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ABSTRACT. -

The asymptotic behaviour of the form factor for non-relativistic bound systems is derived for local and nonlocal two - and three-body potentials, taking into account the Pauli principle. The effect of the Pauli principle is very important, giving rise to the extra factor $q^{-2K_{min}}$, where K_{min} is the minimum degree in the homogeneous polynomials expansion, which grows like $A^{4/3}$ for large particle number A.

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Recently there has been an upsurge of interest in the study of the asymptotic behaviour of the form factor of bound systems[1, 2, 3]. This interest has mostly been connected to hadronic models, and for this reason no attention has been given in theoretical works to the Pauli principle. The asymptotic behaviour of the form factor, however, due to the large energy of the projectiles now available, is interesting even for systems which are (believed to be) essentially nonrelativistic in their rest frame, like nuclei, in which case the Pauli principle is of course most important. This paper is mainly devoted to the study of the effect of the Pauli principle in the asymptotic behaviour of the form factor of nonrelativistic bound systems. Relativistic corrections and anelastic effects connected to excitation of hadronic degrees of freedom can also be very important, but they will be completely neglected to the extent that they cannot be represented by local or nonlocal, two - or many-body non relativistic interactions.

The expression of the form factor is

$$F(q) = \sum_{\alpha_1 \dots \alpha_A} \int d\underline{p}_1 \dots \int d\underline{p}_A \; \delta(\underline{p}_1 + \dots \underline{p}_A)\, \varphi^{*}(\underline{p}_1 \alpha_1, \dots \underline{p}_A \alpha_A)$$

$$\varphi(\underline{p}_1 - \frac{1}{A}\underline{q}, \alpha_1, \dots \underline{p}_{A-1} - \frac{1}{A}\underline{q}, \alpha_{A-1}, \underline{p}_A + \frac{A-1}{A}\underline{q}, \alpha_A), \tag{1}$$

where φ is the nuclear wave function in the momentum representation and α_i the spin-isospin variables of the i-th nucleon. φ obeys of course the generalized Pauli principle.

We shall first consider two-body local potentials, confining ourselves to potentials less singular than r^{-2} at the origin[4] (the Fourier transform decreases faster than q^{-1} for large q). Iterating (A-1) times the Schrödinger equation we obtain the following equation

$$\varphi(\underline{p}_1 \alpha_1, \dots \underline{p}_A \alpha_A) = -G_o(\underline{p}_1, \dots \underline{p}_A) \sum_{\alpha'_1 \dots \alpha'_A} \int d\underline{p}'_1 \dots \int d\underline{p}'_A \; \delta(\underline{p}'_1 + \dots \underline{p}'_A) \cdot$$

$$\cdot K(\underline{p}_1 \alpha_1, \dots \underline{p}_A \alpha_A / \underline{p}'_1 \alpha'_1, \dots \underline{p}'_A \alpha'_A)\, \varphi(\underline{p}'_1 \alpha'_1, \dots \underline{p}'_A \alpha'_A), \tag{2}$$

where G_o is the free Green function for the system.

A detailed decomposition of φ is needed for the kernel of eq. (2) to be connected[5, 6]. This is not necessary here, however, because as we shall show the disconnected part of K does not contribute to the asymptotics of the form factor. So we decompose K into its connected and disconnected parts, $K = K_C + K_D$, each term being of course completely antisymmetric in the primed and nonprimed variables. K_c consists of (the sum of) products of (A-1) interactions times (A-1) free Green functions G_o. All inner integrations are taken away from momentum conservation. Examples of terms of the sum are given in Fig. 1 for A = 4.

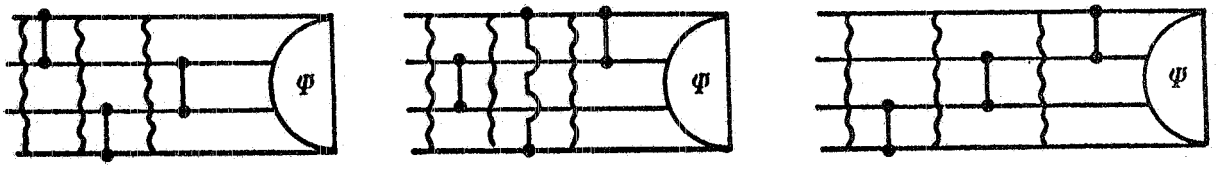

FIG. 1 - Graphs contributing to K_c for A=4. The wavy lines show where the free Green functions should be inserted, the lines with dots at the end points represent the interactions.

It is easy to check that on each wavy line at least one momentum is growing like the momentum transfer q, so that from each Green function we get a factor $1/q^2$, and that in each interaction the argument is growing like q, so that

$$K_c = \mathcal{A}\mathcal{A}' \left\{ \prod_{i=1}^{A-1} \frac{1}{q^2} V(b_i \underline{q} + \underline{Q}_i) \right\}, \tag{3}$$

where V is the Fourier transform of the potential which is an operator in spin-isospin space, the b_i's are numbers, the $\underline{Q}_i$'s are linear combinations of the momenta p_i, p_i', but do not involve the momentum transfer q, and $\mathcal{A}$, $\mathcal{A}'$ are the antisymmetrization operators acting on the nonprimed, primed variables respectively. The effect of the antisymmetrization is to make K_c to behave like a sum of harmonic polynomials of degree greater or equal to K_{min}[7, 8], both in the nonprimed and primed variables. In order to obtain

the polynomial of degree K we must differentiate $\prod_{i=1}^{A-1} \frac{1}{q^2} V(b_i \underline{q} + \underline{Q}_i)$, K times with respect to the $\underline{p}_i$'s and K times with respect to the $\underline{p}'_i$'s. After this, if ψ goes to zero faster than V, we can neglect the $\underline{Q}_i$ in the argument of V and obtain the leading term for $K = K_{min}$

$$(4) \qquad \psi(\underline{p}_1 - \frac{1}{A}\underline{q}, \alpha_1, \ldots \underline{p}_A + \frac{A-1}{A}\underline{q}, \alpha_A) \sim C \frac{d^{2K_{min}}}{dq^{2K_{min}}} \left[\frac{V(q)}{q^2}\right]^{A-1}$$

A sufficient condition for ψ to go to zero faster than V is for V to have a power fall off, in which case

$$(5) \qquad F(q) \sim C\left[\frac{V(q)}{q^2}\right]^{A-1} \frac{1}{q^{2K_{min}}} .$$

Confining to potentials less singular than r^{-2}, we avoid the possibility that C=0. Strictly speaking, however, additional cancellations could occur, so that eq. (5), and the following eqs. (7) and (10) should be reguarded as upper limits. The result of eq. (5) agrees with the result of ref. (3) for the non relativistic three-body case ($K_{min} = 0$), but is in disagreement with the general result of ref. (2) where the Pauli principle is not taken into proper account. The result of ref. (2) is $F(q) \sim [V(q)/q^2]^{A-1} q^{-(l_A + l_B + \delta_A + \delta_B)}$ where $\delta_{A,B}$ is equal to 0 or 1 for normal or unormal parity resp..

We come now to consider K_D. All the terms of K_D contain at least one δ-function of the type $\delta(\underline{p}_i - \underline{p}'_i)$, and a number of inner integrations equal to the number of δ-functions. The graph of Fig. 2, for instance, gives the contribution

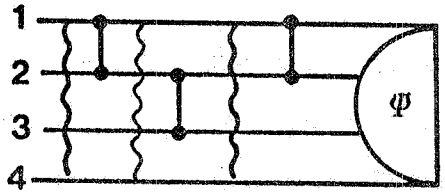

FIG. 2 - Graph contributing to K_D for A=4. The caption is as in FIG. 1.

$$-G_o(\underline{p}_1\underline{p}_2 \ldots \underline{p}_4) \int d\underline{p}'_1 \ldots \int d\underline{p}'_4 \; \delta(\underline{p}'_1 + \ldots \underline{p}'_4) \int d\underline{p} \, V\left[\underline{p}_1 - \underline{p}_2 - (2\underline{p} + \underline{p}_3 + \underline{p}_4)\right] \cdot$$

$$\cdot G_o\left[\underline{p}, -(\underline{p} + \underline{p}_3 + \underline{p}_4), \underline{p}_3, \underline{p}_4\right] V\left[-(\underline{p} + \underline{p}_3 + \underline{p}_4) - \underline{p}_3 - (-\underline{p} - \underline{p}'_3 - \underline{p}_4 - \underline{p}'_3)\right] \cdot$$

$$\cdot G_o\left[\underline{p}, -(\underline{p} + \underline{p}'_3 + \underline{p}_4), \underline{p}'_3, \underline{p}_4\right] V\left[\underline{p} + (\underline{p} + \underline{p}'_3 + \underline{p}_4) - (\underline{p}'_1 - \underline{p}'_2)\right] \delta(\underline{p}_4 - \underline{p}'_4)$$

A term of K_D having n δ-functions behaves asymptotically like the product of (A-1) interactions times (A-1) free Green functions, but contains n inner integrations which take away the momentum transfer from n interactions, while n momenta in $\Psi(\underline{p}'_1, \alpha'_1 \ldots, \underline{p}'_A \alpha'_A)$ are external momenta (containing the momentum transfer). Since Ψ tends to zero faster than the product of n interactions when n momenta go to infinity, K_D does not contribute to the asymptotics.

The case of two-body non local potentials can be treated very much along the same line. If g(q) is the leading form factor of the two-body non-local potential[9], eq. (3) must be replaced by

$$K_C = \mathcal{A}\mathcal{A}' \left\{ \prod_{i=1}^{A-1} \frac{1}{q^2}\, g(b_i \underline{q} + \underline{Q}_i)\; g\,(b'_i \underline{q} + \underline{Q}'_i) \right\}, \tag{6}$$

where the b'_i's and $\underline{Q}'_i$'s are the analog of the b_i's and $\underline{Q}_i$'s. It can be checked that one of the b'_i's is always zero and one of the b_i's is zero for A>2, so that provided Ψ tends to zero faster than g for large q

$$F(q) \sim C\, \frac{d^{2K_{min}}}{dq^{2K_{min}}}\, \frac{1}{q^2} \left[\frac{g^2(q)}{q^2} \right]^{A-2} \sim C \left[\frac{g^2(q)}{q^2} \right]^{A-2} \frac{1}{q^{2(K_{min}+1)}} \tag{7}$$

For A=2 there is an additional power of g(q). A sufficient condition for Ψ to go to zero faster than g is for g to have a power fall off.

In the case of three-body potentials the starting equation for Ψ should be iterated $\left[\frac{A}{2}\right]$ (= integer part of $\frac{A}{2}$) times. It is now necessary to specify the form of the potential in order to extract the asymptotics. As an example we take here

$$V(\underline{p}_1 \underline{p}_2 \underline{p}_3 / \underline{p}'_1 \underline{p}'_2 \underline{p}'_3) = W\left[(\underline{p}_1 - \underline{p}_2) - (\underline{p}'_1 - \underline{p}'_2)\right] W\left[(\underline{p}_2 - \underline{p}_3) - (\underline{p}'_2 - \underline{p}'_3) + \right]$$

$$+ \text{ cyclic permutations,}$$

which gives

$$K_c = \mathcal{A}\mathcal{A}' \left\{ \prod_{i=1}^{[A/2]} \frac{1}{q^2}\, W(c_i \underline{q} + \underline{P}_i)\, W(c'_i \underline{q} + \underline{P}'_i) \right\}, \tag{9}$$

where the c_i's, $\underline{P}_i$'s are the analog of the b_i's, $\underline{Q}_i$'s. It is easy to check that one of c's is zero for A > 4. Moreover for A even there is an inner integration which takes aways the momentum transfer from two W's, so that, again for power fall off of W

$$(10) \qquad F(q) \sim C \frac{d^{2K_{min}}}{dq^{2K_{min}}} \frac{[W(q)]}{q^{A-\delta}}^{A-3+\delta} \sim C\left[\frac{W(q)}{q}\right]^{A-3+\delta} \frac{1}{q^{2K_{min}+3-2\delta}}$$

where $\delta=0$ for A even, $\delta=1$ for A odd. For A=4 there is an additional power of W.

It should be noted that the three-body interaction is likely to dominate over the two-body interaction. This happens in the present example (for enough heavy nuclei) as soon as $W(q) > \frac{V(q)}{q}$, for large q. The generalization to other many-body potentials is obvious.

Summarizing, we have derived the asymptotic behaviour of the form factor of non relativistic bound systems with two-body local and non local and three-body interactions taking the Pauli principle into proper account. The Pauli principle has a very remarkable effect represented by the factor $q^{-2K_{min}}$. Consider that $K_{min}=0$ for He^4, $K_{min}=12$ for O^{16}, $K_{min}=24$ for Ca^{40}, and K_{min} grows like $A^{4/3}$ for large A[8]. The expression for K_{min} is given in ref. (7).

It would be very interesting to establish when the form factor starts being asymptotic, which requires the evaluation of the next term in the asymptotic expansion. We think that the condition for the harmonic polynomials of degree greater than K_{min} to be neglegeable, requires that q should be much greater than the maximum momentum contained in the potential, while extracting q from the Green functions and the harmonic polynomial of degree K_{min}, requires that q shoul be much greater than the Fermi momentum of the system.

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(4) - Potentials singular like r^{-2} at the origin have been considered in ref. (3), and potentials more singular than r^{-2} by Drell et al., ref. (1) for the two -body problem. Potentials singular like $r^{-2(1+\gamma)}$, $0<\gamma<1/2$ will be discussed in a separate paper for the many body problem.

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(9) - g(q) must decrease faster than $q^{-1/2}$ at large q, for the t-matrix to exist (see R. G. Newton - Scattering theory of waves and particles, Mc Graw-Hill Book Co. 1966, p. 275). Moreover, unless g(q) decreases faster than q^{-1} at large q, there may be an asymptotic behaviour different from eq. (7) (see ref. (3)). We shall not discuss this point and we shall assume that g(q) decreases faster than q^{-1}.