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M. Greco, G. Pancheri-Srivastava and Y. Srivastava:
RADIATIVE CORRECTIONS FOR COLLIDING BEAM
RESONANCES.

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ABSTRACT. -

Detailed expressions are presented for radiative corrections to colliding beam experiments in presence of resonances, including interference effects. The derivation of our formulae is accomplished through perturbation theory methods as well as the coherent state formalism. These are then applied to determine the resonance parameters for $\psi(3.1)$ and $\psi(3.7)$.

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1. - INTRODUCTION -

The problem of radiative corrections is well known and understood in quantum electrodynamics⁽¹⁺³⁾. The physical origin lies in the fact that any scattering process involving charged particles in the initial or / and final state cannot take place without the emission of soft photons. The infrared divergence is artificially introduced in the perturbative solution of electrodynamics because of the unphysical separation of real from virtual soft photons, together with the unrealistic counting of the emitted quanta. The correct analysis of the collective soft-photon effect leads to define an observable cross section $(d\sigma)_{\text{obs}}$, for an experiment with energy resolution $\Delta\omega$, which includes the infrared corrections in exponentiated form and is simply related to the lowest order cross section $d\sigma_0$ by⁽⁴⁾

$$(1.1) \quad (d\sigma)_{\text{obs}} \propto \left(\frac{\Delta\omega}{E}\right)^\beta d\sigma_0,$$

where E is the energy of the radiating particle and $\beta \propto (4\alpha/\pi)\ln(2E/m)$ will be better defined later. Eq. (1.1) is a well known result which holds for any process whose bare cross section $d\sigma_0$ is not a rapidly varying function of the energy or momentum transfer. It explicitly shows that the radiative correction depends upon an external parameter $\Delta\omega$, which characterizes a given experiment.

In a scattering process which proceeds via the formation of a resonance, it is not possible to directly apply eq. (1.1), the reason being simply that the resonance cross section is very sensitive to an energy loss in the initial state, while no such effect results from photon emission from the final state. A first analysis of this problem has been performed in ref. (5), where eq. (1.1) has been generalized to

$$(1.2) \quad (d\sigma)_{\text{obs}} \propto \int_0^{\Delta\omega} \frac{d\omega}{\omega} \left(\frac{\omega}{E}\right)^\beta \left\{ \beta_f d\sigma_R(2E) + \beta_i d\sigma_R(2E-\omega) \right\},$$

in the case of resonance formation in the s-channel ($\sqrt{s} = 2E$). β_f and β_i are the radiative parameters of the final and initial state respectively, $\beta \simeq \beta_f + \beta_i$ and $d\sigma_R(\sqrt{s})$ refers to the typical (Breit-Wigner) resonant cross section. Eq. (1.2), which automatically reduces to (1.1) in the case of a smooth cross section, shows already a new and interesting feature, although providing only a partial answer to the problem (see below). In the case of a narrow resonance in fact ($\Gamma \ll \Delta\omega$), the width provides a natural cut-off in damping the integral over the energy loss in the initial state. Unlike eq. (1.1), one finds therefore that the soft photon emission is governed by an intrinsic physical quantity instead of the external parameter $\Delta\omega$, as far as the emission from the initial state is concerned.

Interference effects have been neglected in eq. (1.2), i. e. the overlapping of radiation from the initial and final states. One would also like to have a full answer in the case of interference of a resonance term with a purely QED background. The recent discovery⁽⁶⁾ of very narrow resonances decaying both into leptons and hadrons, provides the natural physical framework for these problems, demanding as well a precise evaluation of the radiative correction factors in order to extract from the experimental data the physical parameters of interest.

The aim of this work is to provide a detailed study of all these radiative effects in e^+e^- scattering and give a precise answer for these questions of immediate experimental interest. A brief account of our main results has already been given⁽⁷⁾.

We consider the infrared factors to all orders in perturbation theory. This is achieved by using two different methods, both providing the same result. In the first case, as shown in section III, we u-

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se a perturbative approach to all orders in the classical currents responsible for soft photon emission. This method is applied to the products of two matrix elements, in order to obtain the radiative corrections factors to a purely resonant cross section as well as to the interference cross section between the resonance and a purely Q.E.D. background. In section IV we apply the coherent states formalism developed in ref. (8), directly to matrix elements which, unlike the usual perturbative expansion of Q.E.D., are finite and factorizable in the infrared factors. This directly follows from a more realistic definition of the final states, which contain an unlimited and undetermined number of soft photons.

The main issue in our results agrees with the qualitative arguments given after eq. (1.2) and can be simply stated by assigning to the initial state a "proper" energy loss $\Delta\omega_i$, with a phase, given by

$$(1.3) \quad \Delta\omega_i = \Gamma/2 \sin \delta_R e^{i\delta_R},$$

where the resonant amplitude is parametrized as $M_R \propto \sin \delta_R e^{i\delta_R}$. The energy loss from the final state $\Delta\omega_f$ coincides on the contrary with the energy resolution $\Delta\omega$ of a given experiment. The overall radiative factors are then simply given in terms of suitable powers of $\Delta\omega_i$ and $\Delta\omega_f$, depending on the type of cross section involved.

Most questions of immediate experimental interest are studied in section V, where we apply our formalism to actual experiments. In particular we discuss the case of experiments of resonance production with a machine resolution σ , such that $\sigma \gtrsim \Gamma$. The partial widths of the recently discovered $\psi(3.1)$ and $\psi(3.7)$ are then extracted from the experimental data. In this connection we give a simple formula which allows a careful estimate of the resonance parameters from a knowledge only of the peak values of the cross sections.

2. - NOTATION AND FORMULAE. -

In this section, we define our notation and collect some relevant formulae.

For a process of the type

$$e^-(p_1) + e^+(p_2) \rightarrow A^-(p_3) + A^+(p_4)$$

we have:

$$(2.1a) \quad s = W^2 = 4E^2 = (p_1 + p_2)^2$$

$$(2.1b) \quad y = W - M$$

$$(2.1c) \quad Z = \cos \vartheta = \hat{p}_1 \cdot \hat{p}_3 = \hat{p}_2 \cdot \hat{p}_4$$

$$(2.2a) \quad \beta_{i,f} = \frac{4\alpha}{\pi} \left[\ln \frac{W}{m_{i,f}} - 1/2 \right]$$

$$(2.2b) \quad \beta_{\text{int}} = \frac{4\alpha}{\pi} \ln (\tan \vartheta / 2)$$

$\Delta\omega$ = Energy resolution of the experiment.

λ = Infrared cut off.

m = electron mass.

A resonance of mass M , total width Γ may conveniently be parametrized in terms of a resonance phase-shift $\delta_R(W)$, defined as

$$(2.3) \quad \tan \delta_R(W) = \frac{\Gamma/2}{-y}$$

Cross sections for the processes $e^+e^- \rightarrow e^+e^-$, $\mu^+\mu^-$ and hadrons can be written as follows:

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$$(2.4) \quad \sigma^c = C_{\text{INFRA}}^{\text{RES}} \sigma_{\text{RES}} (1 + C_F^{\text{RES}}) + C_{\text{INFRA}}^{\text{INT}} \sigma_{\text{INT}} (1 + C_F^{\text{INT}}) + C_{\text{INFRA}}^{\text{QED}} \sigma_{\text{QED}} (1 + C_F^{\text{QED}})$$

$$(2.5a) \quad \left(\frac{d\sigma}{dZ}\right)_{\text{RES}}^{\text{leptonic}} = \left(\frac{\pi\alpha^2}{2s}\right) (1+Z^2) \frac{g^4 M^2/4}{y^2 + (\Gamma/2)^2}$$

$$(2.5b) \quad \left(\frac{d\sigma}{dZ}\right)_{\text{INT}}^{\mu^+\mu^-} = \left(\frac{\pi\alpha^2}{2s}\right) (1+Z^2) \frac{g^2 My}{y^2 + (\Gamma/2)^2}$$

$$(2.5c) \quad \left(\frac{d\sigma}{dZ}\right)_{\text{INT}}^{e^+e^-} = \left(\frac{\pi\alpha^2}{2s}\right) \frac{Z(3+Z^2)}{(1-Z)} \cdot \frac{-g^2 My}{y^2 + (\Gamma/2)^2}$$

$$(2.5d) \quad \left(\frac{d\sigma}{dZ}\right)_{\text{QED}}^{\mu^+\mu^-} = \left(\frac{\pi\alpha^2}{2s}\right) (1+Z^2)$$

$$(2.5e) \quad \left(\frac{d\sigma}{dZ}\right)_{\text{QED}}^{e^+e^-} = \left(\frac{\pi\alpha^2}{2s}\right) \left(\frac{3+Z^2}{1-Z}\right)^2$$

$$(2.5f) \quad \sigma_{\text{RES}}^{\text{hadron}} = \left(\frac{\pi\alpha}{s}\right) \frac{g^2 M\Gamma_h}{y^2 + (\Gamma/2)^2}$$

where, the coupling constant g is defined in terms of the leptonic widths (assumed equal) as

$$(2.6) \quad \Gamma_e \simeq \Gamma_\mu = \frac{\alpha}{3} g^2 M; \quad (\alpha^{-1} \approx 137),$$

and Γ_h is the hadronic width of the resonance.

$C_{\text{INFRA}}^{\text{RES, INT, QED}}$ are the infrared factors associated with each of the respective cross-sections. $C_{\text{F}}^{\text{RES, INT, QED}}$ incorporates the rest of the finite radiative correction and is to be calculated perturbatively. For a very narrow resonance, $\Gamma \ll \Delta\omega$, as shown in the text, C_{INFRA} factors take the form:

$$(2.7a) \quad C_{\text{INFRA}}^{\text{RES}} = \left(\frac{\Delta\omega}{E}\right)^{\beta_f} \left(\frac{\Gamma}{M}\right)^{\beta_i} \frac{\sin[\delta_R(1-\beta_i)]}{[\sin\delta_R]^{1+\beta_i}} \frac{1}{\gamma^{\beta_f} \Gamma(1+\beta_f)}$$

$$(2.7b) \quad \approx \left(\frac{\Delta\omega}{E}\right)^{\beta_f} \left[\frac{y^2 + (\Gamma/2)^2}{(M/2)^2} \right]^{(1/2)\beta_i} \left\{ 1 + \beta_i \left(\frac{y}{\Gamma/2}\right) \delta_R \right\}$$

$$(2.8a) \quad C_{\text{INFRA}}^{\text{INT}} = \left(\frac{\Delta\omega}{E}\right)^{\beta_f + \beta_{\text{int}}} \left(\frac{\Gamma}{M}\right)^{\beta_i} \frac{\cos\delta_R(1-\beta_i)}{(\sin\delta_R)^{\beta_i} \cos\delta_R} \times$$

$$\times \frac{1}{\gamma^{\beta_f + \beta_{\text{int}}} \Gamma(1 + \beta_f + \beta_{\text{int}})}$$

$$(2.8b) \quad \approx \left(\frac{\Delta\omega}{E}\right)^{\beta_f + \beta_{\text{int}}} \left[\frac{y^2 + (\Gamma/2)^2}{(M/2)^2} \right]^{\beta_i/2} \left[1 - \beta_i \frac{\Gamma/2}{y} \delta_R \right]$$

$$(2.9) \quad C_{\text{INFRA}}^{\text{QED}} = \left(\frac{\Delta\omega}{E}\right)^{\beta_i + \beta_f + 2\beta_{\text{int}}} \frac{1}{\gamma^{\beta_i + \beta_f + 2\beta_{\text{int}}} \Gamma(1 + \beta_i + \beta_f + 2\beta_{\text{int}})},$$

where $\ln \gamma = .5772$ is Euler's constant.

β_{int} - dependence in (2.7) and (2.8) is different from that shown in our earlier work⁽⁷⁾. This difference arises due to extra terms from the virtual graphs. (This is further discussed in the text and in Appendix A).

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C_F factors are as follows:

(i) $e^+e^- \rightarrow$ hadrons:

Here one may safely neglect σ_{QED} and σ_{INT} (at least for $\psi(3.1)$ and $\psi'(3.7)$ resonances). Then, we need only C_F^{RES} for hadrons. It is^(5, 9)

$$(2.10) \quad C_F^{\text{had}} \approx \frac{13}{12} \beta_e + \frac{\alpha}{\pi} \left(\frac{\pi^2}{3} - \frac{17}{18} \right)$$

(ii) $e^+e^- \rightarrow \mu^+ \mu^-$:

For this reaction, C_F^{QED} has been calculated in ref. (10-12). The corresponding C_F^{INT} and C_F^{RES} are taken from ref. (13).

$$(2.11) \quad C_F^{\text{QED}} = \frac{13}{12} (\beta_e + \beta_\mu) + \left(\frac{2\alpha}{\pi} \right) \cdot \left(\frac{\pi^2}{3} - \frac{17}{18} \right) + X',$$

$$(2.12) \quad C_F^{\text{INT}} = \frac{11}{12} (\beta_e + \beta_\mu) + \left(\frac{2\alpha}{\pi} \right) \cdot \left(\frac{\pi^2}{3} - \frac{13}{18} \right) + \frac{1}{2} X,$$

where

$$(2.13) \quad X' = - \frac{4\alpha}{\pi} \cdot \frac{1}{(1+Z^2)} \left[Z \left\{ (\ln \sin(\theta/2))^2 + (\ln \cos(\theta/2))^2 \right\} \right. \\ \left. + (\sin^2(\theta/2)) (\ln \cos(\theta/2)) - \cos^2(\theta/2) \ln(\sin(\theta/2)) \right]$$

$$(2.14) \quad C_F^{\text{RES}} = \frac{3}{4} (\beta_e + \beta_\mu) + \left(\frac{2\alpha}{\pi} \right) \left(\frac{\pi^2}{3} - \frac{1}{2} \right) + \left(\frac{4\alpha}{9} \right) \left(\frac{\alpha \Gamma}{\Gamma_e} \right) + Y,$$

where

$$(2.15) \quad Y = - \left(\frac{2\alpha}{3} \right) \left(\frac{\alpha \Gamma}{\Gamma_e} \right) \frac{1}{(1+Z)^2} \left[Z - 2Z^2 (\ln \sin(\theta/2) + \ln \cos(\theta/2)) + \right. \\ \left. + 2(1+Z^2) \ln(\tan(\theta/2)) \right]$$

For all practical purpose the

approximation $C_F^{\text{INT}} \approx \frac{1}{2} (C_F^{\text{QED}} + C_F^{\text{RES}})$ works quite well, since Γ -dependent terms are quite small.

(iii) $e^+e^- \rightarrow e^+e^-$:

C_F^{RES} for this reaction may be read off directly from equation (2.14) substituting $2\beta_e$ for the factor $(\beta_e + \beta_\mu)$ appearing there.

C_F^{QED} is obtained from ref. (12):

$$\begin{aligned}
(2.16) \quad C_F^{\text{QED}} = & \left(\frac{\alpha}{\pi} \right) \left[u^2 - v^2 + w^2 + 2\text{Li}_2\left(\frac{1+Z}{2}\right) - 2\text{Li}_2\left(\frac{1-Z}{2}\right) - \frac{2}{3}\pi^2 \right] + \\
& + \left(\frac{\alpha}{\pi} \right) \frac{1}{(3+Z^2)^2} \left[\frac{u}{3}(3-42Z+42Z^2-14Z^3+11Z^4) - v(5-7Z+3Z^2-Z^3) + \right. \\
& + \frac{1}{3}w(111+21Z+33Z^2+11Z^3) + \frac{u^2}{2}(3+7Z-5Z^2-3Z^3-2Z^4) + \\
& + v^2(3-3Z+Z^2-Z^3) - \frac{w^2}{2}(9+7Z+11Z^2+5Z^3) - 2uvZ(2-Z-Z^3) \\
& - uw(21+3Z+9Z^2-3Z^3+2Z^4) + 2vw(6+5Z+4Z^2+Z^3) - \frac{46}{9}(9+6Z^2+Z^4) \\
& \left. + \frac{\pi^2}{3}(18-15Z+12Z^2-3Z^3+4Z^4) \right],
\end{aligned}$$

where

$$(2.17) \quad u = 2\ln \frac{W}{m}; \quad v = \ln \frac{2E^2(1+Z)}{m^2}; \quad w = \ln \frac{2E^2(1-Z)}{m^2}$$

and the dilogarithm $\text{Li}_2(x)$ is defined as

$$(2.18) \quad \text{Li}_2(x) = - \int_0^x \frac{\ln(1-y)}{y} dy.$$

C_F^{INT} expression is missing. However, as discussed before, the approximation $C_F^{\text{INT}} \approx \frac{1}{2} (C_F^{\text{QED}} + C_F^{\text{RES}})$ ought to work well.

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This completes the list of our formulae. Their derivation, discussion and application follows in the coming sections.

3. - PERTUBATIVE METHOD -

Let us consider the process

$$(3.1) \quad e^-(p_1) + e^+(p_2) \rightarrow A^-(p_3) + A^+(p_4)$$

which we first assume to be described by a smooth function of the energy. In this case process (1), accompanied by the emission of n real soft photons can be described by (2, 3)

$$(3.2) \quad M_{\mu_1 \dots \mu_n}^{(n)} \approx j_{\mu_1}(k_1) \dots j_{\mu_n}(k_n) M(p_1, p_2, p_3, p_4)$$

where $M(p_1, p_2, p_3, p_4)$ is the matrix element for process (1) including all the virtual processes, and

$$j_{\mu}(k) = \frac{ie}{(2\pi)^{3/2}} \left[\frac{p_{1\mu}}{p_1 \cdot k} - \frac{p_{2\mu}}{p_2 \cdot k} - \frac{p_{3\mu}}{p_3 \cdot k} + \frac{p_{4\mu}}{p_4 \cdot k} \right]$$

If process (1) proceeds via a narrow resonance, the matrix element, which describes it, is a rapidly varying function of the energy. Eq. (3.2) then is not valid, since soft photon emission from the initial state drifts the position of the resonance and prevents a complete separation of the infrared terms from the unperturbed matrix element. It is still possible however to have a working analogue of eq. (3.2) in the following way. Let us first consider the emission of just one real soft photon in process (3.1). We can write

$$(3.3) \quad M_{\mu}^{(1)} \simeq I_{\mu}(k)M(W-k) + F_{\mu}(k)M(W)$$

where we have distinguished between initial and final state emission on the basis that one shifts the c.m. energy W to a lower value $W-k$, while the other doesn't. In eq. (3.3) $M(W)$ or $M(W-k)$ is the resonant matrix element inclusive of all virtual processes and

$$I_{\mu}(k) = \frac{ie}{(2\pi)^{3/2}} \left[\frac{p_{1\mu}}{p_1 \cdot k} - \frac{p_{2\mu}}{p_2 \cdot k} \right]$$

$$F_{\mu}(k) = \frac{ie}{(2\pi)^{3/2}} \left[\frac{-p_{3\mu}}{p_3 \cdot k} + \frac{p_{4\mu}}{p_4 \cdot k} \right]$$

We now parametrize the amplitudes $M(W)$ and $M(W-k)$ by writing

$$M(W-k) = \int d\omega \delta(\omega-k) M(W-\omega)$$

$$M(W) = \int d\omega \delta(\omega)M(W-\omega)$$

and use the representation

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt e^{ixt}$$

so that eq. (3.3) becomes

$$(3.4) \quad M_{\mu}^{(1)} \simeq \int d\omega M(W-\omega) \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{i\omega t} \tilde{j}_{\mu}(k, t)$$

with

$$\tilde{j}_{\mu}(k, t) = I_{\mu}(k)e^{-ikt} + F_{\mu}(k)$$

In eq. (3.4) the limits of integration in ω will be fixed later by the re-

quirement that the total energy loss be less or equal to the energy resolution $\Delta\omega$ of the experiment.

We can now iterate eq. (3.4) to obtain the matrix element corresponding to the emission of n real soft photons

$$(3.5) \quad M_{\mu_1 \dots \mu_n}^{(n)} = \int d\omega M(W-\omega) \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{i\omega t} \tilde{j}_{\mu_1}(k_1, t) \dots \tilde{j}_{\mu_n}(k_n, t)$$

The cross section for the n th-order bremsstrahlung is then proportional to

$$\begin{aligned} & \frac{1}{n!} \int d\omega M(W-\omega) \int d\omega' M^+(W-\omega') \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{i\omega t} \int_{-\infty}^{+\infty} e^{-i\omega' t'} \frac{dt'}{2\pi} \times \\ & \times \left[\int_0^{\Delta\omega} \frac{d^3 k}{2k_0} \tilde{j}_{\mu}(k, t) \tilde{j}^{\mu*}(k, t') \right]^n \end{aligned}$$

Summing over n we get the cross-section for process (3.1) accompanied by a radiation loss up to $\Delta\omega$

$$(3.6) \quad \sigma_{\text{RES}}(W, \theta) \propto \int d\omega M(W-\omega) \int d\omega' M^+(W-\omega') \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{i\omega t} \int_{-\infty}^{+\infty} \frac{dt'}{2\pi} e^{-i\omega' t'} e^{h(t, t')}$$

where

$$(3.7) \quad \begin{aligned} h(t, t') &= \int \frac{d^3 k}{2k_0} \tilde{j}_{\mu}(k, t) \tilde{j}^{\mu*}(k, t') = \\ &= \beta_i \int_{\lambda}^{\Delta\omega} \frac{dk}{k} e^{-ik(t-t')} + \beta_{\text{int}} \int_{\lambda}^{\Delta\omega} \frac{dk}{k} (e^{-ikt} + e^{ikt'}) + \beta_f \int_{\lambda}^{\Delta\omega} \frac{dk}{k} \end{aligned}$$

with

λ = infrared energy cut-off

ω = energy resolution of the experiment

$$\beta_i = \frac{k^2}{2} \int d\Omega_k (I_\mu I^\mu)^* \simeq \frac{4\alpha}{\pi} \left(\ln \frac{W}{m_i} - \frac{1}{2} \right)$$

$$\beta_{\text{int}} = \frac{k^2}{2} \int d\Omega_k (I_\mu F^\mu)^* \simeq \frac{4\alpha}{\pi} \ln \left(\text{tg} \frac{\theta}{2} \right)$$

$$\beta_f = \frac{k^2}{2} \int d\Omega_k (F_\mu F^\mu)^* \simeq \frac{4\alpha}{\pi} \left(\ln \frac{W}{m_f} - \frac{1}{2} \right)$$

θ = c.m. scattering angle between equally charged particles.

The photon energy being positive definite, the ω and ω' integrals in eq. (3.6) extend only over positive values (this is also a consequence of the analyticity properties of $h(t, t')$).

In eq. (3.7) we have introduced a photon minimum energy to handle the infrared divergence arising from the description of real photon emission. We can rewrite eq. (3.7) as

$$h(t, t') = (\beta_i + 2\beta_{\text{int}} + \beta_f) \ln \frac{\Delta\omega}{\lambda} + f(t, t')$$

where now $f(t, t')$ is finite as $\lambda \rightarrow 0$ and

$$(3.8) \quad f(t, t') = \beta_i \int_0^{\Delta\omega} \frac{dk}{k} (e^{-ik(t-t')} - 1) + \beta_{\text{int}} \int_0^{\Delta\omega} \frac{dk}{k} (e^{-ikt} + e^{ikt'} - 2).$$

Eq. (3.6) thus becomes

$$(3.9) \quad \sigma_{\text{RES}}(W, \theta) \propto \left(\frac{\Delta\omega}{\lambda} \right)^\beta \int d\omega M(W-\omega) \int d\omega' M(W-\omega') \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{i\omega t} \int_{-\infty}^{+\infty} \frac{dt'}{2\pi} e^{-i\omega' t'} e^{i\omega t} f(t, t')$$

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where $\beta = \beta_i + 2\beta_{\text{int}} + \beta_f$. If we assume $\beta_{\text{int}} = 0$, the t and t' integrations can easily be done. We postpone to the next section and to Appendix A the discussion on the β_{int} dependence of σ_{RES} and proceed by putting $\beta_{\text{int}} = 0$. Then $f(t, t')$ depends only upon the variable $\tau = t = t'$ so that eq. (3.9) can be rewritten as

$$\sigma_{\text{RES}}(W, \theta) \propto \left(\frac{\Delta\omega}{\lambda}\right)^{\beta_i + \beta_f} \int d\omega M(W-\omega) \int d\omega' M^+(W-\omega')$$

$$\int_{-\infty}^{+\infty} \frac{dT}{2\pi} e^{i(\omega-\omega')T} \int_{-\infty}^{+\infty} \frac{d\tau}{2\pi} e^{i\frac{\omega+\omega'}{2}\tau} e^{i\tau} f(\tau).$$

The dT integral gives a $\delta(\omega - \omega')$ and we are left with

$$\sigma_{\text{RES}}(W, \theta) \propto \left(\frac{\Delta\omega}{\lambda}\right)^{\beta_i + \beta_f} \int \frac{d\omega}{\omega} |M(W-\omega)|^2 \int_{-\infty}^{+\infty} \frac{d\tau}{2\pi} e^{i\tau}$$

$$(3.10) \quad e^{\beta_i \int_0^{\Delta\omega/\omega} \frac{dk}{k} (e^{-ik\tau} - 1)}$$

The apparent λ divergence in eq. (3.10) must be cancelled by the virtual photon contributions, i. e. it must be

$$(3.11) \quad |M(W-\omega)|^2 = \left(\frac{\lambda}{E}\right)^{\beta_i + \beta_f} |M_E(W-\omega)|^2$$

with $E = W/2$ and $|M_E(W-\omega)|^2$ containing, upon renormalization, only finite contributions. In $M_E(W-\omega)$, the subscript "E" indicates the scale used to extract the virtual photon contribution to the λ divergence. Inserting eq. (3.11) into eq. (3.10) and performing the $d\tau$ integration as in ref. (4) we finally obtain

$$(3.12) \quad \sigma_{\text{RES}}(W, \theta) = \left(\frac{\Delta\omega}{E}\right)^{\beta_f} N(\beta_i) \frac{\sin(\pi\beta_i)}{\pi} \int_0^{\Delta\omega} \frac{d\omega}{\omega} \left(\frac{\omega}{E}\right)^{\beta_i} \sigma_E(W-\omega, \theta)$$

where $N(\beta_i)$ is a normalization factor which is related to the probability that two or more photons from the initial state combine to give a total energy loss larger than $\Delta\omega$. Since $N(\beta_i) - 1$ is of order β_i^2 we can safely set it equal to 1. It should be noticed that in eq. (3.12) no similar normalization factor for final state emission appears. This reflects the fact that in obtaining this equation we have not applied energy conservation to the emission from the final state. The error involved is obviously of order β_f^2 and hence negligible.

In eq. (3.12) the maximum energy loss from the initial state has been set equal to that from the final state. In an actual e^+e^- colliding beam experiment this, in general, is not correct: in ADONE experiment, for instance, the energy resolution on the initial state is of order 100 MeV (for $W \sim 3$ GeV) while that on the final state may be four or five times larger. However, for resonant process characterized by a width $\Gamma \ll \Delta\omega$ and in the resonance region, the $d\omega$ integral is highly insensitive to the upper limit. To see this, we must first extract the dependence from $\sigma_E(W-\omega, \theta)$. From ref. (5) we have to the next order in α ,

$$(3.13) \quad \sigma_E(W-\omega, \theta) \simeq \frac{A_0 (\Gamma/2)^2 [1 + C_F^{\text{RES}}]}{(W-M-\omega)^2 + (\Gamma/2)^2}$$

where C_F^{RES} (see sect. 2) is a finite contribution of order α , independent of ω . A_0 depends from the specific process under consideration and represents the peak value of the unshifted resonant cross section, i. e.

$$\sigma_0(W, \theta) = \frac{A_0 (\Gamma/2)^2}{(W-M)^2 + (\Gamma/2)^2}$$

We can now insert eq. (3.13) into (3.12) and evaluate the integral. As it is, the integration depends upon the resonance parameters and gi-

ves different results for $\Delta\omega \gg \Gamma$ or $\Delta\omega \ll \Gamma$. In the case of the $\psi(3.1)$ and $\psi(3.7)$ resonances, we can safely set equal to ∞ the upper limit of integration. The error involved is of order

$$\beta_i \frac{(W-M)^2 + (\Gamma/2)^2}{\Delta\omega^2},$$

hence negligible. We thus get, for a very narrow resonance,

$$(3.14) \quad \sigma_{\text{RES}}(W, \theta) = \left(\frac{\Delta\omega}{E}\right)^{\beta_f} \left[\frac{(W-M)^2 + (\Gamma/2)^2}{(M/2)^2} \right]^{\beta_i/2} \frac{\sin[\delta_R(1-\beta_i)]}{\sin\delta_R} \times \\ \times \sigma_0(W, \theta)(1+C_F^{\text{RES}})$$

where δ_R is the resonant phase shift, i. e. $\text{ctg } \delta_R = (M-W)/(\Gamma/2)$.

We now wish to comment briefly on the various factors appearing at the r. h. side of eq. (3.14):

a) C_F^{RES} represents finite virtual photon contributions as well as, eventually, hard bremsstrahlung contributions from the final state. To determine it, one should proceed as follows. A first order expansion of eq. (3.14) gives

$$\sigma_{\text{RES}}(W, \theta) \approx \left[1 + \beta_f \ln \frac{\Delta\omega}{E} + \frac{1}{2} \beta_i \ln \frac{(W-M)^2 - (\Gamma/2)^2}{(M/2)^2} + \right. \\ \left. + \beta_i \frac{W-M}{\Gamma/2} \delta_R + C_F^{\text{RES}} \right] \sigma_0(W, \theta)$$

By comparing the above expansion with the QED perturbative calculation to the same order one can then determine C_F^{RES} . This was done, for instance in ref. (5) for the process $e^+e^- \rightarrow \rho \rightarrow \pi^+\pi^-$. C_F^{RES} for the processes interesting ψ production are given in sect. 2.

b) The factor

$$\left[\frac{(W-M)^2 + (\Gamma/2)^2}{(M/2)^2} \right]^{\beta_i/2}$$

shows that the emission of photons from the initial state is regulated by the width of the resonance. This agrees with our physical intuition: if an electron-positron pair of total c.m. energy $W \simeq M$ radiates an energy larger than Γ the subsequent annihilation will not be observed as a resonant process. The resonant peak cross section is therefore reduced by a factor $(\Gamma/M)^{\beta_i}$.

c) The factor

$$\frac{\sin[\delta_R(1-\beta_i)]}{\sin \delta_R}$$

gives the radiative tail of the resonance. In fact an expansion in β_i ($\beta_i = 0.076$ at $W = 3$ GeV) gives

$$\frac{\sin[\delta_R(1-\beta_i)]}{\sin \delta_R} \simeq 1 + \beta_i \frac{W-M}{\Gamma/2} \delta_R$$

which shows the characteristic radiative tail on the right of the resonance where $\delta_R \gtrsim \pi/2$. This is mostly a single photon effect, unlike the previous one.

Eq. (3.14) gives the radiatively corrected resonant part of the cross-section. When the QED background cannot be neglected, interference effects between the resonant amplitude and the photon channels should be taken into account. In the following we derive the expression for the radiative corrections to the interference part of the cross-section. To obtain the correction factor we must consider the interference between the QED amplitude for process (3.1) accompanied by the

emission of n real soft photons, with the analogous resonant amplitude and then sum upon n . The QED amplitude is given by eq. (3.2), i. e.

$$M_{\mu_1 \dots \mu_n}^{(\text{QED})} \approx j_{\mu_1}(k_1) \dots j_{\mu_n}(k_n) M^{(\text{QED})}(W)$$

while the resonant one by eq. (3.5), i. e.

$$M_{\mu_1 \dots \mu_n}^{(\text{RES})} \approx \int d\omega M^{(\text{RES})}(W-\omega) \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{i\omega t} \tilde{j}_{\mu_1}(k_1, t) \dots \dots \tilde{j}_{\mu_n}(k_n, t)$$

where $M^{(\text{QED})}(W)$ and $M^{(\text{RES})}(W-\omega)$ are the elastic amplitudes for process (3.1) inclusive of all virtual processes. One gets for the radiatively corrected interference cross-section

$$(3.15) \quad \sigma_{\text{INT}}(W, \theta) \propto 2\text{Re} \left\{ \left[M^{(\text{QED})}(W) \right]^+ \int d\omega M^{(\text{RES})}(W-\omega) \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{i\omega t} e^{g(t)} \right\}$$

with

$$g(t) = (\beta_i + 2\beta_{\text{int}} + \beta_f) \ln \frac{\Delta\omega}{\lambda} + (\beta_i + \beta_{\text{int}}) \int_0^{\Delta\omega} \frac{dk}{k} (e^{-ikt} - 1).$$

In $g(t)$ we have separated out the infrared divergence introducing, as before, a minimum photon energy λ . The analyticity properties of $g(t)$ restrict the ω integration upon positive values. For $\omega < \Delta\omega$ eq. (3.15) gives

$$\sigma_{\text{INT}}(W, \theta) \propto 2\text{Re} \left\{ \left[M^{\text{QED}}(W) \right]^+ \left(\frac{\Delta\omega}{\lambda} \right)^\beta \frac{\sin[\pi(\beta_i + \beta_{\text{int}})]}{\pi} \right. \\ \left. \int_0^{\Delta\omega} \frac{d\omega}{\omega} \left(\frac{\omega}{\Delta\omega} \right)^{\beta_i + \beta_{\text{int}}} M^{\text{RES}}(W-\omega) \right\}.$$

As before we put $\beta_{\text{int}} = 0$ and cancel the λ divergence by postulating that (see Appendix A)

$$2\text{Re} \left\{ \left[M^{\text{QED}}(W) \right]^+ M^{\text{RES}}(W-\omega) \right\} = \left(\frac{\lambda}{E} \right)^{\beta_i + \beta_f} 2\text{Re} \left\{ \left[M_E^{\text{QED}}(W) \right]^+ \right. \\ \left. M_E^{\text{RES}}(W-\omega) \right\}$$

so that

$$(3.16) \quad \sigma_{\text{INT}}(W, \theta) = \left(\frac{\Delta\omega}{E} \right)^{\beta_f} \frac{\sin(\pi\beta_i)}{\pi} \int_0^{\Delta\omega} \frac{d\omega}{\omega} \left(\frac{\omega}{E} \right)^{\beta_i} \sigma_E^{\text{INT}}(W-\omega, \theta).$$

We now have to extract the residual ω dependence from $\sigma_E^{\text{INT}}(W-\omega)$.

We can write

$$(3.17) \quad \sigma_E^{\text{INT}}(W-\omega, \theta) \simeq \frac{B_0 \Gamma/2 (W-M-\omega) [1 + C_F^{\text{INT}}]}{(W-M-\omega)^2 + (\Gamma/2)^2}$$

where C_F^{INT} , as before, is a finite contribution of order α , independent of ω . B_0 is defined so that the interference cross-section between the Breit-Wigner and the lowest order QED amplitude is

$$\sigma_0^{\text{INT}}(W, \theta) = \frac{B_0 (W-M) \Gamma/2}{(W-M)^2 + (\Gamma/2)^2}.$$

The insertion of eq. (3.17) into (3.16) gives

20.

$$\sigma_{\text{INT}}(W, \theta) = \left(\frac{\Delta\omega}{E}\right)^{\beta_f} \left[\frac{(W-M)^2 + (\Gamma/2)^2}{(M/2)^2} \right]^{\beta_i/2} \times$$

(3.18)

$$\times \frac{\cos[\delta_R(1-\beta_i)]}{\cos\delta_R} \sigma_o^{\text{INT}}(W, \theta)(1+C_{\text{F}}^{\text{INT}}) \dots$$

having let $\Delta\omega \rightarrow \infty$, as before.

In eq. (3.18) the term

$$\frac{\cos[\delta_R(1-\beta_i)]}{\cos\delta_R} \simeq \frac{-\beta_i \Gamma/2}{W-M} \delta_R + 1$$

shows that the interference is not zero at the peak, where $\delta_R = \pi/2$. This is due to the fact that photon emission from the initial state generates, at the peak, a real part in the resonant amplitude. It is mostly a single photon effect, just like the radiative tail in σ_{RES} .

4. - COHERENT STATES APPROACH. -

4.1. - Pure Q.E.D. processes. -

The application of the coherent states formalism to the problem of the infrared divergence in electrodynamics was developed in ref. (8). For reader's convenience, and with an eye to its generalization for resonant processes given below, we briefly review the results obtained in ref. (8).

As noted above already, the infrared divergence in Q.E.D. is entirely due to the unphysical description of the perturbative solution. The simple observation that the number of soft photons emitted in a reaction can never be measured, immediately leads to the neces

sity of introducing a new definition of final states, closer to physical reality. Consider a process

$$(4.1) \quad a + b \rightarrow c + d + \dots,$$

in which at least some of the particle a , b , c , d , .. involved are charged. What one really observes is the process

$$(4.2) \quad a + b \rightarrow c + d + \dots + \Gamma,$$

where Γ stands for an unlimited number of photons subject only to the condition that the total 4-momentum of the emitted photons is fixed by the energy and momentum resolution of the experimental apparatus.

In the following, for sake of simplicity, we shall assume only an energy resolution $\Delta\omega$. By calling $|f\rangle$ the final state of the reaction (4.1), let us introduce the state vector $|f'\rangle$, defined as

$$(4.3) \quad |f'\rangle = e^{i\Lambda_c} |f\rangle,$$

with

$$(4.4) \quad \Lambda_c = \int d^4x j_\mu(x) A_\mu(x),$$

where $A_\mu(x)$ is the quantized electromagnetic field, and $j_\mu(x)$ the Fourier transform of the classical current $j_\mu(k)$ defined as

$$(4.5) \quad j_\mu(k) = \frac{ie}{(2\pi)^{3/2}} \sum_{i=a,b,\dots} \varepsilon_i \frac{p_\mu^{(i)}}{(p \cdot k)} \quad \text{for } k_0 \leq \Delta\omega$$

$$j_\mu(k) = 0 \quad \text{for } k_0 > \Delta\omega$$

with $\varepsilon_i = +1$ for positive outgoing and negative incoming particles and -1 otherwise. The operator $e^{i\Lambda_c}$ defined in eq. (4.3) is unitary. The state $|f'\rangle$ contains an unlimited and undetermined number of photons, created by the electromagnetic field of the distribution of classical currents (4.5).

It is now possible to define a matrix element $\bar{M} = \langle f | S | i \rangle$ which, as proved in ref. (8) is: (i) finite and does not therefore possess an infrared divergence, (ii) separable in the infrared factor and (iii) directly comparable with the experimental cross section which results proportional to $|\bar{M}|^2$. In particular one has:

$$(4.6) \quad \begin{aligned} \bar{M} &= \langle f | e^{-i\Lambda} c S | i \rangle = \exp \left\{ \frac{1}{2} \int_{\lambda}^{\Delta\omega} \frac{d^3k}{2k} [j_{\mu}(k) j_{\mu}^*(k)] \right\} \langle f | S | i \rangle_{\lambda} = \\ &= \exp \left\{ \frac{\beta}{2} \int_{\lambda}^{\Delta\omega} \frac{dk}{k} \right\} \langle f | S | i \rangle_{\lambda} = \left(\frac{\Delta\omega}{\lambda} \right)^{\beta/2} \langle f | S | i \rangle_{\lambda}, \end{aligned}$$

where

$$\beta = \frac{1}{2} \int d\Omega_k |k|^2 j_{\mu}(k) j_{\mu}^*(k).$$

The auxiliary parameter λ has been introduced consistently both in $\int_{\lambda} [j_{\mu}(k) j_{\mu}^*(k)] d^3k/2k$ and $\langle f | S | i \rangle_{\lambda}$ as the lower limit of the permitted energies of real and virtual photons, and has to be finally set to zero. In this limit, because of the finiteness of \bar{M} , the apparent divergence in $(\Delta\omega/\lambda)^{\beta/2}$ is cancelled by an analogous divergence in $\langle f | S | i \rangle_{\lambda}$. One can therefore write

$$(4.7) \quad \bar{M} = \left(\frac{\Delta\omega}{E} \right)^{\beta/2} M_E,$$

which finally defines a matrix element M_E which is finite, i. e. without any infrared divergence. Of course M_E may have ultraviolet divergence which, upon normalization, will give additive correction factors.

The cross section for process (4.2) is proportional to $|\bar{M}|^2$,

$$(4.8) \quad (d\sigma)_{\text{obs}} \propto |\bar{M}|^2 = \left(\frac{\Delta\omega}{E} \right)^{\beta} |M_E|^2 \propto \left(\frac{\Delta\omega}{E} \right)^{\beta} d\sigma_E,$$

and comparison with perturbation theory finally determines $d\sigma_E$. In e^+e^- scattering ($e^+e^- \rightarrow f$), one has for example

$$(4.9) \quad (d\sigma)_{\text{obs}} = \frac{1}{\gamma\beta} \frac{1}{\Gamma(1+\beta)} \left(\frac{\Delta\omega}{E}\right)^\beta d\sigma_E = \frac{1}{\gamma\beta} \frac{1}{\Gamma(1+\beta)} \left(\frac{\Delta\omega}{E}\right)^\beta d\sigma_0(1+C_F),$$

where $\sqrt{s} = 2E$ is the total energy in the c.m. frame, C_F is the left-over finite correction and $d\sigma_0$ is the lowest order cross section for the process of interest. The extra factor $\gamma^\beta \Gamma(1+\beta)$, where $\ln\gamma = .5772$ is Euler's constant, is found⁽⁸⁾, as also discussed in the Appendix B, by imposing the global condition

$$(4.10) \quad \sum_{n=1}^{\infty} k_0^{(n)} \leq \Delta\omega$$

on the states $|f\rangle$, instead of $k_0^{(n)} \leq \Delta\omega$ for every n , as implied in (4.5). This factor however, for small values of β is $1 - \pi^2\beta^2/12$, so it is very close to 1. This is related to the fact that $[\gamma^\beta \Gamma(1+\beta) - 1]$ represents the probability that two or more photons, each with energy $< \Delta\omega$ combine to give a total energy loss $> \Delta\omega$.

4.2. - Pure resonant processes. -

Equations (4.7-9) are valid provided the "bare" matrix element (without soft photon emission) is not a rapidly varying function of the energy or the momentum transfer. Let us consider now the modifications to our formalism due to the presence of a resonance, for example in the reaction $e^+e^- \rightarrow \text{resonance} \rightarrow f$.

As shown in ref. (5) it can be easily seen that the presence of a resonance, of mass M and width Γ , only modifies the classical current relative to the initial state, as follows:

$$(4.11) \quad j_\mu^R(k) = \frac{(W-M)+i\Gamma/2}{(W-M-k)+i\Gamma/2} I_\mu(k) + F_\mu(k)$$

where, in the notations of the preceding section, $I_\mu(k)$ and $F_\mu(k)$ are the classical currents relative to the initial and final states respectively, in absence of the resonance. The current $j_\mu^R(k)$ is exactly the Fourier transform of

24.

$$(4.12) \quad j^R(x) = j^f(x)\theta(t) + \overline{j^{(i)}}(x),$$

where $\overline{j^{(i)}}(x)$ is given by

$$(4.13) \quad \overline{j^{(i)}}(x) = (-iE^*) \int_0^\infty dt_0 j^{(i)}(x) \theta(-t-t_0) e^{iE^*t_0},$$

with $E^* = W - M + i\Gamma/2$. In other words one takes account of the finiteness of time interval between the creation of the final state at $t = 0$ and the creation of the resonant state at $-t_0 < 0$, having also accounted for the damping induced by the resonance itself.

In analogy with eqs. (4.3-4) one can now formally introduce the quantity Λ_R given by

$$(4.14) \quad \Lambda_R = \int d^4x j_\mu^R(x) A_\mu(x),$$

as the action relative to the distribution of classical currents $j_\mu^R(k)$ of eq. (4.11); with

$$j_\mu(x) = \frac{1}{(2\pi)^4} \int d^4k j_\mu^R(k) e^{-ikx}.$$

Similarly, one can introduce a new final state vector, obtained by operating with $e^{i\Lambda_R}$ on the usual final state $|f\rangle$. In this case, however, the operator $e^{i\Lambda_R}$ is no longer unitary, because $\Lambda_R^+ \neq \Lambda_R$. One is therefore led to define

$$(4.15) \quad |f''\rangle = \frac{1}{\sqrt{N}} e^{i\Lambda_R} |f\rangle,$$

with

$$(4.16) \quad N = \langle e^{-i\Lambda_R^+} e^{i\Lambda_R} \rangle.$$

One has:

$$(4.17) \quad \langle e^{-i\Lambda_R^+} e^{i\Lambda_R} \rangle = e^{1/2 [\Lambda_R^+, \Lambda_R]} \langle e^{-i(\Lambda_R^+ - \Lambda_R)} \rangle,$$

with

$$(4.18) \quad [\Lambda_R^+, \Lambda_R] = \int d^4k \delta(k^2) \theta(k_0) \left\{ j_\mu(k) j_\mu^*(k) - j_\mu(-k) j_\mu^*(-k) \right\},$$

where for simplicity we have dropped the label R from the current $j_\mu^R(k)$. Similarly, by series expansion of the exponential in eq. (4.17), and after contraction of the electromagnetic field operators, one finds:

$$(4.19) \quad \langle e^{-i(\Lambda_R^+ - \Lambda_R)} \rangle = \exp \left\{ -\frac{1}{2} \int d^4k \delta(k^2) \theta(k_0) \left[j_\mu^*(k) - j_\mu(-k) \right] \times \right. \\ \left. \times \left[j_\mu^*(-k) - j_\mu(k) \right] \right\}.$$

From the combined equations (4.17-19) one finally obtains

$$(4.20) \quad N = \exp \left\{ \frac{1}{2} \int d^4k \delta(k^2) \theta(k_0) \left[2j_\mu^*(k) j_\mu(k) - j_\mu^*(-k) j_\mu^*(k) - j_\mu(-k) j_\mu(k) \right] \right\}.$$

All the integrals in the above equations have a finite domain of integration, corresponding to $0 \leq k_0 \leq \Delta\omega$, once $j_\mu^R(k)$ is restricted as in eq. (4.5). The global condition (4.10), imposed on the states (4.15), gives an overall factor in analogy to (4.9), which will be discussed in the Appendix B.

We are able now to define a matrix element $\overline{M}_R = \langle f' | S | i \rangle$ as in eq. (4.6), where the S-matrix does contain now the interaction-Hamiltonian responsible for the creation of the resonance R. We therefore have

$$(4.21) \quad \overline{M}_R = \frac{1}{\sqrt{N}} \langle f' | e^{-i\Lambda_R^+} S | i \rangle.$$

Formally \overline{M}_R is the matrix element of the operator $S' = \exp[-i\Lambda_R^+] S$

between the states $\langle f |$ and $| i \rangle$ which do not include outgoing soft photons; in other words the infinite number of soft photons created by S is destroyed by $\exp [-i\Lambda_R^+]$. In terms of diagrams, some typical lowest order contributions to (4.21) come from the diagrams of Fig. 1, where we have indicated with a small circle the action of the classical

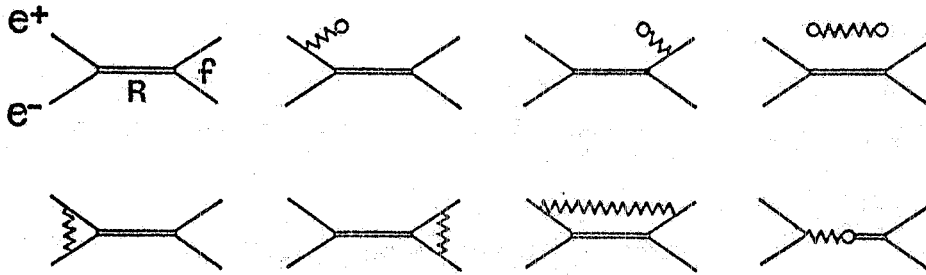


FIG. 1

source.

Without entering into the details of the calculations, which proceed along the same lines of ref. (8), we find:

$$(4.22) \quad \overline{M}_R = \frac{(M_R)_\lambda}{\sqrt{N}} \exp \left\{ \int_\lambda^{\Delta\omega} \frac{dk}{2k} \int |k|^2 d\Omega_k \left[j_\mu^*(k) j_\mu(k) - \frac{1}{2} j_\mu^*(k) j_\mu^*(-k) \right] \right\},$$

where we have dropped, as above, the index R in $j_\mu^R(k)$, and we have consistently introduced, as in eq. (4.6), an auxiliary parameter λ which has to go finally to zero. It has to be noticed that in the limit of very large Γ , the resonant factor in eq. (4.11) goes to 1 and we recover all the previous results for the usual classical current.

From eqs. (4.20-22) one finally obtains

$$(4.23) \quad \overline{M}_R = (M_R)_\lambda \exp \left\{ \frac{1}{2} \int_\lambda d^4 k \delta(k^2) \theta(k_0) \left[j_\mu^*(k) j_\mu(k) - i \operatorname{Im}(j_\mu^*(k) j_\mu^*(k)) \right] \right\}.$$

In order to proceed further we need the explicit expressions for the following integrals:

$$(4.24) \quad I_1 = \int d^4 k \delta(k^2) \theta(k_0) [j_\mu^*(k) j_\mu(k)],$$

$$(4.25) \quad I_2 = \int d^4 k \delta(k^2) \theta(k_0) [j_\mu^*(k) j_\mu^*(-k)],$$

with $\lambda \leq k_0 \leq \Delta\omega$. From eq. (4.11) one easily finds:

$$(4.26) \quad I_1 = \beta \ln \frac{\Delta\omega}{\lambda} - (\beta_i + 2\beta_{\text{int}}) \frac{1}{2} \ln \frac{(\Delta\omega - y)^2 + (\Gamma/2)^2}{y^2 + (\Gamma/2)^2} + \frac{y\beta_i}{(\Gamma/2)} \left[\operatorname{arctg} \left(\frac{\Delta\omega - y}{\Gamma/2} \right) + \operatorname{arctg} \frac{y}{(\Gamma/2)} \right],$$

$$(4.27) \quad I_2 = \beta \ln \frac{\Delta\omega}{\lambda} - (\beta_i + 2\beta_{\text{int}}) \frac{1}{2} \ln \frac{(y - i\Gamma/2)^2 - \Delta\omega^2}{(y - i\Gamma/2)^2},$$

where $y = W - M$ and $\beta = \beta_i + 2\beta_{\text{int}} + \beta_f$.

Substituting eqs. (4.26-27) into (4.23) one gets

$$(4.28) \quad \overline{M}_R = (M_R)_\lambda \left(\frac{\Delta\omega}{\lambda} \right)^{\beta/2} \left\{ \frac{y^2 + (\Gamma/2)^2}{(\Delta\omega - y)^2 + (\Gamma/2)^2} \right\}^{\frac{\beta_i + 2\beta_{\text{int}}}{4}} \exp \left\{ -i \frac{\beta_i + 2\beta_{\text{int}}}{4} \times \left[\operatorname{arctg} \frac{\Gamma y}{(\Gamma/2)^2 - y^2} - \operatorname{arctg} \frac{\Gamma y}{\Delta\omega^2 - y^2 + (\Gamma/2)^2} \right] \right\}.$$

As in eq. (4.6), the apparent divergence in $(\Delta\omega/\lambda)^{\beta/2}$ is cancelled by an analogous divergence in $(M_R)_\lambda$, coming from the infrared virtual contributions. Moreover all the infrared corrections, in the case of a resonant process, are only those contained in eq. (4.28). In fact for fixed y , if we perform the limit $\Gamma \gg \Delta\omega$, all resonant factors in eq. (4.28) go to 1 and we recover the previous result

$$\overline{M}_R = (M_R)_\lambda (\Delta\omega/\lambda)^{\beta/2}$$

of eq. (4.6), valid in the case of a matrix element which is a smooth function of the energy.

In view of the application to the case of very narrow resonances, as the recently discovered⁽⁶⁾ $\psi(3.1)$ and $\psi(3.7)$, let us consider eq. (4.28) in the limit of $\Gamma \ll \Delta\omega$.

To this purpose we have to specify the λ dependence of the virtual photon effects contained in $(M_R)_\lambda$. This is discussed in the Appendix A. From eq. (4.28) and (A-2), in the limit of $\Gamma \ll \Delta\omega$, we finally have:

$$(4.29) \quad \overline{M}_R \underset{\Delta\omega \gg \Gamma}{=} \frac{1}{\gamma^{\beta_f/2} \Gamma^{(1+\beta_f/2)}} (M_R)_E \left(\frac{\Delta\omega}{E}\right)^{\beta_f/2} \left(\frac{\Gamma/2}{E \sin \delta_R}\right)^{\beta_i/2} \times e^{-\frac{\beta_i}{2} \delta_R \operatorname{ctg} \delta_R} e^{-i \frac{\beta_i}{2} \delta_R},$$

where we have defined $M_R \propto e^{i\delta_R} \sin \delta_R$, with $\sin \delta_R = (\Gamma/2) / \sqrt{y^2 + (\Gamma/2)^2}$, and $[\gamma^{\beta_f/2} \Gamma^{(1+\beta_f/2)}]^{-1}$ is the normalization factor discussed in Appendix B. (eq. B.6). For the radiatively corrected cross section we finally have:

$$(4.30) \quad (d\sigma_R)_{\text{obs}} \underset{\Delta\omega \gg \Gamma}{=} \frac{1}{\gamma^{\beta_f} \Gamma^{(1+\beta_f)}} (d\sigma_R)_0 \left(\frac{\Delta\omega}{E}\right)^{\beta_f} \left(\frac{\Gamma/2}{E \sin \delta_R}\right)^{\beta_i} \times e^{-\beta_i \delta_R \operatorname{ctg} \delta_R} (1 + C_F^{\text{RES}}),$$

where the factor $(1+C_F^{\text{RES}})$ stands, as usual, for the left-over finite corrections.

Let us compare this result with the analogous one obtained in the proceeding section (eq. 3.14). One can see that the factor $1-\beta_i \delta_R \text{ctg} \delta_R$, responsible for the radiative tail of the resonance, appears in exponentiated form. This indicates that this exponential factor, which is also present in eq. (4.28), and blows up for y very large, cannot be taken literally, and has to be considered only in the first order approximation. The physical origin of this unpleasant behaviour is due to the fact that, for very large values of y , the emission of a photon with energy $\omega \simeq y$ is not any-more a soft process and imposes strong constraints on the emission of further photons. Such a correlation is not embodied in the coherent approach which uses soft photons emitted independently. A more detailed discussion can be found in the Appendix B.

Eq. (4.30) has to be therefore transformed into:

$$(4.31) \quad (d\sigma_R)_{\text{obs}} = \frac{1}{\gamma^{\beta_f} \Gamma(1+\beta_f)} (d\sigma_R)_0 \left(\frac{\Delta\omega}{E}\right)^{\beta_f} \left(\frac{\Gamma/2}{E \sin\delta_R}\right)^{\beta_i} \times$$

$$\times (1-\beta_i \delta_R \text{ctg} \delta_R) (1+C_F^{\text{RES}}),$$

and can now be used without any restrictions on y . The same considerations apply of course to eq. (4.28).

Let us discuss further our result. Apart from the radiative tail, eq. (4.31) differs from the Q.E.D. result (eq. 4.9) only in the fact that a "proper" energy loss $(\Gamma/2 \sin \delta_R)$ substitutes the external quantity $\Delta\omega$, as far as the emission from the initial state is concerned. A very narrow resonance, in other words, fixes by itself the amount of maximum energy which can be lost from the initial state, in order to still allow the resonance to be produced.

Finally, let us discuss the distribution function $dP(\omega)$ of the emitted radiation for resonant processes. In the normal case one has, as well known⁽⁴⁾,

$$(4.32) \quad dP(\omega) \approx \left(\frac{\omega}{E}\right)^\beta \frac{d\omega}{\omega}.$$

In the resonant case, a similar distribution has not been introduced explicitly. From eq. (4.28) however the desired answer can be obtained in integrated form. In fact one has:

$$(4.33) \quad (d\sigma_R)_{\text{obs}} = \int_0^{\Delta\omega} \frac{dP_R(\omega)}{d\omega} d\omega (d\sigma_R)_0,$$

and therefore from (4.28) one easily finds:

$$(4.34) \quad dP_R(\omega) \approx \left(\frac{\omega}{E}\right)^\beta \left\{ \beta_f + 2\beta_{\text{int}} \frac{y(y-\omega) + (\Gamma/2)^2}{(y-\omega)^2 + (\Gamma/2)^2} + \beta_i \times \right. \\ \left. \times \frac{y^2 + (\Gamma/2)^2}{(y-\omega)^2 + (\Gamma/2)^2} \right\} \frac{d\omega}{\omega},$$

where the approximate equality is due to a linearization in β_i , β_{int} and β_f performed in the bracket of the r. h. s. Of course, in the limit of very large Γ , this reduces to (4.32). Furthermore, for $\beta_f = \beta_{\text{int}} = 0$, we recover the known result^(5, 9, 15)

$$(4.35) \quad (d\sigma_R)_{\text{obs}} \approx \beta_i \int_0^{\Delta\omega} \frac{d\omega}{\omega} \left(\frac{\omega}{E}\right)^{\beta_i} d\delta_R(W-\omega),$$

whereas for $\beta_{\text{int}} = 0$, we obtain the result derived in ref. (5):

$$(4.36) \quad (d\sigma_R)_{\text{obs}} \approx \int_0^{\Delta\omega} \frac{d\omega}{\omega} \left(\frac{\omega}{E}\right)^{\beta_i + \beta_f} \left\{ \beta_f d\sigma_R(W) + \beta_i d\sigma_R(W-\omega) \right\}.$$

4.3. - Interference of a resonant term with a pure Q. E. D. background.

We would like to discuss now the radiative effects arising from the interference of a resonant term with a pure Q. E. D. term. In other words we have in mind processes like $e^+ e^- \rightarrow e^+ e^- (\mu^+, \mu^-, \dots)$ in which the reaction can proceed via two different dynamics. For these processes, of course, the detailed knowledge of the radiative effects is necessary for those experiments studying for example forward-backward asymmetries, dips, and similar subtle effects.

From the point of view discussed in this section, the procedure to follow is quite clear. One starts with the sum of two well defined and finite matrix elements, \bar{M}_{QED} and \bar{M}_R , as

$$(4.37) \quad \bar{M} = \bar{M}_{\text{QED}} + \bar{M}_R = \langle f | e^{-i\Lambda_c} S_{\text{QED}} | i \rangle + \frac{1}{\sqrt{N}} \langle f | e^{-i\Lambda_R^+} S_R | i \rangle.$$

The observed cross section is then proportional to $|\bar{M}|^2$. While $|\bar{M}_{\text{QED}}|^2 \propto (d\sigma_{\text{QED}})_{\text{obs}}$ and $|\bar{M}_R|^2 \propto (d\sigma_R)_{\text{obs}}$, the interference term comes only from that part of $\langle f | e^{-i\Lambda_c}$ which overlaps with $\langle f | e^{-i\Lambda_R^+} / \sqrt{N}$. One is therefore led to define a final state $e^{i\Lambda_R} |f\rangle / \sqrt{N}$ also for the pure QED part of the interaction. We introduce therefore a matrix element $\bar{M}_{\text{QED}}^{\text{INT}}$ defined as

$$(4.38) \quad \bar{M}_{\text{QED}}^{\text{INT}} = \frac{1}{\sqrt{N}} \langle f | e^{-i\Lambda_R^+} S_{\text{QED}} | i \rangle.$$

All the interference effects will then come from $\text{Re} \left\{ \bar{M}_{\text{QED}}^{\text{INT}*} \bar{M}_R \right\}$, as one can see by comparing with perturbation theory.

Proceeding as in the preceding cases, and using the same notation, we find:

$$(4.39) \quad \bar{M}_{\text{QED}}^{\text{INT}} = (M_{\text{QED}})_\lambda \int_\lambda d^4k \delta(k^2) \theta(k_0) \left[j_\mu^{R^*}(k) j_\mu^c(k) - \frac{1}{2} j_\mu^R(k) j_\mu^{R^*}(k) - \frac{1}{2} i \text{Im} (j_\mu^{R^*}(k) j_\mu^{R^*}(-k)) \right],$$

where $j_\mu^R(k)$ is defined in eq. (4.11) and $j_\mu^C(k) = I_\mu(k) + F_\mu(k)$ is the usual classical current without resonant modifications. $(M_{\text{QED}})_\lambda = \langle f | S_{\text{QED}} | i \rangle_\lambda$, has to be identified with that of eq. (4.6), where the index QED was emitted.

For the first integral in the r. h. s. of eq. (4.39) we easily find:

$$(4.40) \quad I_3 = \int d^4k \delta(k^2) \theta(k_0) [j_\mu^{*R}(k) j_\mu^C(k)] = \\ = \beta \ln \left(\frac{\Delta\omega}{\lambda} \right) - (\beta_i + \beta_{\text{int}}) \ln \frac{y - i\Gamma/2 - \Delta\omega}{y - i\Gamma/2}$$

which, together to eqs. (4.24-25) finally leads to:

$$(4.41) \quad \bar{M}_{\text{QED}}^{\text{INT}} = (M_{\text{QED}})_\lambda \left(\frac{\Delta\omega}{\lambda} \right)^{\beta/2} \left\{ \frac{y^2 + (\Gamma/2)^2}{(\Delta\omega - y)^2 + (\Gamma/2)^2} \right\}^{\beta_i/4} \exp * \\ * \left\{ - \frac{y\beta_i}{\Gamma} \left[\text{arctg} \frac{\Delta\omega - y}{\Gamma/2} + \text{arctg} \frac{y}{\Gamma/2} \right] \right\} \exp \left\{ -i \frac{\beta_i + 2\beta_{\text{int}}}{4} * \right. \\ * \left[\text{arctg} \frac{\Gamma y}{(\Gamma/2)^2 - y^2} - \text{arctg} \frac{y\Gamma}{\Delta\omega^2 - y^2 + (\Gamma/2)^2} \right] - i (\beta_i + \beta_{\text{int}}) \text{arctg} * \\ * \left. \frac{\Delta\omega (\Gamma/2)}{(\Delta\omega - y)y - (\Gamma/2)^2} \right\} .$$

As usual the infrared divergence is automatically cancelled from $(M_{\text{QED}})_\lambda$ and $(\Delta\omega/\lambda)^{\beta/2}$, introducing a factor $(M_{\text{QED}})_E (\Delta\omega/E)^{\beta/2}$.

Interference effects come now from the real part of the product $\bar{M}_R \bar{M}_{\text{QED}}^{\text{INT}*}$, where both quantities are finite. Combining eqs. (4.28) and (4.41) we obtain finally, in the limit of $\Delta\omega \gg \Gamma$,

$$\overline{M}_R \overline{M}_{\text{QED}}^{\text{INT}*} = \left[\gamma^{\beta_f + \beta_{\text{int}}} \Gamma(1 + \beta_f + \beta_{\text{int}})^{-1} \right] (M_R)_E (M_{\text{QED}})_E^* \left(\frac{\Delta\omega}{E} \right)^{\beta_f + \beta_{\text{int}}} \quad (4.42)$$

$$\left(\frac{\Gamma/2}{E \sin \delta_R} e^{-i\delta_R} \right)^{\beta_i},$$

where we have also introduced the normalization factor discussed in the Appendix B. This result again explicitly shows the appearance of a factor $\Gamma/2 \sin \delta_R e^{i\delta_R}$ as the "proper" energy loss from the initial state, playing the symmetrical role of $\Delta\omega$ for final state. Notice also that the exponential factor giving rise to the resonance tail, both common to eqs. (4.28) and (4.41), has completely disappeared.

With the standard definition $M_R \propto \sin \delta_R e^{i\delta_R}$, and taking the real part of eq. (4.42) we finally obtain for the interference cross section:

$$(d\sigma^{\text{INT}})_{\text{obs}} = \frac{1}{\gamma^{\beta_f + \beta_{\text{int}}} \Gamma(1 + \beta_f + \beta_{\text{int}})} (d\sigma^{\text{INT}})_0 \left(\frac{\Delta\omega}{E} \right)^{\beta_f + \beta_{\text{int}}} \times \quad (4.43)$$

$$\left(\frac{\Gamma/2}{E \sin \delta_R} \right)^{\beta_i} \times \left\{ \cos(\delta_R \beta_i) + \text{tg} \delta_R \sin(\delta_R \beta_i) \right\} (1 + C_F^{\text{INT}}),$$

where $(d\sigma^{\text{INT}}) \propto 2\text{Re} \left[(M_{\text{QED}})_0 (M_R)_0^* \right]$ and $(1 + C_F^{\text{INT}})$ is, as usual, the left over finite correction. Eq. (4.43) coincides with the result obtained in the preceding section (eq. 3.18), in the limit $\beta_{\text{int}} = 0$.

Let us summarize the results of this section. By extending the formalism of ref. (8) to the case of resonance production, we have shown that the introduction of coherent states provides a very powerful tool for an intuitive and physically significant description of the infrared problems also when the presence of a very narrow resonance introduces further subtleties in the problem. In particular all the results can

be rigorously proved to all orders in α , without any approximation. They explicitly show that in the presence of a narrow resonance a physical parameter, the width Γ , enters in the description of the radiative effects.

These results can be also easily extended to the case of a general reaction $a + b \rightarrow c + d + \dots$, proceeding through the formation of a resonance.

5. - PHENOMENOLOGICAL APPLICATIONS TO $\psi(3.1)$ AND $\psi'(3.7)$.

Some applications of our formulae to the $\psi(3.1)$ and $\psi'(3.7)$ resonances have already been presented in ref. (7). Here we wish to comment upon several aspects of those applications, and discuss some further ones.

As well known, since the resonances of interest are very narrow, one has to integrate over the machine resolution, which is assumed to be

$$(5.1) \quad G(W'-W) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(W'-W)^2}{2\sigma^2}}$$

where σ is the machine dispersion, such that $(\Delta W)_{\text{FWHM}} = 2.3548 \sigma$.

In the case of hadron production, β_{int} and β_f are negligible and therefore one has, for the experimental observed cross section

$$(5.2) \quad \tilde{\sigma}(W) = \int G(W' - W) dW' \sigma(W'),$$

where

$$(5.3) \quad \sigma(W') = \frac{12\pi}{W'^2} \frac{\Gamma_e \Gamma_h}{\Gamma^2} \sin^2 \delta_R(W') \left\{ \frac{\Gamma}{W' \sin \delta_R(W')} \right\}^{\beta_i} \left\{ 1 - \beta_i \delta_R \text{ctg} \delta_R \right\} \left\{ 1 + C_F^{\text{RES}} \right\},$$

with
$$C_F^{\text{RES}} = \frac{13}{12} \beta_i + \frac{\alpha}{\pi} \left(\frac{\pi^2}{3} - \frac{17}{18} \right).$$

Using the above formulae fits to both Frascati and SPEAR data⁽⁶⁾ have already been presented in (7). As stated there a useful formula for the observed cross section at the peak is

$$(5.4) \quad \tilde{\sigma}(M) = \frac{6\pi^2 \Gamma_e \Gamma_h}{\sqrt{2\pi} \sigma M^2 \Gamma} \left(\frac{\Gamma}{M} \right)^{\beta_i} \exp \left(\frac{\Gamma}{2\sqrt{2}\sigma} \right)^2 \left\{ \operatorname{erfcf} \left(\frac{\Gamma}{2\sqrt{2}\sigma} \right) + \frac{\beta_i}{2} \times \right. \\ \left. \times E_1 \left[\left(\frac{\Gamma^2}{8\sigma^2} \right) \right] \right\} (1 + C_F^{\text{RES}}),$$

which is a straight forward consequence of (5.2-3) with the only approximation of setting $[\sin \delta_R(W')]^{-\beta_i} \approx 1$. In eq. (5.4) the second term in the square bracket represents the contribution from the radiative tail. For resonances whose total width is smaller than the machine resolution, a simple expansion of eq. (5.4) in powers of $(\Gamma/2\sqrt{2}\sigma)$ leads to:

$$(5.5) \quad \tilde{\sigma}(M) = \frac{6\pi^2 \Gamma_e \Gamma_h}{\sqrt{2\pi} \sigma M^2 \Gamma} \left(\frac{\Gamma}{M} \right)^{\beta_i} \left(1 + \frac{\Gamma^2}{8\sigma^2} \right) \left\{ 1 - \frac{\Gamma}{\sqrt{2\pi}\sigma} + \beta_i \times \right. \\ \left. \times \left[\ln \frac{2\sqrt{2}\sigma}{\Gamma} - \gamma/2 \right] \right\} (1 + C_F^{\text{RES}}),$$

where $\gamma = .5772$ is Euler's constant. As explicitly shown in ref. (7), eq. (5.5) gives an excellent estimate (at the level of 2%) of the parameter $(\Gamma_e \Gamma_h / \Gamma)$ once $\tilde{\sigma}(M)$ is given and the machine dispersion σ is fixed.

In place of eq. (5.5) Yennie⁽¹⁵⁾ obtains (for $\psi(3.1)$) for $\tilde{\sigma}(M)$

$$(5.6) \quad \tilde{\sigma}(M) \approx \frac{6\pi^2}{\sqrt{2\pi}\sigma} \frac{\Gamma_e \Gamma_h}{M^2 \Gamma} \left(\frac{2\sqrt{2}\sigma}{M} \right)^{\beta_i} (1 + 0.79 \beta_i).$$

While analytically the two expressions disagree, the numerical estimates for $(\Gamma_e \Gamma_h / \Gamma)$ agree to about 5%.

For $\psi(3.1)$ taking the published value⁽⁶⁾ from SPEAR $\tilde{\sigma}(M) \approx 2.3 \mu\text{b}$ and $\sigma = 1 \text{ MeV}$ we obtain

$$(5.7) \quad \frac{\Gamma_e \Gamma_h}{\Gamma} \simeq (4.0 \pm 0.2) \text{ keV},$$

which is in good agreement with our analysis⁽⁷⁾ of the Frascati data⁽¹⁶⁾.

A similar analysis for $\psi(3.7)$ gives

$$(5.8) \quad \frac{\Gamma_e \Gamma_h}{\Gamma} \simeq (2.8 \pm 0.3) \text{ keV},$$

having used $\tilde{\sigma}(M) \approx .75 \mu\text{b}$ ⁽⁶⁾ and $\sigma = 1.15 \text{ MeV}$ ($\Gamma = 1 \text{ MeV}$) or $\sigma \approx 1.4 \text{ MeV}$ ($\Gamma = 0.2 \text{ MeV}$). It is obvious from our formulae that if the peak values are changed, $\Gamma_e \Gamma_h / \Gamma$ changes almost linearly, for a given σ .

Regarding the leptonic modes the extraction of the dynamical parameters is more complicated since the details of the experimental setups have to be taken into account. As shown in our earlier work⁽⁷⁾ no dips due to interference can arise (upon machine integration) in $e^+ e^- \rightarrow e^+ e^-$. Also no significant charge asymmetry results in $e^+ e^- \rightarrow \mu^+ \mu^-$.

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APPENDIX A - DISCUSSION OF VIRTUAL PHOTON EFFECTS -

We discuss briefly the infrared properties of the amplitude M_λ defined in the text.

For a smooth amplitude, e. g. that for a pure QED process $e^+e^- \rightarrow \mu^+\mu^-$, the infrared dependence of M_λ is well known:

$$(A1) \quad M_\lambda = \left(\frac{\lambda}{E}\right)^{\beta/2} M_E = \left(\frac{\lambda}{E}\right)^{\beta/2} M_0 \sqrt{1+C_F},$$

where M_0 is the lowest order matrix element and C_F denotes finite contributions (vertex corrections, vacuum polarization, ...), which are to be computed perturbatively. E is defined as $\sqrt{s}/2$ and λ is the minimum energy cut-off. If one wishes to perform this calculation in terms of a photon mass λ' , the conversion rules may be found from ref. (3).

Notice however that λ^2 does not always scale with s . For example in electron-electron elastic scattering in certain expressions λ^2 scales with t , where t is the momentum transfer⁽¹⁴⁾.

Now we turn to the case of interest, i. e. e^+e^- annihilation through a resonance. In this case one finds that as far as β_i and β_f parts are concerned λ^2 still scales with s whereas the β_{int} part scales with the resonant propagator. Explicitly:

$$(A.2) \quad (M_R)_\lambda = (M_R)_E \left(\frac{\lambda}{E}\right)^{(\beta_i+\beta_f)/2} \left\{ \frac{\lambda}{\sqrt{y^2+(\Gamma/2)^2}} e^{i\delta_R} \right\}^{\beta_{\text{int}}}$$

where again $(M_R)_E$ differs from the pure Breit-Wigner (M_R) by finite virtual corrections.

We were led to the above form for the β_{int} dependence upon being informed of explicit second order calculations by Altarelli, Ellis and Petronzio⁽¹³⁾.

In the limit of large $\Delta\omega$ (valid for narrow resonances) the β_{int} dependence in (A2) exactly cancels its counter part found from the real photon emission, as given in the text. This demonstration proceeds upon the following recognition.

Following Yennie, Frautschi and Suura⁽³⁾, we define

$$(A3) \quad B_{\text{int}} = \frac{-ie^2}{(2\pi)^3} \int \frac{d^4k}{k^2 - \lambda'^2} \left\{ \frac{(2p_1 - k)_\mu}{2p_1 \cdot k - k^2} - \frac{(2p_2 - k)_\mu}{2p_2 \cdot k - k^2} \right\} \left\{ \frac{(2p_3 - k)_\mu}{2p_3 \cdot k - k^2} - \frac{(2p_4 - k)_\mu}{2p_4 \cdot k - k^2} \right\} S(k)$$

and

$$(A4) \quad \tilde{B}_{\text{int}} = -\pi \int_{k_0 < \Delta\omega} \frac{d^3k}{\sqrt{k^2 + \lambda'^2}} j_\mu^{(i)} j_\mu^{(f)*} S(k)$$

where B_{int} and \tilde{B}_{int} denote the virtual and real soft photons contributions.

$j_\mu^{(i)}$ and $j_\mu^{(f)}$ are the classical currents defined in the text (eq. 4.5) and $S(k)$ is the resonant factor (eq. 4.11). As shown in ref. (3) the infrared contributions in B_{int} arise from the δ -function part of the photon propagator:

$$\frac{1}{k^2 - \lambda'^2 + i\epsilon} = \text{P.V.} \frac{1}{k^2 - \lambda'^2} - i\pi\delta(k^2 - \lambda'^2).$$

Then a straight forward calculation shows that the infrared factors of B_{int} and \tilde{B}_{int} cancel exactly in the limit of large $\Delta\omega$ ($\Delta\omega \gg \Gamma$). This argument can be generalized to all orders. Hence the exponentiated form of the virtual infrared factors is completely determined, (A2), since the corresponding real photon contributions have been explicitly obtained in the text.

The above arguments do not apply to the β_i term, the reason being that the expression similar to (A3) does not have the resonant factor $S(k)$, while the corresponding (A4) does. Thus, the infrared singularity, as $\lambda' \rightarrow 0$, is indeed cancelled (because $S(k) \rightarrow 1$ as $k \rightarrow 0$), but the scale of the virtual contributions is provided by the total energy \sqrt{s} .

APPENDIX B - DISCUSSION OF THE ENERGY CONSERVATION CONSTRAINTS. -

We discuss here the implications coming from the exact conservation of energy in the coherent states method in order to obtain the correct normalization factors as well as to show the correct behaviour of the factor responsible for the radiative tail of the resonance, which in eqs. (4.38-30) appears in exponentiated form.

The exact procedure of imposing the global condition (4.10) on the final states has been discussed in ref. (8). We shall adopt here a slightly different, but completely equivalent procedure, by operating on the various matrix elements.

For the classical current, with no resonance present, one has (see eq. 4.6)

$$\bar{M} = \sum_{n=0}^{\infty} \left\{ \frac{1}{2} \int_{\lambda}^{\Delta\omega} \frac{d^3 k_1}{2k_1} [j_{\mu}(k_1) j_{\mu}^*(k_1)] \dots \int_{\lambda}^{\Delta\omega} \frac{d^3 k_n}{2k_n} \right. \\ \left. x [(j_{\mu}(k_n) j_{\mu}^*(k_n))] \right\}^n \langle f | S | i \rangle_{\lambda}, \quad (\text{B1})$$

i.e. $k_1, \dots, k_n \leq \Delta\omega$, when the energy is not fully conserved. Energy conservation can be achieved by introducing a factor

$$I(\Delta\omega) = \begin{cases} 1 & \text{for } \sum k_n \leq \Delta\omega \\ 0 & \text{for } \sum k_n > \Delta\omega \end{cases}$$

which leads to:

$$(B.2) \quad \bar{M}^{\text{cons}} = \frac{1}{2\pi} \int_{-\Delta\omega}^{\Delta\omega} dx \int_{-\infty}^{\infty} e^{-i\sigma x} d\sigma \exp \left\{ \frac{1}{2} \int_{\lambda}^{\infty} \frac{d^3 k}{2k} [j_{\mu}(k) j_{\mu}^*(k)] e^{i\sigma k} \right\} \langle f | S | i \rangle_{\lambda}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\sigma}{\sigma} \sin(\sigma \Delta\omega) \exp \left\{ \frac{\beta}{2} \int_{\lambda}^{\infty} \frac{dk}{k} e^{i\sigma k} \right\} \langle f | S | i \rangle_{\lambda}$$

and finally to:

$$(B3) \quad \bar{M}^{\text{cons}} = \left(\frac{\Delta\omega}{\lambda}\right)^{\beta/2} \frac{1}{\gamma^{\beta/2}} \frac{1}{\Gamma(1+\beta/2)} M_{\lambda} = \left(\frac{\Delta\omega}{E}\right)^{\beta/2} \frac{1}{\gamma^{\beta/2}} \frac{1}{\Gamma(1+\beta/2)} M_E,$$

with the same notations of equations (4.7) and (4.9). Because of the smallness of β one has $[\Gamma(1+\beta/2)]^2 \approx \Gamma(1+\beta)$, and therefore a small correction factor $[\gamma^{\beta} \Gamma(1+\beta)]^{-1}$ which properly normalizes the cross section. As discussed in the text, the smallness of the correction factor reflects the small probability of emitting two or more photons which combine to give a total energy loss $\Delta\omega$.

When there is a resonance present, the same procedure immediately leads to

$$(B4) \quad \bar{M}_R^{\text{cons}} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\sigma}{\sigma} \sin(\sigma \Delta\omega) \exp \left\{ \frac{1}{2} \int_{\lambda}^{\infty} \frac{d^3 k}{2k} j_{\mu}(k) j_{\mu}^*(k) e^{i\sigma k} \right\} (M_R)_{\lambda},$$

where $j_{\mu}(k) \equiv j_{\mu}^R(k)$, and we have dropped for simplicity the phase factor in eq. (4.23). One has:

$$(B5) \quad \int_{\lambda}^{\infty} \frac{d^3 k}{2k} j_{\mu}(k) j_{\mu}^*(k) e^{i\sigma k} = \beta E_1(-i\lambda\sigma) - \frac{i\gamma}{\Gamma} \beta_i \left\{ e^{i\sigma(y+i\Gamma/2)} E_1[i\sigma(y+i\Gamma/2)] - e^{i\sigma(y-i\Gamma/2)} E_1[i\sigma(y-i\Gamma/2)] \right\} - \frac{\beta_i + 2\beta_{\text{int}}}{2} \left\{ e^{i\sigma(y+i\Gamma/2)} E_1[i\sigma(y+i\Gamma/2)] + e^{i\sigma(y-i\Gamma/2)} E_1[i\sigma(y-i\Gamma/2)] \right\}.$$

In view of the complexity of eq. (B5), it is not possible to perform analytically the extra integration in eq. (B4), for any $\Delta\omega$, Γ and y . However, for very narrow resonances, a compact answer is fortunately achieved by expanding the r.h. s. of (B5) in terms of $(y+i\Gamma/2)/\Delta\omega$.

One finally obtains:

$$(B6) \quad \bar{M}_R^{\text{cons}} = \frac{1}{\gamma^{\beta_f/2}} \frac{1}{\Gamma(1+\beta_f/2)} \left(\frac{\Delta\omega}{\lambda}\right)^{\beta_f/2} \left\{ \frac{\sqrt{y^2+(\Gamma/2)^2}}{\lambda} e^{-i\delta_R} \right\}^{\beta_i/2 + \beta_{\text{int}}} \times e^{-\frac{\beta_i \delta_R}{2} \text{ctg} \delta_R} (M_R) \lambda,$$

which coincide with eq. (4.29) apart from the normalization factor $\gamma^{\beta_f/2} \Gamma(1+\beta_f/2)$.

In the same limit of a very narrow resonance, one obtains a similar result for $(\bar{M}_{\text{QED}}^{\text{INT}})^{\text{cons}}$. Explicitly we find:

$$(B7) \quad (\bar{M}_{\text{QED}}^{\text{INT}})^{\text{cons}} = \frac{1}{\gamma^{\beta_f/2 + \beta_{\text{int}}}} \frac{1}{\Gamma(1+\beta_f/2 + \beta_{\text{int}})} \left(\frac{\Delta\omega}{\lambda}\right)^{\beta_f/2 + \beta_{\text{int}}} \left\{ \frac{\sqrt{y^2+(\Gamma/2)^2}}{\lambda} e^{i\delta_R} \right\}^{\beta_i/2} e^{\beta_i/2 \delta_R \text{ctg} \delta_R} (M_{\text{QED}}) \lambda$$

From the above results it therefore follows that the normalization factors depend upon the same coefficients which appear as powers of $\Delta\omega$. This is very satisfactory from a physical point of view, because of the probabilistic meaning associated with those factors.

Finally let's turn to the problem of the factor responsible for the radiative tail of the resonance, which appears in eqs. (B6) and (4.28-30) in exponentiated form, which in the perturbative approach (see eq. 3.14) has the correct $1/y$ behaviour. It is easy to convince

ourself that this feature only depends upon the expansion used in the r.h.s. of eq. (B5), to obtain eq. (B6). By performing the limit y in (B5), for example, the correct $1/y$ behaviour can be shown to arise naturally.

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