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A. Turrin: EFFECT OF CROSSING A DEPOLARIZING RESONANCE IN CYCLIC ACCELERATORS. -

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ABSTRACT -

The Froissart and Stora method is applied to investigate the spin motion of spin $\frac{1}{2}$ particles in crossing a single resonance with arbitrary initial conditions of vertical polarization. The results of rigorous calculations are expressed by a simple formula already found by some Authors by approximate methods.

In their 1960 paper[1] Froissart and Stora discuss the dynamics of spin $\frac{1}{2}$ particles polarization in crossing depolarization resonances during acceleration in circular accelerators.

These Authors calculate the asymptotic values of the occupation numbers $g(t)$ and $f(t)$ for the states $+\frac{1}{2}$ and $-\frac{1}{2}$ of spin $\frac{1}{2}$ systems under the influence of a perturbing resonant magnetic field constant in amplitude and frequency ω_R when the precession angular frequency $\omega_p(t)$ sweeps linearly through the resonance line. Starting with the special initial conditions at $t = -\infty$

$$g(-\infty) = 0 \, , \qquad \left| f(-\infty) \right| = 1 \, ,$$

i.e. considering an initially fully polarized beam, they give for the transition probability at $t = +\infty$ the expression

$$\left| g(+\infty) \right|^2 = 1 - e^{-\frac{\pi \omega^2}{2\Gamma}} \quad . \tag{1}$$

In Eq. (1) ω is the strength of the perturbation and Γ is the time rate of change of the difference $\omega_p(t) - \omega_R = \dot{\chi}$, i.e. $\frac{d}{dt}\left[\omega_p(t) - \omega_R\right] = \ddot{\chi} = \Gamma$. Γ is assumed constant in the neighbourhood of the resonance line, centered at $t = 0$ (if Γ is small the resonance is crossed slowly and the probability of spin flip is increased).

By the above-assumed initial conditions, the corresponding initial and asymptotic expressions of the vertical component of the polarization vector

$$S_z = 1 - 2 \left| g \right|^2 \tag{2}$$

become $\qquad S_z(-\infty) = 1$

and

$$S_z(+\infty) = 2\, e^{-\frac{\pi \omega^2}{2\Gamma}} - 1\,. \tag{3}$$

In the following, the Froissart and Stora procedure will be applied to demonstrate that, assuming $-1 \leq S_z(-\infty) \leq 1$, the expression of $S_z(+\infty)$ becomes, after a single resonance is passed through,

$$S_z(+\infty) = S_z(-\infty)\left(2\, e^{-\frac{\pi \omega^2}{2\Gamma}} - 1\right), \tag{4}$$

as one may expect „a priori" .

This very simple formula is routinely used by many Authors, even if his validity has not been demonstrated directly up to now. Approximate analytic calculations [(2),(4)] have been made about, leading all to result expressed by Eq. (4). However, for the sake of completeness of the Froissart and Stora treatment, one would like to have an exact derivation of Eq. (4), as will be done in the present note.

In a reference frame which is attached to the particle and which rotates about the main field direction with an angular velocity $\dot{\chi} = \omega_p(t) - \omega_R$, the Schroedinger time-dependent equation leads to the pair of equations

$$\dot{f} = -i\,\frac{\omega}{2}\; g\, e^{-i\chi} \tag{5a}$$

$$\dot{g} = -i\,\frac{\omega}{2}\; f\, e^{i\chi}\quad . \tag{5b}$$

These coupled differential equations can readily be separated by differentiation. Thus,

$$\ddot{g} - i\dot{\chi}\,\dot{g} + \left(\frac{\omega}{2}\right)^2 g = 0 \tag{6a}$$

$$\ddot{f} + i\dot{\chi}\,\dot{f} + \left(\frac{\omega}{2}\right)^2 f = 0 \quad . \tag{6b}$$

Assuming now (see Ref. [1]) a constant rate of change of the difference $\dot{\chi} = \omega_p(t) - \omega_R$, i.e. assuming $\dot{\chi} = \Gamma t$, Eq.s (6a) and (6b) become

$$\ddot{g} - i\Gamma t\,\dot{g} + \left(\frac{\omega}{2}\right)^2 g = 0 \tag{7a}$$

$$\ddot{f} + i\Gamma t\,\dot{f} + \left(\frac{\omega}{2}\right)^2 f = 0 \quad . \tag{7b}$$

Making the change of independent variable $z = \frac{i}{2}\Gamma t^2$ in Eq. (7a) and $u = -\frac{i}{2}\Gamma t^2$ in (7b) we obtain

$$z\frac{d^2g}{dz^2} + \left(\frac{1}{2} - z\right)\frac{dg}{dz} - ag = 0 \tag{8a}$$

$$u\frac{d^2f}{du^2} + \left(\frac{1}{2} - u\right)\frac{df}{du} + af = 0 \quad , \tag{8b}$$

where the dimensionless complex parameter $a = \frac{i\omega^2}{8\Gamma}$ has been introduced.

Both Eqs. (8a) and (8b) have the standard form of the confluent hypergeometric equation[5]. Solutions for such equations are well known[5]; they can be written in the form

$$g = g(0)\,F\left(a\left|\frac{1}{2}\right|z\right) + \delta z^{\frac{1}{2}}\,F\left(\frac{1}{2}+a\left|\frac{3}{2}\right|z\right) \tag{9a}$$

$$f = \gamma z^{\frac{1}{2}} F\left(\frac{1}{2} - a \left| \frac{3}{2} \right| -z\right) + f(0)\, F\left(-a \left| \frac{1}{2} \right| -z\right) \quad , \tag{9b}$$

where g(0), f(0), γ and δ are integration constants and where the F's are confluent hypergeometric functions[(5)]. *

Because of the coupled nature of the original equations (5a) and (5b) only two of the four coefficients g(0), f(0), γ and δ can be chosen arbitrarily.

In terms of the new independent variable z , Eq. s(5a) and (5b) transform into the equations

$$z^{\frac{1}{2}} \frac{df}{dz} = a^{\frac{1}{2}} g\, e^{-z} \tag{5c}$$

$$z^{\frac{1}{2}} \frac{dg}{dz} = a^{\frac{1}{2}} f\, e^{z} \quad . \tag{5d}$$

Putting solutions (9a) and (9b) into Eqs. (5c) and (5d) one obtains (e. g. at z = 0)

$$\gamma = 2\, a^{\frac{1}{2}}\, g(0)$$

$$\delta = 2 a^{\frac{1}{2}}\, f(0) \ ,$$

so that we can write Eq. s(9a) and (9b) in the final form

$$g = g(0)\, F\left(a \left| \frac{1}{2} \right| z\right) + f(0)\, 2a^{\frac{1}{2}} z^{\frac{1}{2}} F\left(\frac{1}{2} + a \left| \frac{3}{2} \right| z\right) \tag{9c}$$

$$f = g(0)\, 2a^{\frac{1}{2}} z^{\frac{1}{2}} F\left(\frac{1}{2} - a \left| \frac{3}{2} \right| -z\right) + f(0)\, F\left(-a \left| \frac{1}{2} \right| -z\right) \quad . \tag{9d}$$

* -- The notation $F(a|b|z)$ is used for $M(a, b, z)$ of Ref. 5 .

The asymptotic behaviour of $g_{t\to+\infty}$ and $f_{t\to+\infty}$ is found by inserting in Eqs.(9c) and (9d) the asymptotic expansions of the F functions[5]. The connection between $(g_{t\to+\infty}, f_{t\to+\infty})$ and $(g(0), f(0))$ may then be expressed by the linear transformation

$$g_{t\to+\infty} = m_{11}\, g(0) + m_{12}\, f(0) \qquad (10a)$$

$$f_{t\to+\infty} = m_{21}\, g(0) + m_{22}\, f(0) \quad , \qquad (10b)$$

where

$$m_{11} = \pi^{\frac{1}{2}} \frac{|z|^{-a}\, e^{ia\frac{\pi}{2}}}{\Gamma(\frac{1}{2}-a)} \qquad m_{12} = i\pi^{\frac{1}{2}} a^{\frac{1}{2}} \frac{|z|^{-a}\, e^{ia\frac{\pi}{2}}}{\Gamma(1-a)}$$

$$m_{21} = \pi^{\frac{1}{2}} a^{\frac{1}{2}} \frac{|z|^{a}\, e^{ia\frac{\pi}{2}}}{\Gamma(1+a)} \qquad m_{22} = \pi^{\frac{1}{2}} \frac{|z|^{a}\, e^{ia\frac{\pi}{2}}}{\Gamma(\frac{1}{2}+a)} \qquad (11)$$

and the Γ's are gamma functions.

On the other hand it can be seen from Eqs.(9c) and (9d) that the functions $z^{\frac{1}{2}}$ F change sign when the variable t goes from negative to positive values. Therefore, the asymptotic behaviour of $g_{t\to-\infty}$ and $f_{t\to-\infty}$ is

$$g_{t\to-\infty} = m_{11} g(0) - m_{12}\, f(0) \qquad (12a)$$

$$f_{t\to-\infty} = -m_{21} g(0) + m_{22} f(0) \quad . \qquad (12b)$$

Solving Eq. s(12a) and (12b) for g(0) and f(0) and substituting in Eq. s (10a) and (10b) one gets the relationship between g and f at $t \to \pm \infty$:

$$g_{t\to+\infty} = 2m_{11}m_{12}\, f_{t\to-\infty} + e^{-2|a|\pi} g_{t\to-\infty} \quad . \qquad (13)$$

Note that the determinant of the matrix m_{ij} is identically unity (use of the functional relations $\Gamma(1+a)\,\Gamma(1-a) = |a|\pi/\sinh(|a|\pi)$ and $\Gamma(\frac{1}{2}+a)\,\Gamma(\frac{1}{2}-a) = \pi/\cosh(|a|\pi)$ has been made).

We now put $g_{t\to-\infty} = |g_{t\to-\infty}|e^{i\psi}$, $f_{t\to-\infty} = \left(1-|g_{t\to-\infty}|^2\right)^{\frac{1}{2}} ie^{i\psi}$, develop the squared modulus $|g_{t\to+\infty}|^2$ and use Eq. (2) .

We get for $S_{z_{t\to+\infty}}$

$$S_{z_{t\to+\infty}} = S_{z_{t\to-\infty}}(2\, e^{-4|a|\pi} - 1) + 2(1 - e^{-4|a|\pi})^{\frac{1}{2}} e^{-2|a|\pi}(1 - S^2_{z_{t\to-\infty}})^{\frac{1}{2}} \cos(\varphi) \qquad (14)$$

where

$$\varphi = \frac{\pi}{4} - 2|a|\ln|z| - \arg\Gamma(\tfrac{1}{2} - i|a|\pi) - \arg\Gamma(1-i|a|\pi) \quad ,$$

i.e., after the resonance is crossed, the time average value of the vertical component of the polarization vector is expressed by Eq. (4).

The final expression (14) of $S_z(+\infty)$ is seen to be in agreement with formula (3.12) of Ref. [3] which gives the connection between $S_{z(t<0)}$ and $S_{z(t>0)}$ across the region of t where the resonance is effective.

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