COMITATO NAZIONALE PER L'ENERGIA NUCLEARE
Laboratori Nazionali di Frascati

LNF-75/32(P)

S. Ferrara, R. Gatto and A. F. Grillo: PROPERTIES OF PARTIAL-WAVE AMPLITUDES IN CONFORMAL INVARIANT FIELD THEORIES. -

Servizio Documentazione
dei Laboratori Nazionali di Frascati del CNEN
Casella Postale 70 - Frascati (Roma)

IL NUOVO CIMENTO VOL. 26 A, N. 3 1 Aprile 1975

Properties of Partial-Wave Amplitudes in Conformal Invariant Field Theories.

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(ricevuto il 19 Novembre 1974)

Summary. — Analyticity properties of partial-wave amplitudes of the conformal group $O_{D,2}$ (D not necessarily integer) in configuration space are investigated. The presence of Euclidean singularities in the Wilson expansion in conformal invariant field theories is discussed, especially in connection with the program of formulating dynamical bootstrap conditions coming from the requirement of causality. The exceptional case of $D=2$ is discussed in detail.

1. – Introduction,

In previous papers [1,2] conformal partial-wave expansions in configuration space have been studied and their connection to the Wilson operator expansion has been clarified. In particular the problem of exhibiting the partial waves for the four-point function has been solved. This problem has also

(1) S. FERRARA, R. GATTO, A. F. GRILLO and G. PARISI: *Nucl. Phys.*, **49** B, 77 (1972).
(2) S. FERRARA, R. GATTO, A. F. GRILLO and G. PARISI: *Nuovo Cimento*, **19** A, 667 (1974).

been recently considered by many authors [3-5]. In particular POLYAKOF [5] has proposed different choices of partial waves, of different analyticity properties, and obtained dynamical constraints (self-consistency conditions) in conformal invariant theories. Similar dynamical constraints have also been suggested by us [6]. The aim of the present paper is to further investigate the singularity structure of conformal partial-wave amplitudes. We shall also present some calculational improvements for conformally invariant theories.

We find that, for any value of the dimension of space-time $D > 2$, Euclidean singularities are present in partial-wave amplitudes [2] (except for peculiar relations between the dimension of the operators involved), this fact reflecting the lack of convergence of conformal operator expansion at large distances.

Our partial-wave amplitudes, to be considered as related to Wightman functions in Minkowski space, are unanbiguously defined using the conformal ansatz for the operator expansion [7], regarded as a Taylor expansion near the tip of the light-cone, which verifies the Wilson dimensional rule. We stress that these requirements avoid confusion with different possible ansatz for partial-wave amplitudes [3-5]. Because of the exhibited singularity structure, which reflects itself into a violation of causality in single partial waves, it is clear that dynamics must provide a mechanism of cancellation of such singularities. Whether such a problem has any nontrivial solution is still at present an open question. Free-field theories provide at least an example.

The paper is organized as follows. In Sect. **2** we give the singularity structure of partial waves. We then show the connection to the problem of the singularities of double hypergeometric functions. Connection with operator product expansion is elucidated, especially concerning the exceptional points when causality is restored. In Sect. **3** the peculiar role of $D = 2$ (D being the dimension of space-time) is discussed. The paper ends with coucluding remarks and an Appendix where the derivation of the main formulae needed in the text is indicated.

2. – Properties of conformal partial-wave amplitudes.

In this Section we study the nature of Euclidean singularities in the conformal partial waves in D dimensions. We confine ourselves to the contribution of a conformal Lorentz scalar operator in the product $A(x)B(y)$, generaliza-

(3) G. MACK: to appear in *Renormalization and Symmetry in Field Theory*, edited by E. CAIANIELLO (New York, N.Y.); V. DOHREV, G. MACK, V. PETROVA, S. PETROVA and I. T. TODOROV: Dubna preprint E2-7977 (1974) and *Springer Lecture Notes in Physics* (to appear).

(4) A. A. MIGDAL: Landau Institute preprint (1972).

(5) A. M. POLYAKOF: Landau Institute preprint, Cernogolovsk (1973).

(6) S. FERRARA, R. GATTO and A. F. GRILLO: *Ann. of Phys.*, **76**, 161 (1973).

(7) S. FERRARA, R. GATTO and A. F. GRILLO: *Lett. Nuovo Cimento*, **4**, 115 (1972).

tion to higher spins being straightforward [2]. It is known [2,7] that the requirement of conformal invariance on the Wilson expansion gives the unique ansatz

$$A(x)B(0)=\left(\frac{1}{x^2}\right)^{\frac{1}{2}(\Sigma_{AB}-l)}\frac{\Gamma(l)}{\Gamma((l+\Delta_{AB})/2)\,\Gamma((l-\Delta_{AB})/2)}\cdot \tag{2.1}$$
$$\cdot\int_0^1 d\lambda\,\lambda^{\frac{1}{2}(l+\Delta_{AB})-1}(1-\lambda)^{\frac{1}{2}(l-\Delta_{AB})-1}\,{}_0F_1\left(l+1-\frac{D}{2},\;\frac{-x^2}{4}\lambda(1-\lambda)\,\Box_x\right)O(\lambda x)\,,$$

where $O(x)$ is a scalar operator which contributes to the Wilson expansion of $A(x)B(0)$. $\Sigma_{AB}=l_A+l_B$, $\Delta_{AB}=l_A-l_B$, where l_A, l_B, l are the scale dimensions of A, B, O respectively ($l=\frac{1}{2}D-1$) for massless fields. Normalization has been chosen in such a way that, on the tip of the light-cone, the product behaves as $A(x)B(0)\sim x^{l-\Sigma_{AB}}O(0)$.

With the aid of (2.1) the contribution of the local operator $O(x)$ to the four-point function [1]

$$\langle 0|A(x)B(y)C(z)D(t)|0\rangle=[(x-y)^2]^{\frac{1}{2}(\Sigma_{AB}-\Delta_{AB})}\cdot \tag{2.2}$$
$$\cdot[(x-z)^2]^{-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}[(x-t)^2]^{-\frac{1}{2}(\Delta_{AB}-\Delta_{CD})}[(z-t)^2]^{-\frac{1}{2}(\Sigma_{CD}-\Delta_{AB})}f(\varrho,\eta)\,,$$
$$\varrho=\frac{(x-t)^2(z-y)^2}{(x-y)^2(z-t)^2}\,,\qquad \eta=\frac{(x-z)^2(y-t)^2}{(x-y)^2(z-t)^2}\,,$$

can be evaluated. For the function $f(\varrho,\eta)$ one gets [1,2]

$$f_0(\varrho,\eta)=\frac{\Gamma(l)}{\Gamma(\frac{1}{2}(l+\Delta_{AB}))\,\Gamma(\frac{1}{2}(l-\Delta_{AB}))}\cdot \tag{2.3}$$
$$\cdot\int_0^1 d\sigma\,\sigma^{\frac{1}{2}(\Delta_{AB}-\Delta_{CD})-1}(1-\sigma)^{-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})-1}\left(\frac{\varrho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{-\frac{1}{2}(l+\Delta_{CD})}\cdot$$
$$\cdot{}_2F_1\left(\frac{1}{2}(l+\Delta_{CD}),\;\frac{1}{2}(l-\Delta_{CD});\;l+1-\frac{D}{2};\;\left(\frac{\varrho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{-1}\right).$$

We remark that this function depends on D in an analytic way. Partial waves can thus be defined also for nonintegral values of D.

If, in general, we call f_{n,l_n} the partial wave associated to a Lorentz irreducible tensor $(n/2, n/2)$, the four-point function $f(\varrho,\eta)$ can be written in the form

$$f(\varrho,\eta)=\sum_{n,l_n}f_{n,l_n}(\varrho,\eta)\,, \tag{2.4}$$

where n, l_n are the conformal quantum numbers of all possible local operators which are simultaneously present in the expansion of $A(x)B(y)$ and of $C(z)D(t)$.

The general formula (2.3) is sufficient to discuss Euclidean singularities. In the region where ϱ, η are both positive, the function ${}_2F_1$ exhibits in fact a branch point (or a pole) when its last argument is 1. To show this it is convenient to use the following identity [8]:

$$
{}_2F_1\left(\frac{1}{2}(l-\Delta_{CD}),\ \frac{1}{2}(l+\Delta_{CD});\ l+1-\frac{D}{2};\ \left(\frac{\varrho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{-1}\right)=
$$

$$
=\left[1-\left(\frac{\varrho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{-1}\right]^{1-D/2} {}_2F_1\left(\frac{1}{2}(l-\Delta_{CD})+1-\frac{D}{2},\right.
$$

$$
\left.\frac{1}{2}(l+\Delta_{CD})+1-\frac{D}{2};\ l+1-\frac{D}{2};\ \left(\frac{\varrho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{-1}\right).
$$

In fact, for $D>2$ the hypergeometric function on the right-hand side of this relation has no singularities when its last argument is finite and the Euclidean singularity is directly exhibited by the factorized power on the right-hand side.

This singularity is a pole for even space-time dimensions and a cut for odd (or noninteger) space-time dimensions. This singularity originates an unwanted spacelike anomalous singularity which violates the locality properties of the Wightman functions. However from the previous formula one can realize that there is a sequence of exceptional points, namely at

$$
l=|\Delta_{AB,CD}|-n_{\text{even}}\,, \tag{2.5}
$$

where such singularities disappear. This is because the left-hand side of the previous identity (*i.e.* the kernel of the integral representation) reduces to a polynomial in this case, due to well-known properties of the hypergeometric functions. Moreover we note that these points are always in a finite number; in fact one has $l>\frac{1}{2}D-1$, because of the positivity requirement.

It is interesting to further observe that restoration of causality at these exceptional points has a very simple interpretation by means of operator expansion. Without any loss of generality suppose $l_B>l_A$ so that the critical values are at $l=\Delta_{AB,CD}-n_{\text{even}}$. From (2.1) it follows that, at these values, only a finite number of derivatives of $O(x)$ contributes to the right-hand side, thus causality is straightforward in this case. A more direct proof is gained with the six-dimensional version of (2.1). In terms of six-dimensional co-ordinates (as defined in ref. [6,7]) (2.1) can be written as

$$
A(\eta)\,B(\eta')=D^h(\eta,\eta')\,O(\eta') \qquad (\eta^2=\eta'^2=0)\,, \tag{2.6}
$$

where $h=\frac{1}{2}(\Delta_{AB}-l)$ and $D(\eta,\eta')$ is an invariant differential operator proportional to a (conformal) linear combination of $(x-x')\,\partial'$ and $(x-x')^2\,\Box_{x'}$.

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[8] *Bateman Manuscript Project*: *Higher Transcendental Functions*, Vol. **1**, 2.9(1).

The exceptional values given by (2.5) then correspond to h integer, giving in (2.6) a polynomial of order h in the derivatives of the operator O.

The singularities of the functions defined by (2.3) can be recognized to be those associated to the double hypergeometric functions. In fact (2.3) is an integral representation of a double hypergeometric function of the F_4-type, whose analyticity properties are well known (2).

It is evident that in the more interesting case, when Δ_{AB} $(\Delta_{CD})=0$, these exceptional points cannot be present and Euclidean singularities are always present. In all these cases different partial waves must conspire in such a way as to cancel this causal behaviour. Self-consistency of the theory, if there is a conformal invariant solution, must provide this mechanism. Already free-field theory is an example of a nontrivial cancellation of Euclidean singularities. In fact the conformal partial-wave expansion of $\varphi(x)\,\varphi(0)$ in free-field theory is nonlocal but the entire Wightman function is indeed local, being the symmetric superposition of the three disconnected graphs. This means that the partial waves of $:\varphi^2:$, $\theta_{\mu\nu}$ (the stress tensor) and of the higher order, twist $D-2$ (conserved), tensors, must conspire in such a way to reproduce the two crossed disconnected graphs (the direct disconnected graph is given by the identity operator in the Wilson expansion).

In free-field theory such problems of conspiration of singularities have been explicitly solved (at least in 4 dimensions) with the help of the Klein-Gordon equation (9). In fact the general solution of the Klein-Gordon equation for four-point function can be written as

$$f(\varrho,\eta)=\int\limits_0^1 \mathrm{d}\sigma\,\frac{h(\sigma)}{\sigma^2+\sigma(\eta-\varrho-1)+\varrho}\,, \tag{2.7}$$

where $h(\sigma)=h(1-\sigma)$. $h(\sigma)=1$ for $:\varphi^2:$ and in general $h(\sigma)$ is a polynomial of order n in σ for the n-th–order tensor. Violation of causality in single partial waves is due to the fact that $h_n(\sigma)$, associated to the conserved operators $O_{\alpha_1\ldots\alpha_n}(x)$, is smeared in the interval from 0 to 1. What happens is that the infinite sum gives a function concentrated at the end-points. As a mathematical example note that

$$\sum_{n\,\text{even}}(2n+1)\,P_n(\sigma)=\delta(\sigma)+\delta(1-\sigma)\,. \tag{2.8}$$

Inserting (2.8) into (2.7) one just reproduces the two crossed disconnected graphs.

An example of self-consistent solution in interacting conformal invariant theory has been given in ref. (5) with a somewhat different definition of partial-wave expansion, using the ε-expansion technique. Whether such a mechanism can be realized in more realistic theories in an open problem.

(9) S. FERRARA and M. TESTA: *Phys. Lett.*, **49** B, 95 (1974).

As a final remark of this Section we give the explicit expression for the partial waves of the scalar operator in free-field theory for any dimension D of space-time.

This expression manifestly exhibits the location of Euclidean singularities for the partial waves of the $O_{D,2}$ group:

$$f_{:\varphi^2:,D}(\varrho,\eta) = (\sqrt{\Delta})^{1-\frac{1}{2}D} Q_{\frac{1}{2}D-1}\left(\frac{\varrho+\eta-1}{\sqrt{\Delta}}\right), \tag{2.9}$$

where $\Delta = \varrho^2+\eta^2+1-2(\varrho+\eta+\varrho\eta)$. The derivation of eq. (2.9) is given in the Appendix. Formula (2.9) can be checked for $D=4$ reproducing a known result [10].

3. – Two-dimensional space-time.

The 2-dimensional field theories are known to have some peculiarities as far as conformal properties are concerned [11]. These peculiarities are in general of geometrical origin, and this has also a counterpart in the singularity structure of the four-point function.

The conformal partial waves for the four-point function itself have a remarkably simpler form for $D=2$. In fact if we use the light-cone variables

$$\begin{cases} x_+ = \frac{1}{2}(x^{(0)}+x^{(1)}), \\ x_- = \frac{1}{2}(x^{(0)}-x^{(1)}), \end{cases} \tag{3.1}$$

and the associated harmonic ratios

$$\xi = \frac{(x_- - y_-)(z_- - t_-)}{(x_- - t_-)(z_- - y_-)}, \qquad \omega = \frac{(x_+ - y_+)(z_+ - t_+)}{(x_+ - t_+)(z_+ - y_+)} \tag{3.2}$$

such that

$$\varrho = (\omega\xi)^{-1}, \qquad \eta = \frac{\xi(\omega-1)}{\omega(\xi-1)}, \tag{3.3}$$

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(10) S. Ferrara, R. Gatto, A. F. Grillo and G. Parisi: in *Scale and Conformal Symmetry in Hadron Physics*, edited by R. Gatto (New York, N. Y., 1973).

(11) L. Castell: *Comm. Math. Phys.*, **17**, 127 (1970); M. Hartagsu, R. Seiler and B. Schroer: *Phys. Rev. D*, **5**, 2519 (1972); S. Ferrara, R. Gatto and A. F. Grillo: *Nuovo Cimento*, **12** A, 959 (1972); J. Kupah, W. Rühl and C. C. Yunn: Kaiserlautern preprint (1974).

the form of the formula (2.1) simplifies for $D=2$ into (see the Appendix)

$$(3.4)\qquad f(\omega,\xi)=[(1-\omega)(1-\xi)]^{\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}(\xi\omega)^{\frac{1}{2}(l-\Delta_{AB})}\cdot$$
$$\cdot {}_2F_1(\tfrac{1}{2}(l+\Delta_{AB}),\tfrac{1}{2}(l+\Delta_{CD});l;\xi)\,{}_2F_1(\tfrac{1}{2}(l+\Delta_{AB}),\tfrac{1}{2}(l+\Delta_{CD});l;\omega)\,,$$

and for equal external dimensions

$$(3.5)\qquad f(\omega,\xi)=(\xi\omega)^{\frac{1}{2}l}\,{}_2F_1\left(\frac{l}{2},\frac{l}{2};\,l;\,\xi\right){}_2F_1\left(\frac{l}{2},\frac{l}{2};\,l;\,\omega\right).$$

It is now evident that, according to the general rule stated in the previous Section, these functions have singularities at $\omega=1$, $\xi=1$. However these are not Euclidean singularities; in fact, since $1-\omega$ and $1-\xi$ are again harmonic ratios, the points $\xi=1$, $\omega=1$ correspond to the crossed light-cones. It is important to note that the disappearing of the Euclidean singularities is a direct consequence of the factorization of the 4-point function which is due to the factorization of the conformal group for $D=2$: $O_{2,2}=O_{2,1}\otimes O_{2,1}$.

The previous observation can be better understood from the operator expansion which for $D=2$ assumes the factorized form [2]

$$(3.6)\qquad A(x_+,x_-)B(0,0)=\int_0^1\!\!\int_0^1 \mathrm{d}\tau\,\mathrm{d}\lambda\, f(\tau)f(\lambda)\,O(\lambda x_+,\tau x_-)\,.$$

By commuting with a third operator $C(y_+,y_-)$ we have

$$(3.7)\qquad [A(x_+x_-)B(0,0),\,C(y_+y_-)]=0\,,$$
$$\text{if}\ (x_+-y_+)(x_--y_-)<0\,,\qquad y_+y_-<0\,,$$

while on the right-hand side

$$[O(\lambda x_+,\tau x_-),\,C(y_+y_-)]=0\,,\quad \text{if}\ (\lambda x_+-y_+)(\tau x_--y_-)<0\,,$$

and it is in fact true that these conditions are equivalent because of the integration limits.

Causality is therefore fulfilled for $D=2$ for each conformal tower, a fact which reminds one of the light-cone limit for $D>2$, and is not verified in the general expansion (2.1).

From this argument it also follows that in two-dimensional conformal invariant theories the partial waves associated to any n-point Green's function should satisfy the correct analyticity properties.

4. – Concluding remarks.

In the present paper we have further investigated the properties of partial-wave amplitudes in conformal invariant field theories, as defined through the Wilson operator expansion inserted into the Wightman functions. We have shown that such partial-wave amplitudes do not have in general correct causality properties in Minkowski space so that dynamics must provide a conspiracy mechanism for the elimination of such singularities. Such a conspiracy problem can be regarded as a bootstrap condition for the theory (at least in the case of Lagrangian field theories) perhaps related to the bootstrap equations or to the renormalization group equations [12].

The peculiar role of two space-time dimensions has been particularly stressed, especially in connection to the absence of Euclidean singularities for that case.

Finally some improved formulae for the conformal partial waves of $O_{D,2}$ have been reported.

Appendix

In the first part of this Appendix we derive a modification of the conformal partial wave (2.1) more useful for our purposes.

According to ref. [2] (eq. (3.2)) eq. (2.3) can be rewritten directly as a double hypergeometric function

$$\text{(A.1)}\qquad f_0(\varrho,\eta)=\Gamma(l)\eta^{\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}\varrho^{-\frac{1}{2}(l+\Delta_{CD})}\left[\frac{\Gamma(-\frac{1}{2}(\Delta_{AB}+\Delta_{CD}))}{\Gamma(\frac{1}{2}(l+\Delta_{AB}))\Gamma(\frac{1}{2}(l-\Delta_{AB}))}\cdot\right.$$
$$\cdot F_4\left(\frac{1}{2}(l+\Delta_{CD}),\,\frac{1}{2}(l+\Delta_{AB});\; l+1-\frac{D}{2};\;\frac{1}{2}(\Delta_{AB}+\Delta_{CD})+1;\;\frac{1}{\varrho};\;\frac{\eta}{\varrho}\right)+$$
$$\left.+\left(\frac{\eta}{\varrho}\right)^{\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}(\Delta_{AB}\leftrightarrow-\Delta_{AB},\,\Delta_{CD}\leftrightarrow-\Delta_{CD})\right].$$

This expression is not suitable for the (important) case of equal dimensions and moreover, when restricted to $D=2$, does not give directly the factorized form (3.4). To obtain the improved form of (A.1) we use the following

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([12]) A. A. Migdal: *Phys. Lett.*, **37** B, 98 (1971); M. D'Eramo, L. Peliti and G. Parisi: *Lett. Nuovo Cimento*, **2**, 878 (1971); C. Mack and K. Symanzik: *Comm. Math. Phys.*, **27**, 247 (1972); G. Mack and I. T. Todorov: *Phys. Rev. D*, **8**, 1764 (1973).

identity [13]:

$$(\text{A.2})\qquad F_4(\alpha,\beta,\gamma,\gamma';x,y)=$$

$$=\frac{\Gamma(\gamma')\Gamma(\beta-\alpha)}{\Gamma(\gamma'-\alpha)\Gamma(\beta)}(-y)^{-\alpha}F_4\left(\alpha,\alpha+1-\gamma',\gamma,\alpha+1-\beta;\frac{x}{y},\frac{1}{y}\right)+$$

$$+\frac{\Gamma(\gamma')\Gamma(\alpha-\beta)}{\Gamma(\gamma'-\beta)\Gamma(\alpha)}F_4\left(\beta+1-\gamma',\beta,\gamma;\beta+1-\alpha;\frac{x}{y},\frac{1}{y}\right).$$

This identity allows us to write (A.1) in the form

$$(\text{A.3})\qquad f_0(\varrho,\eta)=(-1)^{\frac{1}{2}(l+\Delta_{CD})}\eta^{-\frac{1}{2}(l-\Delta_{AB})}\frac{\Gamma(l)\Gamma(1-\frac{1}{2}(l+\Delta_{AB}))}{\Gamma(1+\frac{1}{2}(\Delta_{CD}-\Delta_{AB}))\Gamma(\frac{1}{2}(l-\Delta_{CD}))}\cdot$$

$$\cdot\left[F_4\left(\frac{1}{2}(l+\Delta_{CD}),\frac{1}{2}(l-\Delta_{AB}),l+1-\frac{1}{2}D,1+\frac{1}{2}(\Delta_{CD}-\Delta_{AB});\frac{1}{\eta},\frac{\varrho}{\eta}\right)+\right.$$

$$+\left(-\frac{\varrho}{\eta}\right)^{-\frac{1}{2}(l-\Delta_{AB})}\frac{\Gamma(\frac{1}{2}(\Delta_{AB}+\Delta_{CD}))\Gamma(1+\frac{1}{2}(\Delta_{CD}-\Delta_{AB}))}{\Gamma(\frac{1}{2}(l+\Delta_{CD}))\Gamma(1+\frac{1}{2}(\Delta_{CD}-l))}\cdot$$

$$\cdot(-1)^{-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}\frac{\sin\pi(1-\frac{1}{2}(l+\Delta_{AB}))}{\sin\pi(1+\frac{1}{2}(\Delta_{CD}-l))}\cdot$$

$$\left.\cdot F_4\left(\frac{1}{2}(l-\Delta_{CD}),\frac{1}{2}(l-\Delta_{AB}),l+1-\frac{1}{2}D,1-\frac{1}{2}(\Delta_{AB}+\Delta_{CD});\frac{1}{\varrho},\frac{\eta}{\varrho}\right)\right].$$

When $\Delta_{AB}=\Delta_{CD}=0$ (A.3) simplifies into

$$(\text{A.4})\qquad f_0(\varrho,\eta)=(-1)^{\frac{1}{2}l}\eta^{-\frac{1}{2}l}\frac{\Gamma(l)\Gamma(1-\frac{1}{2}l)}{\Gamma(\frac{1}{2}l)}\cdot$$

$$\cdot\left[F_4\left(\frac{1}{2}l,\frac{1}{2}l,l+1-\frac{1}{2}D,l;\frac{1}{\eta};\frac{\varrho}{\eta}\right)+\right.$$

$$\left.+(-1)^{-\frac{1}{2}l}\left(-\frac{\varrho}{\eta}\right)^{-\frac{1}{2}l}F_4\left(\frac{1}{2}l,\frac{1}{2}l,l+1-\frac{1}{2}D,l;\frac{1}{\varrho},\frac{\eta}{\varrho}\right)\right].$$

For $D=2$, using the following identity [14]:

$$(\text{A.5})\qquad F_4(\alpha,\gamma+\gamma'-\alpha-1,\gamma,\gamma';x(1-y),y(1-x))=$$

$$={}_2F_1(\alpha,\gamma+\gamma'-\alpha-1,\gamma;x)\,{}_2F_1(\alpha,\gamma+\gamma'-\alpha-1;\gamma';\zeta)$$

([13]) *Bateman Manuscript Project: Higher Trascendental Functions*, Vol. **1**, 5.11(9).
([14]) *Bateman Manuscript Project: Higher Trascendental Functions*, Vol. **1**, 5.10(5).

and the definitions of Sect. **3** one gets

$$(A.6)\qquad f_0(\varrho,\eta)=\Gamma(l)\eta^{\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}\varrho^{-\frac{1}{2}(l+\Delta_{CD})}\cdot$$
$$\cdot\left[\frac{\Gamma(\frac{1}{2}(\Delta_{AB}+\Delta_{CD}))}{\Gamma(\frac{1}{2}(l-\Delta_{AB}))\Gamma(\frac{1}{2}(l-\Delta_{CD}))}\,{}_2F_1\left(\frac{1}{2}(l+\Delta_{AB}),\frac{1}{2}(l+\Delta_{CD});l,\xi\right)\cdot\right.$$
$$\cdot{}_2F_1\left(\frac{1}{2}(l+\Delta_{AB}),\frac{1}{2}(l+\Delta_{CD});1+\frac{1}{2}(\Delta_{AB}+\Delta_{CD});1-\omega\right)+$$
$$+[(1-\omega)(1-\xi)]^{-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}\frac{\Gamma(\frac{1}{2}(\Delta_{AB}+\Delta_{CD}))}{\Gamma(\frac{1}{2}(l+\Delta_{CD}))\Gamma(\frac{1}{2}(l+\Delta_{AB}))}\cdot$$
$$\cdot{}_2F_1\left(\frac{1}{2}(l-\Delta_{AB}),\frac{1}{2}(l-\Delta_{CD});l;\xi\right)\cdot$$
$$\left.\cdot{}_2F_1\left(\frac{1}{2}(l-\Delta_{AB}),\frac{1}{2}(l-\Delta_{CD});1-\frac{1}{2}(\Delta_{AB}+\Delta_{CD});1-\omega\right)\right],$$

which, by use of a relation between hypergeometric functions ([15]) leads to eqs. (3.4) and (3.5).

In this second part of the Appendix we compute the partial wave associated to the scalar operator in free-field theory for any D.

Consider for instance formula (2.3) for $\Delta_{AB}=\Delta_{CD}=0$, $l=D-2$:

$$(A.7)\qquad f_{:\varphi^2:}(\varrho,\eta)=\frac{\Gamma(D-2)}{\Gamma(\frac{1}{2}D-1)^2}\int_0 d\sigma\frac{1}{\sigma(1-\sigma)}\left(\frac{\varrho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{1-\frac{1}{2}D}\cdot$$
$$\cdot{}_2F_1\left(\frac{1}{2}D-1,\frac{1}{2}D-1;\frac{1}{2}D-1;\left(\frac{\varrho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{-1}\right)=$$
$$=\frac{\Gamma(D-2)}{\Gamma(\frac{1}{2}D-1)^2}\int_0^1 d\sigma[\sigma(1-\sigma)]^{\frac{1}{2}D-2}[\varrho(1-\sigma)+\eta\sigma-\sigma(1-\sigma)]^{1-\frac{1}{2}D},$$

which is nothing but the integral representation for a double hypergeometric function ([16])

$$(A.8)\qquad (\sigma_1\sigma_2)^{1-\frac{1}{2}D}F_1\left(\frac{1}{2}D-1,\frac{1}{2}D-1,\frac{1}{2}D-1,D-2;\frac{1}{\sigma_1},\frac{1}{\sigma_2}\right),$$

where σ_1, σ_2 are the roots of the binomial $\sigma^2+\sigma(\eta-\varrho-1)+\varrho$. By means of a further reduction formula ([17]), (A.8) can be rewritten as

$$(A.9)\qquad (\sigma_1\sigma_2)^{1-\frac{1}{2}D}\left(1-\frac{1}{\sigma_2}\right)^{1-\frac{1}{2}D}{}_2F_1\left(\frac{1}{2}D-1,\frac{1}{2}D-1;D-2;\frac{\sigma_2-\sigma_1}{\sigma_1(\sigma_2-1)}\right).$$

([15]) *Bateman Manuscript Project*: *Higher Trascendental Functions*, Vol. **1**, 2.10(1).
([16]) *Bateman Manuscript Project*: *Higher Trascendental Functions*, Vol. **1**, 5.8.2(5).
([17]) *Bateman Manuscript Project*: *Higher Trascendental Functions*, Vol. **1**, 5.10(1).

(A.9) can be simplified in terms of Legendre functions using some known identities [18]

$$f_{\varphi^2:D}(\varrho, \eta) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}(D-1))}{\Gamma(\frac{1}{2}D-1)} (\sqrt{\Delta})^{1-\frac{1}{2}D} Q_{\frac{1}{2}D-2}\left(\frac{\varrho+\eta-1}{\sqrt{\Delta}}\right), \tag{A.10}$$

where

$$\Delta = \varrho^2 + \eta^2 + 1 - 2(\varrho + \eta + \varrho\eta).$$

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[18] *Bateman Manuscript Project*: *Higher Trascendental Functions*, Vol. **1**, 3.2(5), 3.2(10).

● RIASSUNTO

Si studiano le proprietà di analiticità delle ampiezze di onda parziale del gruppo conforme $O_{D,2}$ (con D anche non intero). Si discute la presenza di singolarità euclidee sull'espansione di Wilson in teorie invarianti rispetto al gruppo conforme, in special modo in relazione alla possibile formulazione di condizioni di bootstrap dinamico che seguono dalla causalità. Si esamina in dettaglio il caso eccezionale $D = 2$.

Свойства парциальных амплитуд в конформно инвариантных теориях поля.

Резюме (*). — Исследуются свойства аналитичности парциальных амплитуд конформной группы $O_{D,2}$ (причем, D не является обязательно целым числом) в конфигурационном пространстве. Обсуждается наличие эвклидовых сингулярностей в разложении Вильсона в конформно инвариантных теориях поля. Особое внимание уделяется формулировке условий динамического бутстрапа, возникающих из требования причинности. Подробно исследуется слулай $D=2$.

(*) *Переведено редакцией.*