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Positivity restriction on anomalous dimensions

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Arguments of positivity and conformal invariance in the Gell-Mann–Low limit imply positive anomalies γ_N [γ_N = (scale dimension)$-2-N$] for the operators $O_{\alpha_1\cdots\alpha_N}$ ($N \neq 0$) which occur in the light-cone expansion of two currents.

It is well known that the softness of the trace of the energy-momentum tensor implies that the Gell-Mann–Low limit[1] of most renormalizable theories is not only scale- but also conformal-invariant.[2] It is generally accepted that such a limit corresponds to a bona fide field theory,[3] the asymptotic theory.[4] We shall here derive, by a very simple procedure, general constraints on the scale dimensions of the symmetric traceless tensors $O_{\alpha_1\cdots\alpha_N}(x)$ (which in particular occur in the light-cone expansion) from the positivity conditions of the asymptotic theory. Such scale dimensions are known to govern the short-distance behavior of the Green functions of the massive theory.[5]

Let us consider the two-point functions

$$W_{\alpha_1\cdots\alpha_N;\beta_1\cdots\beta_N}(x) = \langle 0|O_{\alpha_1\cdots\alpha_N}(x)O_{\beta_1\cdots\beta_N}(0)|0\rangle, \quad (1)$$

where $O_{\alpha_1\cdots\alpha_N}(x)$ are irreducible conformal tensors (i.e., Lorentz tensors $(\frac{1}{2}N, \frac{1}{2}N)$ satisfying $[O_{\alpha_1\cdots\alpha_N}(0), K_\lambda]=0$, where K_λ generates the special conformal transformations) of twist (= scale dimension minus spin) $\tau_N = d_N - N$ ($N \geq 1$, d_N = scale dimension). Conformal symmetry gives uniquely[6]

$$W_{\alpha_1\cdots\alpha_N;\beta_1\cdots\beta_N}(x) = (\text{const})\,(x^2)^{-d_N} S M_{\alpha_1\beta_1}(x)\cdots M_{\alpha_N\beta_N}(x) - (\text{traces}), \quad (2)$$

where S indicates symmetrization and

$$M_{\alpha\beta}(x) = 2\frac{x_\alpha x_\beta}{x^2} - g_{\alpha\beta}\,.$$

We perform the decomposition

$$W_{\alpha_1\cdots\alpha_N;\beta_1\cdots\beta_N}(x) = \sum_{k=0}^{N} w_k(\tau_N) \times W^{(N-k)}_{\alpha_1\cdots\alpha_N;\beta_1\cdots\beta_N}(x), \quad (3)$$

where $W^{(N-k)}_{\alpha_1\cdots\beta_N}(x)$ are homogeneous distributions satisfying

$$\partial_{\alpha_1}\cdots\partial_{\alpha_{k+1}} W^{(N-k)}_{\alpha_1\cdots\alpha_N;\beta_1\cdots\beta_N}(x) = 0\,, \quad 0 \leq k \leq N-1 \quad (4)$$

(i.e., they contain spin values down to $N-k$) and $w_k(\tau_N)$ are coefficients which will be studied in the following.

The spin structure of the homogeneous distributions $W^{(N-k)}_{\alpha_1\cdots\beta_N}(x)$ implies that they are positive distributions for $\tau_N \geq 1-k$. A necessary condition for positivity from Eq. (2) is then $\tau_N \geq 1$. Such a weaker condition can be obtained from scale invariance alone.[7]

To investigate further the restrictions from positivity we study the zeros of $w_k(\tau_N)$ in Eq. (3). At such zeros the kth divergence of $O_{\alpha_1\cdots\alpha_N}(x)$ vanishes:

$$\partial_{\alpha_1}\cdots\partial_{\alpha_k} O_{\alpha_1\cdots\alpha_N}(x) = 0\,. \quad (5)$$

Eq. (5) is satisfied at those τ_N such that[8]

$$[\partial_{\alpha_1}\cdots\partial_{\alpha_k} O_{\alpha_1\cdots\alpha_k}(0),\ K_\lambda] = 0 \quad (6)$$

It then follows that for both the quadratic and quartic Casimir operators C_{I}, C_{III}, on the conformal representation[8] of $O_{\alpha_1\cdots\alpha_N}$, one has

$$C_{\mathrm{I,III}}(d_N, N) = C_{\mathrm{I,III}}(d_N+k, N-k) \quad (7)$$

(we recall that C_{II} vanishes on the representation[8]). Equation (7) has the only solution[9]

$$\tau_N = 3-k\,. \tag{8}$$

We note that the locations of such "conformal" zeros are above those of the "kinematical" zeros (at $\tau_N \leq 1-k$) due to the divergences of the Fourier transforms of the distributions $W^{(N-k)}_{\alpha_1\cdots\beta_N}(x)$. For $W^{(N-k)}_{\alpha_1\cdots\beta_N}(x)$ such that its Fourier transform is a homogeneous distribution with an over-all normalization independent of τ_N, one has that $w_{N-k}(\tau_N)$ must be proportional to

$$(\tau_N-2)[(\tau_N-2+k)\Gamma(\tau_N-1)]^{-1}\,. \tag{9}$$

For $\tau_N \geq 2$, $w_{N-k}(\tau_N) \geq 0$, whereas for $\tau_N < 2$, $w_{N-k}(\tau_N) < 0$. A necessary and sufficient condition for positivity is then $\tau_N \geq 2$.[10,11] Since none of the coefficients $w_{N-k}(\tau_N)$ change sign for $\tau_N \geq 2$, positivity is ensured by positivity at $\tau_N = 2, 3, 4, \ldots$, which correspond to the case of free fields. In conclusion we have shown that the operators $O_{\alpha_1\cdots\alpha_N}(x)$ $(N \geq 1)$, which occur in the light-cone expansion of two electromagnetic and weak currents, have scale dimensions[12,13] $d_N \geq 2+N$ (provided conformal symmetry and positivity hold in the asymptotic theory[14]), implying that [with exclusion of the scalar contribution $O(x)$ to the operator expansion] $j_\mu(x)j_\nu(0)$ is never more singular, at $x^2=0$, and the Green's function in Eq. (2) never less singular than in the free theory.

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[1]M. Gell-Mann and F. Low, Phys. Rev. 95, 1300 (1954).

[2]B. Schroer, Nuovo Cimento Lett. 2, 867 (1971); G. Parisi, Phys. Lett. 39B, 643 (1972). Unfortunately, such proofs do not hold for gauge theories.

[3]See for instance the contributions by B. Schroer and G. Mack in *Scale and Conformal Symmetry in Hadron Physics*, edited by R. Gatto (Wiley, New York, 1973).

[4]Also called skeleton theory by Wilson [K. Wilson, Phys. Rev. 179, 1499 (1969)].

[5]C. G. Callan, Phys. Rev. D 2, 1541 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970).

[6]S. Ferrara, R. Gatto, A. F. Grillo, and G. Parisi, Nuovo Cimento Lett. 4, 115 (1972), where a result by Schreier [E. J. Schreier, Phys. Rev. D 3, 980 (1971)] for vector currents is generalized.

[7]The structure of $W^{(N-k)}_{\alpha_1\cdots\alpha_N;\beta_1\cdots\beta_N}(x)$ is

$$[S(\partial_{\alpha_1}\cdots\partial_{\alpha_{N-k}}\partial_{\beta_1}\cdots\partial_{\beta_{N-k}}g_{\alpha_{N-k+1}\beta_{N-k+1}}\cdots g_{\alpha_N\beta_N} - \text{traces})]\,(1/x^2)^{\gamma_N+k}$$

(S stands for symmetrization), so it is a positive distribution for $\tau_N + k \geq 1$. In this connection see R. J. Crewther, Sun-Sheng Shei, and Tung-Mow Yan, Phys. Rev. D 8, 3396 (1973).

[8]S. Ferrara *et al*. [in *Scale and Conformal Symmetry in Hadron Physics*, edited by R. Gatto (Ref. 3), Eq. (6)] express the fact that the transversality condition (5) must be a conformal-invariant statement. As $\partial_{\alpha_1}\cdots\partial_{\alpha_k}O_{\alpha_1\cdots\alpha_N}(x)$ is an operator of order $N-k$ and dimension d_N+k belonging to the same representation of $O_{\alpha_1\cdots\alpha_N}(x)$, Eq. (7) follows.

[9]Equation (8) can also be deduced, equally simply, from the manifestly conformal-covariant formalism [see Eq. (41) in Ref. 8, p. 106].

[10]A weaker result has been recently obtained by Migdal [A. A. Migdal, Landau Institute report, 1972 (unpublished)], who has been able to prove positivity for $\tau_N = 2$ and $\tau_N \geq 3$.

[11]Related conclusions are also obtained from different assumptions and procedures by W. Rühl, Commun. Math. Phys. 30, 287 (1973).

[12]We have checked that this condition is verified in all the available models which lead to explicit expressions for the anomalous dimensions [for a general review and references see K. G. Wilson and J. B. Kogut, lecture notes, Institute for Advanced Study, Princeton, 1972 (unpublished)].

[13]It is of interest to note that as a consequence of our result the contribution of the isovector second-rank tensor to the Cottingham formula for the p-n electromagnetic mass difference has an unambiguous sign which agrees with the choice by A. Bietti and G. Parisi [Phys. Lett. 43B, 207 (1973)].

[14]We remark that our results do not imply nor are implied by the positivity restrictions which follow from the positivity of the structure functions in deep-inelastic e-p scattering. The latter in fact implies $\tau_{N+2} \geq \tau_N (N \geq 2)$, which corresponds to our result only if one assumes $\tau_2 \geq 2$. This would be true only for the isoscalar part of the operator expansion of currents under the assumption that $O^2_{\alpha_1\alpha_2}(x)$ is the stress tensor. However, our result is more general as it does not need to be related to operator expansions. In this connection see O. Nachtmann, Nucl. Phys. B63, 237 (1973).