{io}^2} \frac{(z_i - 1/a_i)}{(z_i - 1/b_i)} \left[\psi(z_i) - \psi(1/b_i)\right] , \end{aligned} \tag{14}$$

and

$$\begin{aligned} \Delta(q^2) &= \Delta(0) - 2q^2 \sum_i \sum_{n=0}^{\infty} \left(\frac{e\, m_{in}}{f_{in}}\right)^2 \frac{1}{(m_{in}^2 - q^2)^2} = \\ &= \Delta(0) + \sum_i \frac{2 e^2}{b_i f_{io}^2} \left\{ \frac{(z_i - 1/a_i)^2}{(z_i - 1/b_i)} \psi'(z_i) + \frac{\left(\frac{1}{a_i} - \frac{1}{b_i}\right)\left(z_i - \frac{1}{a_i}\right)}{\left(z_i - \frac{1}{b_i}\right)^2} \left[\psi(z_i) - \psi(1/b_i)\right] \right\} \end{aligned} \tag{15}$$

where

$$z_i = 1/a_i - q^2/(a_i m_{io}^2)$$

and

$$\Delta(0) = -2 e^2 \left(\frac{1}{f_\varrho^2 b_\varrho} + \frac{1}{f_\omega^2 b_\omega} + \frac{1}{f_\varphi^2 b_\varphi} + \frac{1}{f_\psi^2 b_\psi} \right).$$

This gives for the asymptotic value of R

(16) $$R = 12\pi^2 \left(\frac{4}{3}\frac{1}{b_\varrho f_\varrho^2} + \frac{1}{b_\psi f_\psi^2}\right),$$

where to simplify the notation we have used the approximate equality:

(16') $$f_\omega^2 b_\omega \simeq 2 f_\varphi^2 b_\varphi \simeq 9 f_\varrho^2 b_\varrho .$$

From eqs. (14) and (15) it is easy to see that the trace identity (6) is always satisfied, as asserted earlier. In the large q^2 limit we find

(17) $$\pi(q^2) \rightarrow -\sum_i \frac{e^2}{b_i f_{io}^2}\left\{ \ln z_i - \frac{1}{z_i}\left[\left(\frac{1}{a_i} - \frac{1}{b_i}\right)\ln z_i - \frac{1}{2}\right] - \psi(1/b_i) \times \right.$$
$$\left. \times \left[1 - \left(\frac{1}{a_i} - \frac{1}{b_i}\right)\right] + 0\left(\frac{1}{z_i^2}\right)\right\},$$

and

(18) $$\Delta(q^2) \rightarrow \sum_i \frac{2e^2}{b_i f_{io}^2}\frac{1}{z_i}\left\{\left(\frac{1}{2} - \frac{2}{a_i} + \frac{1}{b_i}\right) + \left(\frac{1}{a_i} - \frac{1}{b_i}\right)\left[\ln z_i - \psi\left(\frac{1}{b_i}\right)\right] + 0\left(\frac{1}{z_i^2}\right)\right\}$$

From eq. (18) we see that the necessity but not sufficient condition for $\Delta(q^2)$ to superconverge is that $a_i = b_i$. Given this condition, eq. (11) for this model reads:

(19) $$\int_{s_{oi}}^{\infty} ds\left[\operatorname{Im}\pi_i(s) - \frac{\alpha R_i}{3}\right] = \frac{\alpha}{3} R_i\left[s_{oi} - m_{oi}^2\left(1 - \frac{b_i}{2}\right)\right],$$

where we have decomposed $\operatorname{Im}\pi(s) = \sum_i \operatorname{Im}\pi_i(s\ )\,\theta(s-s_i)$. In the spirit of local duality the effective threshold are given by $s_{oi} = m_{oi}^2 - \Delta m_i^2/2 = m_{oi}^2 - b_i m_{oi}^2/2$, which leads to a zero for the r. h. s. of eq. (19) which thus reduces to eq. (1). Local saturation, in this sense, leads us to eq. (16). Since R is an asymptotic quantity, it is tempting to apply symmetry arguments to it. In fact, as stated earlier, the SU(3) relations (16') are better satisfied than for the coupling constants themselves. This is borne out when consider the ψ-component as well. In SU(4) this gives $f_\varrho^2\, b_\varrho = (8/9)\, f_\psi^2\, b_\psi$, which leads to the pleasing result

$$(20)\quad \Gamma(\psi \to e^+e^-) = \frac{8}{9}\,\frac{m_\psi}{m_\varrho}\,\frac{b_\psi}{b_\varrho}\,\Gamma(\varrho \to e^+e^-) \simeq .75\ \Gamma(\varrho \to e^+e^-) \simeq 4.8\ \mathrm{KeV},$$

in very good agreement with experiments. Also, in this spirit, the SU(4) value of R is $R = \frac{6\pi^2}{f_\varrho^2}\left(1+\frac{1}{9}+\frac{2}{9}+\frac{8}{9}\right) \simeq 4.2$ instead of the canonical 10/3, obtained in the coloured SU(4) scheme.

We next consider axial currents. Defining

$$(21)\quad \begin{aligned} \Delta^A_{\mu\nu}(q) &= \int d^4x\, d^4y\, e^{iqy} \langle 0/T(\theta(x)\, A_\mu(y)\, A_\nu(0))/0\rangle = \\ &= (q_\mu q_\nu - g_{\mu\nu} q^2)\, \Delta^c(q^2) + g_{\mu\nu}\, \Delta^{nc}(q^2), \end{aligned}$$

and

$$(22)\quad \begin{aligned} \pi^A_{\mu\nu}(q) &= i\int d^4x\, e^{iqx} \langle 0/T(A_\mu(x)\, A_\nu(0))/0\rangle = \\ &= (q_\mu q_\nu - g_{\mu\nu} q^2)\, \pi^c(q^2) + q_\mu q_\nu\, \pi^{nc}(q^2), \end{aligned}$$

we write the general trace relations:

$$(23a)\quad \Delta^c(q^2) = -\frac{R}{8\pi^2} - 2q^2 \frac{\partial}{\partial q^2}\left[\pi^c(q^2) + \pi^{nc}(q^2)\right],$$

and

$$(23b)\quad \Delta^{nc}(q^2) = \Delta^{nc}(0) + q^2\, \Delta^{nc\,\prime}(0) - 2q^4 \frac{\partial}{\partial q^2}\, \pi^{nc}(q^2).$$

The canonical result (for spin 1/2 quark model) is[2]

$$(24)\quad \Delta^{nc}(0) = -\sum_i \frac{m_i^2}{\pi 2}, \qquad \Delta^{nc\,\prime}(0) = 0,$$

where m_i's are the quark masses. As discussed later our results will be slightly different from those of eq. (24).

As in the vector case, using eqs. (23) we write the dispersion relations

$$(25) \qquad \Delta^c(q^2) + \frac{R}{8\pi^2} = -\frac{2}{\pi} q^2 \int_{s_0}^{\infty} \frac{[\mathrm{Im}\,\pi^c(s) + \mathrm{Im}\,\pi^{nc}(s)]}{(s-q^2)^2} ds ,$$

and

$$(26) \qquad \frac{\Delta^{nc}(q^2) - \Delta^{nc}(0)}{q^2} = \Delta^{nc\,\prime}(0) - \frac{2}{\pi} q^2 \int_{s_0}^{\infty} \frac{\mathrm{Im}\,\pi^{nc}(s)}{(s-q^2)^2} ds .$$

On dimensional grounds (see eq. (22)) we expect $\pi^c(q^2)$ and $\pi^{nc}(q^2)$ to behave in the same way for large q^2 ($\pi^c(q^2)$, $\pi^{nc}(q^2) \sim \ln(-q^2)$). On the other hand $\Delta^c(q^2)$ is expected to vanish generally, as in the vector case, whereas at least $q^{-2}\,\Delta^{nc}(q^2)$ should go to zero. Through eq. (23.b), this already implies $\Delta^{nc\,\prime}(0) \neq 0$, in conflict with the canonical result (24). Such non canonical behaviour is verified in the model discussed below.

In analogy to the vector case, we saturate the currents with infinite families of particles lying on parallel Regge trajectories, with common shape $\alpha' \simeq 1/2m_\varrho^2$. Explicitly

$$(27) \qquad \mathrm{Im}\,\pi^c(s) = \pi \frac{m_A^2}{f_A^2} \sum_n \delta(s - m_{A_n}^2),$$

and

$$(28) \qquad \mathrm{Im}\,\pi^{nc}(s) = \pi f_\pi^2 \sum_n \delta(s - m_{\pi_n}^2).$$

From eq. (25) we then obtain:

$$(29) \quad \Delta^c(q^2) = -\frac{1}{f_\varrho^2} - \frac{m_A^2}{f_A^2} \frac{q^2}{2m_\varrho^4} \psi'\left(\frac{m_A^2 - q^2}{2m_\varrho^2}\right) - f_\pi^2 \frac{q^2}{2m_\varrho^4} \psi'\left(\frac{m_\pi^2 - q^2}{2m_\varrho^2}\right),$$

which for large q^2 reads:

$$(30)\quad \Delta^c(q^2) \rightarrow \left[-\frac{1}{f_\varrho^2} + \frac{1}{m_\varrho^2}\frac{m_A^2}{f_A^2} + \frac{f_\pi^2}{m_\varrho^2} \right] - \frac{1}{q^2}\left[\frac{m_A^2}{f_A^2}\left(1 - \frac{m_A^2}{m_\varrho^2}\right) + f_\pi^2\left(1 - \frac{m_\pi^2}{m_\varrho^2}\right)\right] + 0\left(\frac{1}{q^4}\right).$$

The vanishing of $\Delta^c(q^2)$ therefore gives the Weinberg's first sum rule(4)

$$(31)\quad \frac{m_\varrho^2}{f_\varrho^2} = \frac{m_A^2}{f_A^2} + f_\pi^2 .$$

Since the threshold in this case is of order m_π^2, $\Delta^c(q^2)$ should superconverge in the limit of $m_\pi \rightarrow 0$. From eq. (30) this gives the further relation

$$(32)\quad \frac{m_A^2}{f_A^2}\left(1 - \frac{m_A^2}{m_\varrho^2}\right) + f_\pi^2 = 0 ,$$

which combined with (31) leads to the KSFR relation(5)

$$(33)\quad m_\varrho^2 = 2 f_\varrho^2 f_\pi^2 .$$

In deriving these equations we have used chiral symmetry only in the asymptotic limit, that is to say (i) $R_A = R_V = 1/f_\varrho^2$ and (ii) $\Delta^V(q^2)$ and $\Delta^c(q^2)$ both superconverge because of the smallness of the pion mass (see discussion after eq. (11)).

From eqs. (26) and (28) we have for $\Delta^{nc}(q^2)$

$$(34)\quad \Delta^{nc}(q^2) = \Delta^{nc}(0) + q^2 \Delta^{nc'}(0) - f_\pi^2 \frac{q^4}{2m_\varrho^4}\, \psi'\left(\frac{m_\pi^2 - q^2}{2m_\varrho^2}\right).$$

Demanding that $q^{-2}\Delta^{nc}(q^2) \rightarrow 0$ as $q^2 \rightarrow \infty$ we obtain

(35) $$\Delta^{nc'}(0) = -\frac{f_\pi^2}{m_\varrho^2} .$$

If we make the stronger demand that $\Delta^{nc}(q^2) \to 0$ we get

(36) $$\Delta^{nc}(0) = f_\pi^2 \left(1 - \frac{m_\pi^2}{m_\varrho^2}\right).$$

The last two equations are in disagreement with the canonical result (24). Notice that $\Delta^{nc}(0)$ has the opposite sign to that in eq. (24).

Our arguments can also be applied to the strangeness-changing current propagators. We saturate the vector current with the K^{x} series, and since this current is not conserved we also have contribution from $J^P = 0^+$ or $\varkappa$ series. Preceding as before, i. e. demanding that the corresponding $\Delta^c(q^2) \to 0$ as $q^2 \to \infty$ and using $R^S = R^V = 1/f_\varrho^2$ we find a result similar to eq. (31):

(37) $$\frac{m_\varrho^2}{f_\varrho^2} = \frac{m_{k^x}^2}{f_{k^x}^2} + f_\varkappa^2 .$$

A more interesting result is obtained when we consider the corresponding axial currents. We find:

(38) $$\frac{m_\varrho^2}{f_\varrho^2} = \frac{m_{k_A}^2}{f_{k_A}^2} + f_k^2 ,$$

and the superconvergence result

(39) $$\frac{m_{k_A}^2}{f_{k_A}^2} \left(1 - \frac{m_{k_A}^2}{m^2}\right) + f_k^2 \left(1 - \frac{m_k^2}{m_\varrho^2}\right) = 0.$$

From (38) and (39) we obtain for the kaon decay constant

(40) $$f_k^2 = \left(\frac{m_\varrho^2}{f_\varrho^2}\right) \frac{m_{k_A}^2 - m_\varrho^2}{m_{k_A}^2 - m_k^2} = 2 f_\pi^2 \frac{m_{k_A}^2 - m_\varrho^2}{m_{k_A}^2 - m_k^2} .$$

Using for the axial k_A mass $m_{k_A} = 1.24$ GeV[7] we obtain

(41) $$\frac{f_k}{f_\pi} = 1.2,$$

which is a very satisfactory result. This ratio is rather stable with respect to changes in m_{k_A}. For example even if we identify k_A with the (1280 -1400 MeV) meson[7], we find for this ratio a maximum value of 1.26.

To summarize we have constructed a specific model for the currents and the θ-J-J vertices, using saturation with infinite number of mesons lying on parallel Regge trajectories. In addition to the scaling results, the model provides an explicit realization of various duality and superconvergence sum rules. Many of the old current algebra results which have been obtained using single particle dominance arise naturally from the scaling assumption and the softness of the θ-operator.

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