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LNF-75/4(P)
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B. D'Ettorre Piazzoli, G. Mannocchi, S. Melone, P. Picchi and R. Visentin: APERTURE AND COUNTING RATE OF RECTANGULAR TELESCOPES FOR SINGLE AND MULTIPLE PARALLEL PARTICLES. -

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B. D'Ettorre Piazzoli[x], G. Mannocchi[x], S. Melone[o], P. Picchi and R. Visentin: APERTURE AND COUNTING RATE OF RECTANGULAR TELESCOPES FOR SINGLE AND MULTIPLE PARALLEL PARTICLES. -

INTRODUCTION. -

Since 1971 two spark chamber telescopes with rectangular cross section have been operated in the Mt. Cappuccini Station (Torino) at a mean depth of 58 hg/cm^2 s. r. The triggering modes are organized to detect groups of penetrating particles. A similar set up is now operating also in the Mt. Blanc Station (vertical depth 4300 hg/cm^2 s. r.) with the aim to measure stopping and throughgoing muons[1]. In both cases we have the problem to relate the counting rate to the geometry of the apparatus and to the parameters which characterize the radiation.

The present work is divided in two parts. In the first part (I) we show that the evaluation of the aperture for single particles of rectangular telescopes may be reduced to a double integration. The second part (II) provides formulas - used by our group[2] - for calculation of the absolute intensities of events of a given multiplicity, recorded under some typical triggering conditions by an arrangement of one or more telescopes.

I. 1. - Aperture for single particles. -

At first we consider the aperture of a rectangular telescope for single particle (Fig. 1). For the optical apparatus - like ours - fiducial marks provide to define the useful volume (acceptance volume) smaller than the telescope volume; this procedure allows to eliminate edge effects. The counting rate is[3]

(x) Laboratorio di Cosmogeofisica del CNR (Torino)
(o) Istituto di Fisica - Università di Ancona

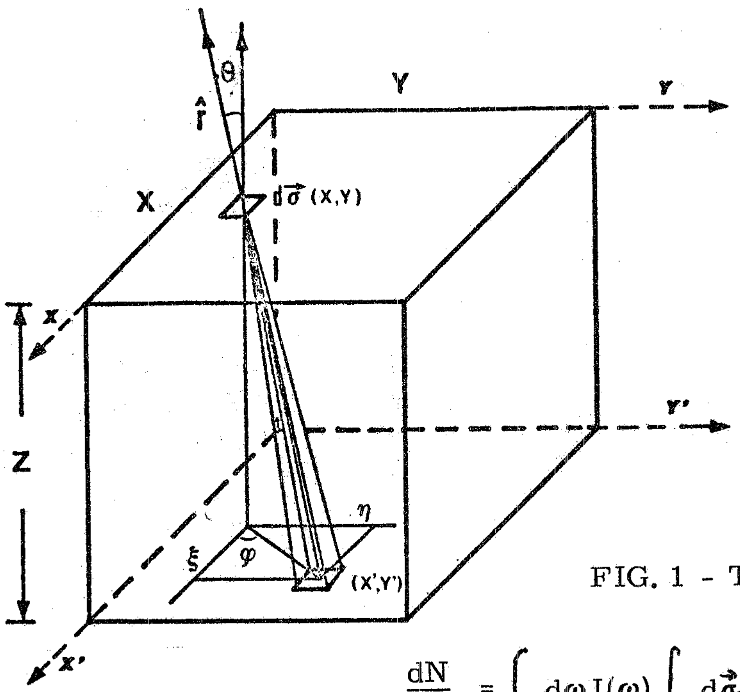

FIG. 1 - Telescope geometry.

$$\frac{dN}{dt} = \int_{\Omega} d\omega\, I(\omega) \int_S d\vec{\sigma}\cdot\hat{r}$$

where

$I(\omega)$ = angular distribution of the detected particles ($cm^{-2}\ sr^{-1}\ s^{-1}$)
$d\omega$ = $d\cos\theta\, d\varphi$ = element of solid angle (θ polar, φ azimuthal angle)
$d\vec{\sigma}\cdot\hat{r}$ = effective element of area looking into ω
S, Ω = total area and solid angle defined by telescope geometry.

By assuming the usual zenith angle distribution $I(\omega) = I_o \cos^n\theta$ we have

$$\frac{dN}{dt} = I_o A_n$$

where

$$A_n = \int_{\Omega} \cos^n\theta\, d\omega \int_S d\vec{\sigma}\cdot\hat{r} = \frac{1}{Z^2}\int_0^X \int_0^X \int_0^Y \int_0^Y \cos^{n+4}\theta\, dx\, dx'\, dy\, dy'$$

is the so-called telescope aperture. The last expression is obtained under assumption of rectangular telescopes of dimensions X, Y, Z. It follows from the relations

$$d\vec{\sigma}\cdot\hat{r} = \cos\theta\, d\sigma = \cos\theta\, dx\, dy$$
$$d\omega = \frac{\cos^3\theta}{Z^2}\, dx'\, dy'$$

with

$$\cos\theta = \frac{Z}{\left[Z^2 + (x-x')^2 + (y-y')^2\right]^{1/2}}$$

Here we point out the possibility to reduce the evaluation of A_n to a double integration by defining the directional response function[3, 4]

$$S(\omega) = \int_S d\vec{\sigma} \cdot \hat{r}$$

This is the available area perpendicular to the direction of the particle $\omega(\theta, \varphi)$. With the definitions of Fig. 1 one has:

(1a)
$$S(\omega) = \cos\theta \left[X - \left|Z \operatorname{tg}\theta \cos\varphi\right|\right] \cdot \left[Y - \left|Z \operatorname{tg}\theta \sin\varphi\right|\right]$$

Applying the transformation

(1b)
$$\xi = Z \operatorname{tg}\theta \cos\varphi \qquad -X \leqslant \xi \leqslant X$$
$$\eta = Z \operatorname{tg}\theta \sin\varphi \qquad -Y \leqslant \xi \leqslant Y$$

with

(1c)
$$\cos\theta = \frac{Z}{\left[Z^2 + \xi^2 + \eta^2\right]^{1/2}}$$
$$d\omega = \frac{\cos^3\theta}{Z^2} \, d\xi \, d\eta$$

we find the expression for the counting rate of particles in the direction ω:

$$\frac{dN}{d\omega dt} = I_o \cos^{n+1}\theta \left[X - \left|Z \operatorname{tg}\theta \cos\varphi\right|\right] \cdot \left[Y - \left|Z \operatorname{tg}\theta \sin\varphi\right|\right]$$

and the aperture

$$A_n = \int_\Omega \cos^n\theta \, S(\omega) \, d\omega = \frac{1}{Z^2} \int_{-X}^{+X} \int_{-Y}^{+Y} \cos^{n+4}\theta \left[X - |\xi|\right]\left[Y - |\eta|\right] d\xi \, d\eta$$

$$= \frac{4}{Z^2} \int_0^X \int_0^Y \cos^{n+4}\theta (X - \xi)(Y - \eta) \, d\xi \, d\eta$$

$$= 4Z^{n+2} \int_0^X \int_0^Y \frac{(X - \xi)(Y - \eta)}{\left[Z^2 + \xi^2 + \eta^2\right]^{(n+4)/2}} \, d\xi \, d\eta$$

where we have used the fact that the integrand is an even function of ξ and η. The explicit solutions of Eq. (2) in the cases n = 0, 1, 2, 3, 4, are

$$A_o = Z^2 \log \frac{R_x^2 R_y^2}{R^2 Z^2} - 2Z \left[X \,\mathrm{arctg}\, (\frac{X}{Z}) + Y \,\mathrm{arctg}\, (\frac{Y}{Z}) \right] + 2YR_x \,\mathrm{arctg}\, (\frac{Y}{R_x}) +$$

$$+ 2XR_y \,\mathrm{arctg}\, (\frac{X}{R_y}).$$

$$A_1 = \frac{4}{3} Z (R - R_x - R_y + Z) - \frac{2}{3} XY \left[\arcsin (\frac{Z^2 - X^2}{R_x^2} - \frac{2Z^2X^2}{(X^2+Y^2)R_x^2}) + \right.$$

$$\left. + \arcsin (\frac{Z^2 - Y^2}{R_y^2} - \frac{2Z^2Y^2}{(X^2+Y^2)R_y^2}) \right]$$

$$A_2 = \frac{1}{2} \left\{ (R_y^2 + Y^2) \frac{X}{R_y} \,\mathrm{arctg}\, (\frac{X}{R_y}) + (R_x^2 + X^2) \frac{Y}{R_x} \,\mathrm{arctg}\, (\frac{Y}{R_x}) - \right.$$

$$\left. - Z \left[X \,\mathrm{arctg}\, (\frac{X}{Z}) + Y \,\mathrm{arctg}\, (\frac{Y}{Z}) \right] \right\}$$

$$A_3 = \frac{4Z^5}{15} \left\{ \frac{1}{R^3} - \frac{1}{R_x^3} - \frac{1}{R_y^3} + \frac{1}{Z^3} - X^2 \left[\frac{1}{Z^2 R_x} (\frac{1}{R_x^2} + \frac{2}{Z^2}) - \frac{1}{R_y^2 R} (\frac{1}{R^2} + \frac{2}{R_y^2}) \right] - \right.$$

$$\left. - Y^2 \left[\frac{1}{Z^2 R_y} (\frac{1}{R_y^2} + \frac{2}{Z^2}) - \frac{1}{R_x^2 R} (\frac{1}{R^2} + \frac{2}{R_x^2}) \right] \right\} -$$

$$- \frac{2}{5} \left\{ XY \left[\arcsin (1 - \frac{2X^2R^2}{R_x^2(X^2 + Y^2)}) + \arcsin (1 - \frac{2Y^2R^2}{R_y^2(X^2 + Y^2})) \right] + \right.$$

$$+ \frac{Z^3X^2Y^2}{R} \left(\frac{1}{R_x^2} \left[\frac{X^2 - 3R_x^2}{R_x^2 Z^2} - \frac{1}{3} (\frac{2}{R^2} + \frac{1}{R_x^2}) \right] + \frac{1}{R_y^2} \cdot \right.$$

$$\left. \left. \cdot \left[\frac{Y^2 - 3R_y^2}{R_y^2 Z^2} - \frac{1}{3} (\frac{2}{R^2} + \frac{1}{R_y^2}) \right] \right) \right\}.$$

$$A_4 = \frac{Z^6}{6}\left\{\left(\frac{1}{R^4} - \frac{1}{R_x^4} - \frac{1}{R_y^4} - \frac{1}{Z^4}\right) - X\left[\frac{3}{2}\left(\frac{1}{Z^5}\,\mathrm{arctg}\,\frac{X}{Z} - \frac{1}{R_y^5}\,\mathrm{arctg}\,\frac{X}{R_y}\right) + \right.\right.$$

$$\left. + \frac{X}{Z^2}\left(\frac{1}{R_x^4} + \frac{3}{2Z^2R_x^2}\right) - \frac{X}{R_y^2}\left(\frac{1}{R^4} + \frac{3}{2R_y^2R^2}\right)\right] -$$

$$- Y\left[\frac{3}{2}\left(\frac{1}{Z^5}\,\mathrm{arctg}\,\frac{Y}{Z} - \frac{1}{R_x^5}\,\mathrm{arctg}\,\frac{Y}{R_x}\right) + \frac{Y}{Z^2}\left(\frac{1}{R_y^4} + \frac{3}{2Z^2R_y^2}\right) - \right.$$

$$\left.\left. - \frac{Y}{R_x^2}\left(\frac{1}{R^4} + \frac{3}{2R_x^2R^2}\right)\right]\right\} + \frac{Z^2}{6}\,XY\left[\frac{X(8R_x^4 + 7Z^2R_x^2 - 3Z^2X^2)}{2Z^2R_x^5}\,\mathrm{arctg}\,\frac{Y}{R_x} + \right.$$

$$+ Y\,\frac{(8R_y^4 + 7Z^2R_y^2 - 3Z^2Y^2)}{2Z^2R_y^5}\,\mathrm{arctg}\,\frac{X}{R_y} +$$

$$\left. + \frac{XY\,R_x^2R_y^2(R_y^2 + R_x^2)(2Z^2 + 7R^2) - 3XY\,R^2\,(X^2R_y^4 + Y^2R_x^4)}{2R_x^4R_y^4R^4}\right]$$

where

$$R_x^2 = X^2 + Z^2$$
$$R_y^2 = Y^2 + Z^2$$
$$R^2 = X^2 + Y^2 + Z^2$$

II. - APERTURE FOR PARALLEL PARTICLES. -

We report a technique to evaluate the counting rate of rectangular telescopes for parallel particles. The detection of the EAS muon component is the task of the apparatus in the Mt. Cappuccini Station[1,2]. It consists of two spark chamber telescopes, parallelepiped shaped; they allow the reconstruction in the real space of the trajectories with an angular resolution $< .5^{o}$ and, consequently, the selection of the events of parallel penetrating particles (muons). Some well-defined triggering conditions allow to record groups of muons with a multiplicity $i \geq 2$.

The absolute rates of different multiplicities are sensitive to the features of the shower, like size and radial distributions, as well as to the geometry of the apparatus. That has to be taken into account when we compare the results of different authors.

At first we consider the case of neighbouring detecting areas so that their distance from the shower axis may be considered the same. Then the counting rate for i particles from direction ω is given by

$$\text{(3)} \qquad \frac{dN(i)}{d\omega dt} = \int_0^\infty 2\pi \, RdR \int_{n_{min}}^{n_{max}} dn \, f(n) \cos^\varrho \theta \, \frac{dP_i}{d\omega} (R, n, \omega)$$

where $f(n, \omega) = f(n) \cos^\varrho \theta$ is the frequency of showers with size n to the angle ω and $(dP_i/d\omega)(R, n, \omega)$ is the probability to detect i of the n particles of a shower whose axis falls at a distance R from the detector array. In writing (3) the integration on R is carried out over the shower front following the hypothesis of neighbouring detectors. Thus the effective radial muon distribution is independent of the azimuthal angles. This is not the case of detectors scattered on a large area which we shall consider later.

In the evaluation of the probability function enters the lateral structure function $\varrho(R, n)$ normalized as

$$\int_0^\infty \varrho(R, n) \, 2\pi \, RdR = 1$$

which gives the fractional density of the considered EAS component at a distance R from the axis. The distance R is unambiguously defined if the variation of $\varrho(R, n)$ within the effective detecting area is negligible. The essence of the problem is the computation of $dP_i/d\omega$. This probability function depends on

a) the statistical fluctuations of the lateral density
b) the geometry of the apparatus
c) the triggering requirements and acceptance criteria.

As to the first point we use the binomial statistics, assuming no correlation between the n particles of the same shower. This should be a good approximation to the real situation. Hence we have

$$\text{(4)} \qquad \frac{dP_i}{d\omega} (R, n, \omega) = \binom{n}{i} \left[\varrho(R, n) \, \Gamma(\omega) \right]^i \left[1 - \varrho(R, n) \, \Gamma(\omega) \right]^{n-i}$$

for a single telescope of the effective area $\Gamma(\omega)$ in the direction $\omega(\theta, \varphi)$. In more complicated cases, as considered in the following, we will use the multinomial distribution. As to the points b) and c) we shall take into account the geometry of the apparatus and the triggering conditions used at the Mt. Cappuccini and Mt. Blanc Station as well as the acceptance criteria applied by us in the analysis of data. These assumptions define the value of the effective area $\Gamma(\omega)$. We shall consider the response of a single telescope as well as the coincidence counting rate for two contiguous telescopes.

The rate is obtained by integrating on ω Eq. (3) with the help of the transformations (1)

$$\frac{dN}{dt}(i) = \int_0^\infty 2\pi R dR \int_{n_{min}}^{n_{max}} dn \ f(n) \int_\Omega \cos^\varrho \theta \frac{dP_i}{d\omega} d\omega$$

(5)

$$= \int_0^\infty 2\pi \, R dR \int_{n_{min}}^{n_{max}} dn \ f(n) 4Z^{\varrho+1} \int_0^{\xi_M} \int_0^{\eta_M} \left(\frac{dP_i}{d\omega}\right) \frac{d\xi \, d\eta}{(Z^2+\xi^2+\eta^2)^{(\varrho+3)/2}}$$

where now the probability function is expressed in terms of the (cartesian) coordinates ξ and η. Z is the height of the single telescope or of one of the two telescopes, and ξ_M, η_M are the maximum allowed values of ξ, η.

II. 1. - Single Telescope. -

Let us consider first the response of a spark chamber (or other visualizing device) telescope with two rectangular detector C_1 and C_2. There detectors and the fiducial marks (see Fig. 2) define the useful volume $V(X, Y, Z)$. The response depends on the acceptance criteria adopted to select the events; we apply the two criteria such that, i) the apparatus has a full efficiency for any configuration of particles which satisfies the appropriate conditions and, ii) that the aperture can be accurately calculated. The first criterion is that of "strict acceptance" which demands that a particle in order to be considered should traverse the whole useful volume of the telescope. Hence the particles accepted cross the counters C_1, C_2 and all the K spark chambers so that the effective area is $S(\omega)$, Eq. (1a). Thus the rate

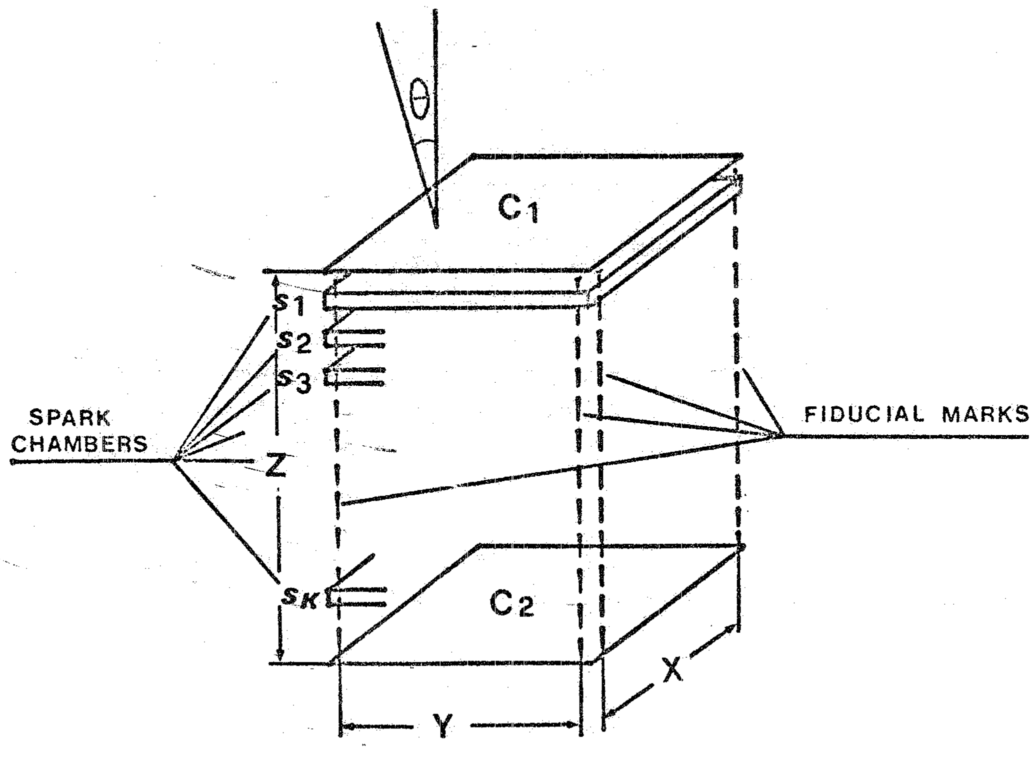

FIG. 2 - A typical spark chamber telescope.

$\frac{dN}{dt}(i)$ for such events is given by Eq. (5) with the probability function (4) where $\Gamma(\omega) = S(\omega)$, while $\xi_M = X$, $\eta_M = Y$. The second criterion is that of "large acceptance" which is obtained retaining the above conditions for at least one particle and allowing the other remaining particles to traverse also partially the useful volume but to cross at least $m \leqslant k$ spark chambers. A possible situation is schematically shown in Fig. 3 for a projected view. The useful area $A(\omega)$ for these particles is greater than $S(\omega)$ and is given by

$$A(\omega) = A(\xi, \eta) = \cos\theta \left[X + \Delta h \frac{\xi}{Z} \right] \left[Y + \Delta h \frac{\eta}{Z} \right]$$

with

$$\Delta h = h_{k-m+1} - h_m$$

where h_n is the distance from the top counter of the n-th spark chamber (downward numbered).

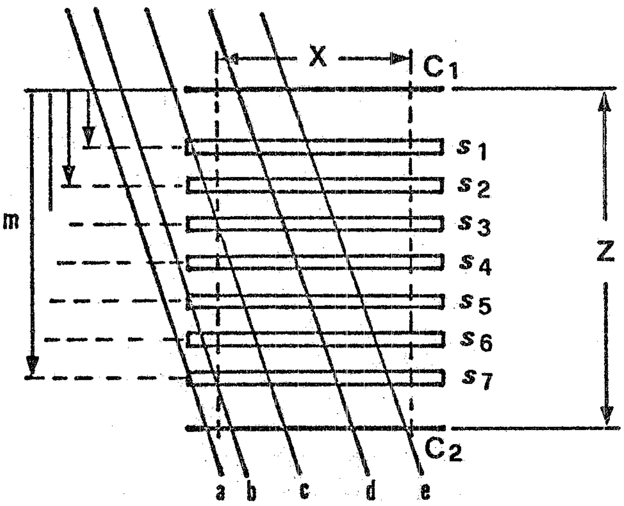

FIG. 3 - Classification of a typical event (projected view). Strict acceptance criterion: only the particles d and e are accepted. Large acceptance criterion with m=3: also the particle c is counted.

The function $A(\omega)$ is a generalization of $S(\omega)$: in fact the application of the strict acceptance criterion gives $A(\omega) = S(\omega)$ because requires, by definition, $\Delta h = -Z$.

Thus the probability function for i particles with $j \geqslant 1$ on the area $S(\omega)$ and the other $k = i-j$ over the remaining area $A(\omega) - S(\omega)$ is given by

$$\frac{dP_i}{d\omega}(R,n,\omega) = \sum_{\substack{j+k=i \\ j \geqslant 1}} \frac{dP_{jk}}{d\omega}(R,n,\omega)$$

where the sum is performed over all the allowed configurations, and

$$\frac{dP_{jk}}{d\omega}(R,n,\omega) = \frac{n!}{j!\,k!\,(n-j-k)!}\left[\varrho(R,n)S(\omega)\right]^j \left[\varrho(R,n)(A(\omega)-S(\omega))\right]^k \left[1 - \varrho(R,n)A(\omega)\right]^{n-j-k}$$

Performing the summation one obtains:

$$\frac{dP_i}{d\omega}(R,n,\omega) = \binom{n}{i}\varrho^i(R,n)\left\{A^i(\omega) - \left[A(\omega)-S(\omega)\right]^i\right\}\left[1-\varrho(R,n)A(\omega)\right]^{n-i} \tag{6}$$

The limit of Eq. (6) for $A(\omega) \rightarrow S(\omega)$ is, of course, Eq. (4). The integration limits for ξ and η are again X and Y.

II. 2. - More Sophisticated Triggers. -

In order to avoid, especially at shallow depths, the high rate from single particles, a higher coincidence is required. At the Mt. Capuccini station the trigger, for each telescope, is provided by a fourfold coincidence between the counters C_1, C_2, C_3, C_4 (Fig. 4). Such an arrangement defines, besides $V(X,Y,Z)$, also the useful volumes $V_1 = V_2(X/2, Y, Z)$ and the corresponding effective areas

$$S_1(\omega) = S_2(\omega) = \cos\theta(X/2 - \xi)(Y - \eta)$$

where the maximum values for ξ and η are

$$\begin{aligned} \xi_M &= X/2 \\ \eta_M &= Y \end{aligned} \tag{7}$$

The triggering conditions and the criterion of strict acceptance requires $j \geqslant 1$ particles crossing C_1 and C_3, $k \geqslant 1$ crossing C_2 and C_4 - to satisfy the trigger - and the other $l = i-j-k$ through any pair of upper and lower counters. Thus the probability function for an event of multiplicity i is given by

$$\frac{dP_i}{d\omega}(R,n,\omega) = \sum_{\substack{m+l=i \\ m \geqslant 2}} \sum_{\substack{j+k=m \\ j,k \geqslant 1}} \frac{dP_{jkl}}{d\omega}(R,n,\omega)$$

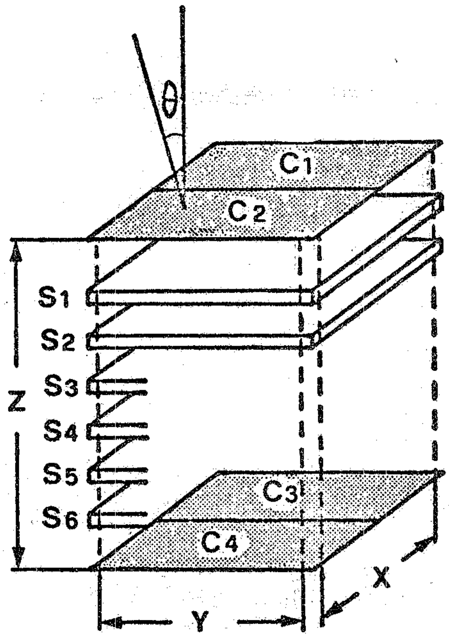

FIG. 4 - The schematic structure of the telescopes in the Mt. Cappuccini Station.

where

$$\frac{dP_{jkl}}{d\omega}(R,n,\omega) = \frac{n!}{j!k!l!(n-j-k-l)!}\left[\varrho(R,n)S_1(\omega)\right]^j\left[\varrho(R,n)S_2(\omega)\right]^k$$

$$\left[\varrho(R,n)\left[S(\omega)-(S_1(\omega)+S_2(\omega))\right]\right]^l\left[1-\varrho(R,n)S(\omega)\right]^{n-j-k-l}$$

We find - for $S_1(\omega) = S_2(\omega)$

$$\frac{dP_i}{d\omega}(R,n,\omega) = \binom{n}{i}\varrho^i(R,n)\left\{S^i(\omega)+\left[S(\omega)-2S_1(\omega)\right]^i-2\left[S(\omega)-S_1(\omega)\right]^i\right\}\left[1-\varrho(R,n)S(\omega)\right]^{n-i} \qquad (8)$$

The rate for i particles is again obtained inserting the expression (8) in Eq. (5) with the integration limits given by Eq. (7). The application of the criterion of large acceptance - that is $l = i-j-k$ particles over the area $A(\omega)$ - demands the substitution $S(\omega) \rightarrow A(\omega)$.

II. 3. - Two Telescopes. -

We refer to two telescopes which are parallel each other but not necessarily identical. As it has been mentioned previously, we suppose the relative distance so small that the value of $\varrho(R,n)$ may be considered constant on the whole area but such that the acceptance areas are not overlapping. A typical arrangement is sketched in Fig. 5. The trigger is given by a coincidence $C_1 \cdot C_2 \cdot C_3 \cdot C_4$. The useful volumes, $V(X, Y, Z)$ and $V'(X', Y', Z')$, defines the acceptance areas

$$S(\omega) = \cos\theta\,(X - \xi)(Y - \eta)$$

$$S'(\omega) = \cos\theta\,(X' - \xi')(Y' - \eta') = \cos\theta\,(X' - \xi \frac{Z'}{Z})(Y' - \eta \frac{Z'}{Z})$$

where the maximum values for ξ and η are

$$\xi_M = \min\left\{X, X' \frac{Z}{Z'}\right\}$$

(9)

$$\eta_M = \min\left\{Y, Y' \frac{Z}{Z'}\right\}$$

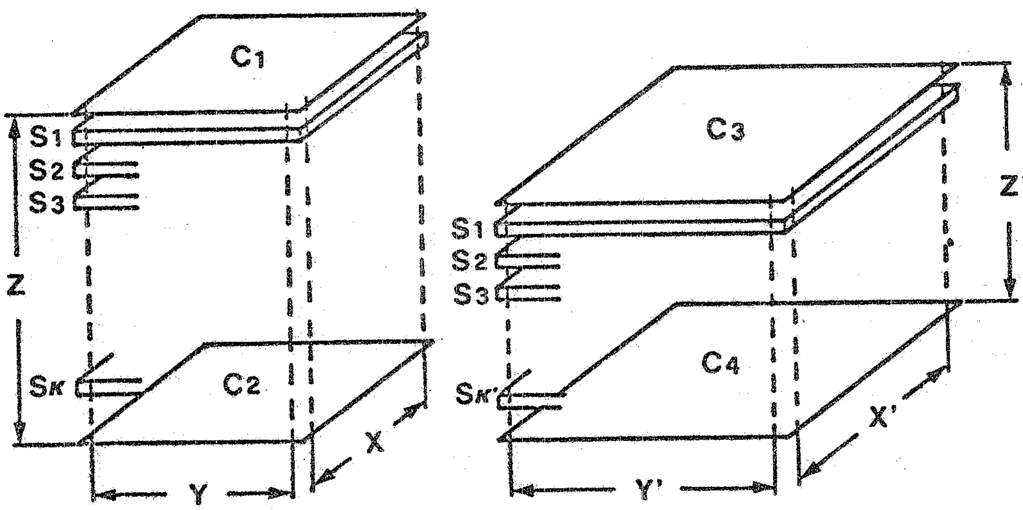

FIG. 5 - Geometry for two neighbouring telescopes in coincidence.

The criterion of strict acceptance demands $j \geqslant 1$ particles in V and $k \geqslant 1$ in V', so that we have for a coincidence event of multiplicity i

$$\frac{dP_i}{d\omega}(R,n,\omega) = \sum_{\substack{j+k=i \\ j,k \geqslant 1}} \frac{dP_{jk}}{d\omega}(R,n,\omega)$$

with

$$\frac{dP_{jk}}{d\omega}(R,n,\omega) = \frac{n!}{j!k!(n-j-k)!}\left[\varrho(R,n)S(\omega)\right]^j\left[\varrho(R,n)S'(\omega)\right]^k$$
$$\left[1-\varrho(R,n)(S(\omega)+S'(\omega))\right]^{n-j-k}$$

The result:

$$\frac{dP_i}{d\omega}(R,n,\omega) = \binom{n}{i}\varrho^i(R,n)\left[(S(\omega)+S'(\omega))^i - S^i(\omega) - S'^i(\omega)\right]$$
$$\left[1-\varrho(R,n)(S(\omega)+S'(\omega))\right]^{n-i}$$

is symmetric respect to $S(\omega)$ and $S'(\omega)$. The integration limits for ξ and η in Eq. (5) are defined by Eq. (9). For two identical telescopes, $S(\omega) = S'(\omega)$, the result simplifies as follows:

$$\frac{dP_i}{d\omega}(R,n,\omega) = (2^i-2)\binom{n}{i}\left[\varrho(R,n)S(\omega)\right]^i\left[1-2\,\varrho(R,n)S(\omega)\right]^{n-i}$$

and $\xi_M = X$, $\eta_M = Y$.

Similarly to the previous cases the criterion of large acceptance is satisfied whenever $a \geqslant 1$ particles traverse S_1, $b \geqslant 1$ S_2 - and the other particles are distributed over the remaining useful area, that is r on $A(\omega) - S(\omega)$ and s on $A'(\omega) - S'(\omega)$. The definition of the large acceptance areas is the usual one:

$$A(\omega) = \cos\theta\left(X+\Delta h\frac{\xi}{Z}\right)\left(Y+\Delta h\frac{\eta}{Z}\right)$$

$$A'(\omega) = \cos\theta\left(X'+\Delta h'\frac{\xi'}{Z'}\right)\left(Y'+\Delta h'\frac{\eta'}{Z'}\right) = \cos\theta\left(X'+\Delta h'\frac{\xi}{Z}\right)\left(Y'+\Delta h'\frac{\eta}{Z}\right)$$

We have thus:

$$\frac{dP_i}{d\omega}(R,n,\omega) = \sum_{\substack{j+k=i \\ j\geqslant 1 \\ k\geqslant 1}}\ \sum_{\substack{a+r=j \\ a\geqslant 1}}\ \sum_{\substack{b+s=k \\ b\geqslant 1}} \frac{dP_{abrs}}{d\omega}(R,n,\omega)$$

where

$$\frac{dP_{abrs}}{d\omega}(R,n,\omega) = \frac{n!}{a!b!r!s!(n-a-b-r-s)!}\left[\varrho(R,n)S(\omega)\right]^{a}\left[\varrho(R,n)S'(\omega)\right]^{b}$$

$$\left[\varrho(R,n)(A(\omega)-S(\omega))\right]^{r}\left[\varrho(R,n)(A'(\omega)-S'(\omega))\right]^{s}$$

$$\left[1-\varrho(R,n)(A(\omega)+A'(\omega))\right]^{n-(a+b+r+s)}$$

We find

$$\frac{dP_i}{d\omega}(R,n,\omega) = \binom{n}{i}\varrho^{i}(R,n)\left\{\left[A(\omega)+A'(\omega)\right]^{i}-\left[A(\omega)+A'(\omega)-S(\omega)\right]^{i}-\right.$$

$$\left.-\left[A(\omega)+A'(\omega)-S'(\omega)\right]^{i}+\left[A(\omega)-S(\omega)+A'(\omega)-S'(\omega)\right]^{i}\right\}\left[1-\varrho(R,n)\,(A(\omega)+A'(\omega))\right]^{n-i}$$

The integration limits for ξ and η are again given by the relation (9).

The limit for identical telescopes - with the same spark chamber disposition, so that $\Delta h = \Delta h'$ - is easily obtained putting $S(\omega)=S'(\omega)$ and $A(\omega)=A'(\omega)$. Moreover, the limits $A(\omega)\to S(\omega)$ and $A'(\omega)\to S(\omega)$ give the strict acceptance expressions.

II. 4. - The General Case. -

The more general case is that of many detectors arranged over a large area to sample the shower at different distances from the axis. At first our calculation refers, like in previous sections, to rectangular telescopes parallel each other whose acceptance areas are not overlapping.

Now the geometrical disposition is such that the azimuthal symmetry is lost even if the muon radial distribution retains it on the front of the shower. Thus we must relate the telescope distances in the laboratory to the effective distances on the shower front. Referring to Fig. 6 we define:

R, ϕ = polar coordinates which single out on the horizontal plane the impact point A of the axis of a shower from the direction $\omega(\theta, \varphi)$. It is useful to connect the reference system to a telescope;

R_j, ϕ_j = polar coordinates of the other telescopes on the horizontal plane;

r_j', α_j = polar coordinates of the telescopes around A on the horizontal plane. The azimuthal angle is measured from the vertical plane containing the shower axis. They are given by

$$r'_j = \left[R^2 + R_j^2 - 2R\,R_j \cos(\phi - \phi_j)\right]^{1/2}$$

$$\alpha_j = \varphi - \beta_j \pmod{\pi}$$

with

$$\beta_j = \operatorname{arctg} \frac{R_j \cos\phi_j - R\cos\phi}{R_j \sin\phi_j - R\sin\phi} \qquad j \neq 0$$

$$\beta_o = \phi \quad \text{for the "reference telescope"}$$

Then the effective distance from the axis of the particles striking the i-th telescope is

$$r_j = r'_j\, f_j\,(\theta, \alpha_j)$$

$$f_j(\theta, \alpha_j) = \sqrt{\cos^2\theta + \sin^2\alpha_j \sin^2\theta}$$

which is a function of R, ϕ, θ, φ.

The counting rate of events recorded at the angle $\omega(\theta, \varphi)$ is given by

$$(10) \qquad \frac{dN}{d\omega\, dt} = \int_o^{\infty} R\cdot dR \int_o^{2\pi} d\phi \int_{n_{min}}^{n_{max}} f(n)\, dn \cos^{\varrho+1}\theta \,\frac{dP}{d\omega}(r_j, n, \omega)$$

where $f(n)\cos^{\varrho}\theta\, dn\, d\phi\, R\, dR \cos\theta$ is the flux of shower (coming from the direction $\omega(\theta, \varphi)$), with the size between n and n+dn, and the axis which falls in the area $d\phi\, R\, dR$. The probability function $(dP/d\omega)$ is defined by the particular configuration of the event. It is not difficult to show that for one or more neighbouring telescopes so that all r_j can be considered identical, one obtains from here Eq. (3) (see Appendix).

The integral counting rate is

$$\frac{dN}{dt} = \int_o^{\infty} R dR \int_o^{2\pi} d\phi \int_{n_{min}}^{n_{max}} f(n)\, dn\, Z^{\varrho+2} \int_{-\xi_M}^{+\xi_M} \int_{-\eta_M}^{+\eta_M} \left(\frac{dP}{d\omega}\right) \frac{d\xi\, d\eta}{\left[Z^2 + \xi^2 + \eta^2\right]^{(\varrho+4)/2}}$$

where Z is the height of the reference telescope and ξ_M, η_M are defined as in Eq. (9).

For instance we consider the coincidence of i muons on $V_o(X, Y, Z)$, 1 on $V_1(X_1, Y_1, Z_1)$ and k on $V_2(X_2, Y_2, Z_2)$, such events being now recorded in the Mt. Cappuccini Station extended by adding a third telescope about 30 meters distant from the principal array(1). We have for the probability function

$$\frac{dP_{ilk}}{d\omega}(r_j, n, \omega) = \frac{n!}{i!1!k!(n-i-1-k)!}\left[\varrho_o(R,n)S_o(\omega)\right]^i\left[\varrho_1(R,n)S_1(\omega)\right]^1$$

$$\left[\varrho_2(R,n)S_2(\omega)\right]^k\left[1 - \varrho_o(R,n)S_o(\omega) - \varrho_1(R,n)S_1(\omega) - \varrho_2(R,n)S_2(\omega)\right]^{n-i-1-k}$$

with

$$S_o(\omega) = \cos\theta\,(X - |\xi|)(Y - |\eta|)$$
$$S_1(\omega) = \cos\theta\,(X_1 - |\xi|\,(Z_1/Z)(Y_1 - |\eta|\,(Z_1/Z))$$
$$S_2(\ \ ) = \cos\theta\,(X_2 - |\xi|\,(Z_2/Z)(Y_2 - |\eta|\,(Z_2/Z))$$
$$\varrho_j(R,n) = \varrho\,(r_j, n)$$

where r_j depend on ξ and η through the azimuthal angle φ (and zenith θ), being

$$\varphi = \mathrm{arctg}\,(\eta/\xi)$$

The integration limits are

$$\xi_M = \min\left\{X, X_1\frac{Z}{Z_1}, X_2\frac{Z}{Z_2}\right\}$$

$$\eta_M = \min\left\{Y, Y_1\frac{Z}{Z_1}, Y_2\frac{Z}{Z2}\right\}$$

Relaxing the condition of telescopes with parallel side we add only a little complication. For a fourth telescope $Y_3(X_3, Y_3, Z_3)$ turned of an angle γ - which may be always chosen between 0 and $\pi/2$ as shown in Fig. 6 - the effective area is

$$S_3(\omega) = \cos\theta(X_3 - |\xi\cos\gamma + \eta\sin\gamma|\frac{Z_3}{Z})(Y_3 - |\eta\cos\gamma - \xi\sin\gamma|\frac{Z_3}{Z})$$

while the angular integration becomes

$$\int_{-\eta_M}^{\eta_M} d\eta \int_{\xi_{min}(\eta)}^{\xi_{max}(\eta)} d\xi$$

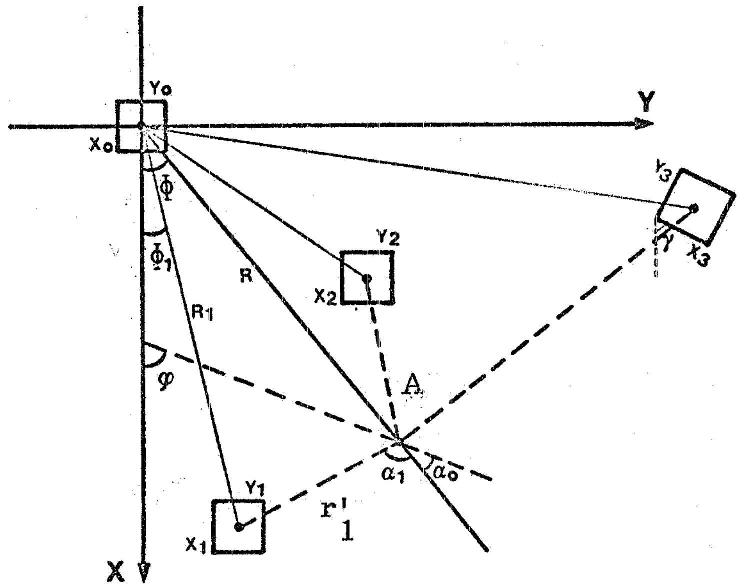

FIG. 6 - Geometric parameters in the general case of many telescopes on a large area (top view).

with

$$\eta_M = \min \left\{ Y, Y_1 \frac{Z}{Z_1}, \; Y_2 \frac{Z}{Z_2}, \; (X_3 \sin\gamma + Y_3 \cos\gamma) \frac{Z}{Z_3} \right\}$$

$$\xi_{min}(\eta) = \max \left\{ -X, \; -X_1 \frac{Z}{Z_1}, \; -X_2 \frac{Z}{Z_2}, \; \xi_{3m} \right\}$$

$$\xi_{max}(\eta) = \min \left\{ X, \; X_1 \frac{Z}{Z_1}, \; X_2 \frac{Z}{Z_2}, \; \xi_{3M} \right\}$$

where

$$\xi_{3m}(\eta) = \max \left\{ \frac{-\eta \sin\gamma - X_3 \, Z/Z_3}{\cos\gamma}, \; \frac{\eta\cos\gamma - Y_3 \, Z/Z_3}{\sin\gamma} \right\}$$

$$\xi_{3M}(\eta) = \min \left\{ \frac{-\eta \sin\gamma + X_3 \ Z/Z_3}{\cos\gamma} , \frac{\eta \cos\gamma + Y_3 \ Z/Z_3}{\sin\gamma} \right\}$$

A similar procedure can be easily applied to any number of telescopes.

CONCLUSIONS. -

We have developed a general approach to calculate, for rectangular telescopes, the absolute frequency of particles by using the shower size distribution $f(n, \omega)$, the lateral structure function $\varrho(R, n)$ and a reasonable assumption about the statistical fluctuations. The geometry of the detecting arrangement is a crucial factor in determining the absolute frequencies and our general method allow a clear comparison between the results of different apparatus in terms of the aforementioned parameters provided that well defined acceptance criteria are adopted.

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APPENDIX. -

Let the m telescopes be neighbouring so that $r_o = r_1 = \ldots r_m$. Then

$$\frac{dP}{d\omega}(r_o, r_1 \ldots r_m; n, \omega) = \frac{dP}{d\omega}(r_o, n, \omega) = \frac{dP}{d\omega}(R \cdot f_o(\theta, \alpha_o); n, \omega)$$

We obtain for Eq. (10)

$$\frac{dN}{d\omega dt} = \int_{n_{min}}^{n_{max}} f(n)\, dn \int_o^{2\pi} d\phi \int_o^{\infty} R\, dR \frac{dP}{d\omega}(R \cdot f_o(\theta, \alpha_o);\ n, \omega) \cos^{\varrho+1}\theta$$

$$= \int_{n_{min}}^{n_{max}} f(n)\, dn \int_o^{2\pi} \frac{d\phi}{f_o^2(\theta, \alpha_o)} \int_o^{\infty} R\, dR \frac{dP}{d\omega}(R, n, \omega) \cos^{\varrho+1}\theta$$

$$= \int_{n_{min}}^{n_{max}} f(n)\, dn\, 2\pi \int_o^{\infty} R\, dR \frac{dP}{d\omega}(R, n, \omega) \cos^{\varrho}\theta \equiv \text{Eq. (3)}$$

since

$$\int_o^{2\pi} \frac{d\phi}{f_o^2(\theta, \alpha_o)} = \frac{2\pi}{\cos\theta}$$