To be submitted to
Jnl. of Math. Phys.

COMITATO NAZIONALE PER L'ENERGIA NUCLEARE
Laboratori Nazionali di Frascati

LNF-74/39(P)
10 Luglio 1974

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ABSTRACT. -

An N-body system with interparticle forces that are attractive at short range collapses in the limit $N \to \infty$, namely in this limit the ground-state energy per particle diverges to negative infinity. If instead the forces are sufficiently repulsive at short range, in the limit $N \to \infty$ saturation occurs, leading to a finite ground-state energy per particle ε. This quantity depends, among other things, on the "coupling constant" g (entering as a factor that multiplies the interaction), and the above remark clearly implies that it is defined in the $N \to \infty$ limit only for positive values of g (although, for finite N, it is defined both for positive and negative g). This fact is generally taken to imply that the function $\varepsilon(g)$ is nonalatic in g at $g = 0$; and therefore that the perturbative expansion of $\varepsilon(g)$, being a power expansion in g, is necessarily nonconvergent (although it might be asymptotic).

The purpose of this paper is to demonstrate the lack of cogency of this argument. It is therefore concluded that nonconvergence of the perturbative expansion for $\varepsilon(g)$ is thus far an unproven hypothesis. The lack of cogency of analogous current arguments concerning the equilibrium density of many-body systems is also pointed out.

Let us consider the N-body system characterized by the hamiltonian

$$H = (2m)^{-1} \sum_{i=0}^{N} p_i^2 + g \sum_{i>j=1}^{N} v(r_{ij}), \tag{1}$$

where of course $\vec{p}_i = -i\hbar\nabla_i$ and $r_{ij} = |\vec{r}_i - \vec{r}_j|$. Let $E(N;\rho, g)$ be the ground-state energy of this system when confined within a container of volume $V = N/\rho$, namely the lowest eigenvalue of the equations

$$H\,\chi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = E(N;\rho, g)\,\chi(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N) \;, \tag{2}$$

$$\chi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = 0 \text{ if } \vec{r}_1 \text{ or } \vec{r}_2 \text{ or } \ldots \vec{r}_N \text{ on surface } S \text{ of volume } V \,. \tag{3}$$

Under these conditions the quantity

$$\rho = N/V \tag{4}$$

represents the mean particle density of the system in the container; this quantity coincides with the actual density of the system only if this fills uniformly the whole container.

If the potential $v(r)$ is integrable at infinity and is sufficiently repulsive (positive) at short range, as we assume hereafter, for $g>0$ the system exhibits saturation(1), namely the ("thermodynamical", i. e. macroscopic) limit

$$\varepsilon(\rho, g) = \lim_{N\to\infty} \left[E(N;\rho, g)/N\right], \qquad g > 0 \;, \tag{5}$$

exists and is finite. This limiting value represents the energy per particle in a macroscopic sample; it is certainly negative if the system is self-bound (as we assume hereafter(2)) and if its equilibrium (average) density $\bar{\rho}(g)$ exceeds the mean density in the container ρ, so that the (macroscopic) system occupies only part of the (macroscopic) container. Moreover, under such conditions, the energy per particle is clearly independent of ρ. If instead $\rho > \bar{\rho}(g)$, the system is compressed within the container, and therefore $\varepsilon(\rho, g)$ does depend on ρ, being presumably an increasing function of ρ. Thus, as a function of ρ, $\varepsilon(\rho, g)$ has the behavior indicated in Fig. 1[3]. The quantity

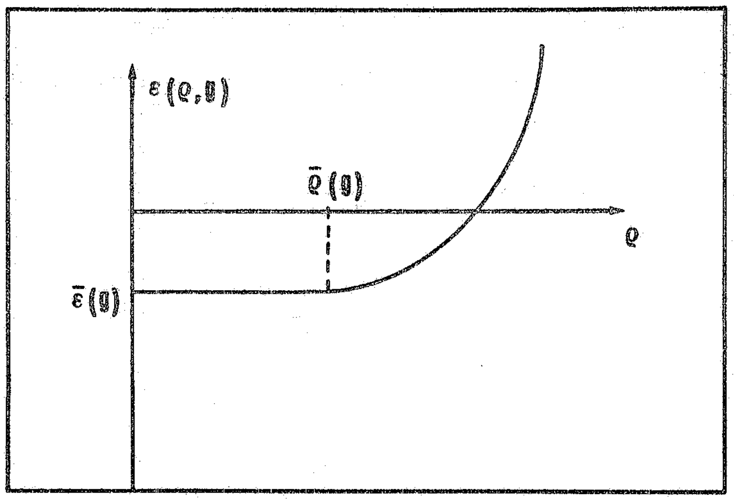

FIG. 1 - Qualitative behavior of the energy per particle $\varepsilon(\rho, g)$ in a macroscopic system with saturating forces $(g>0)$, as a function of the mean density ρ in the container. For additional explanations,see text.

$\bar{\varepsilon}(g)$ (see Fig. 1) is then the energy per particle in a self-bound system of macroscopic size, and $\bar{\rho}(g)$ (see Fig. 1) is the corresponding density (or average density, if the equilibrium configuration is not homogeneous, as is for instance the case for crystals). A basic task of many-body theory is the evaluation of $\bar{\varepsilon}(g)$ and $\bar{\rho}(g)$.

In principle the quantities $\bar{\varepsilon}(g)$ and $\bar{\rho}(g)$ could be defined without any reference to a container; one should take the $N \to \infty$ limit in the problem characterized by the N-body Schroedinger equation (2), with no boundary conditions besides the asymptotic restrictions implied by the requirement that the many-body wave function χ be normalizable (in the center-of-mass frame). This is of course equivalent to the treatment outlined above, but with $\rho = 0$. There are, however, two reasons why the consideration of a container is useful, even though the corresponding problem in somewhat more complex, as indicated by the dependence of $\varepsilon(\rho, g)$ upon ρ, that is clearly not analytic at $\rho = \bar{\rho}(g)$ (see Fig. 1).

The first reason is connected with the use of a perturbative approach; the presence of a container avoids the technical complication that would otherwise be associated with the continuous spectrum of the kinetic energy part of the hamiltonian. Indeed, for regular potentials[4], $E(N; \rho, g)$ is then an analytic function of g at $g = 0$, so that the power expansion

$$E(N; \rho, g) = \sum_{n=0}^{\infty} E_n(N; \rho)\, g^n \tag{6}$$

has a nonvanishing radius of convergence (for N and V finite). Thus, for sufficiently weak forces, perturbation theory is then convergent - although the radius of convergence might tend to zero as N tends to infinity (see below).

A second, and related, reason, has to do with the nonexistence, in the absence of a container, of any normalizable eigenstate of H for very small values of g; while to discuss the analyticity in g at $g = 0$, it is of course just on such values of g that attention is focussed[5].

For $g < 0$, the interaction $gv(r)$ becomes attractive at short range, and as a consequence the system, in the thermodynamical limit, collapses[6], namely

$$\left[E(N; \rho, g)/N\right] \xrightarrow[N \to \infty]{} -\infty, \qquad g < 0 . \tag{7}$$

This fact is generally taken to imply that the function $\varepsilon(\rho, g)$ of eq. (5) is not analytic in g at $g = 0$, and therefore that any perturbative approach to the computation of $\varepsilon(\rho, g)$ (and, a fortiori, $\bar{\varepsilon}(g)$ and $\bar{\rho}(g)$) involves the handling of nonconvergent (although possibly asymptotic) expansions. The purpose of this communication is to point out that such conclusion lacks altogether cogency. We also note that, in spite of the nonanalytic ρ dependence of $\varepsilon(\rho, g)$ displayed in Fig. 1, the functions

$\bar{\varepsilon}(g)$ and $\bar{\rho}(g)$, namely the quantities of physical interest, might well be analytic (and even entire), and so might be the function $\bar{\varepsilon}(\rho, g)$ defined to coincide with $\varepsilon(\rho, g)$, eq. (5), for $g > 0$ and $\rho > \bar{\rho}(g)$, and by analytic continuation for other values of g and ρ; of course the values of $\bar{\varepsilon}(\rho, g)$ when g and ρ do not satisfy the above inequalities would then have no direct physical interpretation.

The simplest way to prove our point is to display a counterexample, namely to produce an explicit function $E(N; \rho, g)$ (artificially concocted), that does have the properties (5) (with $\varepsilon(\rho, g)$ having the behavior shown in Fig. 1), (6) and (7), and yet yields functions $\bar{\varepsilon}(\rho, g)$, $\bar{\varepsilon}(g)$ and $\bar{\rho}(g)$ that are analytic (indeed entire). Having in mind a detailed recent analysis, in the framework of the many-fermion problem, of this question[7], in which the conclusion that we consider unwarranted plays a crucial role, we require moreover that the function $E(N;\rho, g)$ have all the additional properties, that have been shown to hold in this case. This allows to conclude, also in context of the physically most interesting case, that nonconvergence of the perturbative expansion for the energy per particle, and the density, of the ground state of an N-body system in the $N \to \infty$ limit, is thus far an unproven conjecture.

These additional properties are, in the first place, the finiteness, in the $N \to \infty$ limit, of the coefficients $E_n(N, \rho)$ of eq. (6) divided by N:

$$\lim_{N \to \infty} \left[E_n(N, \rho)/N \right] = \tilde{\varepsilon}_n(\rho) \quad . \tag{8}$$

This property is a precondition for raising the issue of the convergence of the expansion

$$\sum_{n=0}^{\infty} \tilde{\varepsilon}_n(\rho) g^n = \tilde{\varepsilon}(\rho, g) \quad . \tag{9}$$

As we shall presently show, it may happen that this series converges; and the function $\tilde{\varepsilon}(\rho, g)$ it defines may (but need not) coincide with the function $\bar{\varepsilon}(\rho, g)$ defined above.

The second additional property of the coefficients $E_n(N, \rho)$ is validity of the bound[8]

$$\left| E_n(N, \rho) \right| < \Gamma N n! (\tilde{\Lambda})^n \qquad (10)$$

with Γ and $\tilde{\Lambda}$ independent of N.

Consider now the example

$$E(N;\rho, g) = -(1+z) N g^2 \left\{ g^2 - (\rho - g)^2 / \left[1 + \exp(-N + N^2(g-\rho)) \right] \right\} +$$
$$+ z N g^2 \left\{ g^2 - (\rho - g)^2 / \left[1 + \exp(-N + N^2(g-\rho)) \right] \right\} / \left[1 + \exp(N - gN^2) \right] -$$
$$- g^2 N^2 / \left[1 + \exp(N + gN^2) \right] . \qquad (11)$$

It is easily seen that it satisfies the inequality (10) and the condition (7), that it yields through (5)

$$\varepsilon(\rho, g) = g^2 \rho(\rho - 2g) \qquad \text{for } \rho \geq \bar{\rho}(g) = g , \qquad (12a)$$

$$\varepsilon(\rho, g) = -g^4 = \bar{\varepsilon}(g) \qquad \text{for } \rho \leq \bar{\rho}(g) = g , \qquad (12b)$$

and that the coefficients $\tilde{\varepsilon}_n(\rho)$ of eq. (8) are finite and yield, through eq. (9),

$$\tilde{\varepsilon}(\rho, g) = (1 - z) g^2 \rho(\rho - 2g) . \qquad (13)$$

Note that the function $\tilde{\varepsilon}(\rho, g)$ is entire and coincides, if $z = 0$, with the function $\varepsilon(\rho, g)$ for $\rho \geq \bar{\rho}(g) = g > 0$ (so that $\tilde{\varepsilon}(\rho, g) = \bar{\varepsilon}(\rho, g)$). Indeed we have constructed our example so that, in this case ($z = 0$), $\bar{\varepsilon}(g) = -g^4$ and $\bar{\rho}(g) = g$ could be recovered from $\tilde{\varepsilon}(\rho, g)$ by applying the usual prescriptions, i.e. minimizing with respect to ρ.

Although we have built into the expression (11) of $E(N; \rho, g)$ some properties that are presumably also valid in realistic cases, it should be emphasized that the only purpose of this counterexample is to invalidate the claim that, using only the information discussed above, it is possible to prove that the energy per particle of the N-body system in the $N \to \infty$ limit is nonanalytic in the coupling constant g[7]. The question of plausibility - wheter it is likely that in realistic cases one or the other possibility prevail - is outside of the scope of the present paper[9]. It is, however, worth calling attention to the mechanism whereby, in the example given here, singularities of the N-body system ground-state energy $E(N;\varrho, g)$ approach, as N increases, the origin of the complex g plane, and indeed accumulate there (and also on the positive real axis, in the complex ϱ plane, at the point corresponding to the actual density $\bar{\varrho}(g)$ of the macroscopic system), but disappear in the $N \to \infty$ limit[10]. In the example displayed these singularities are of polar type[11]; it is clear that analogous examples with branch points instead of poles could be invented just as easily.

Acknowledgments

One of us (FC) would like to thank professor G.A. Baker, Jr. for extensive correspondence on the subject of this paper.

Footnotes and References. -

(1) -For conditions on the potential function $v(r)$ that are sufficient to guarantee saturation, see: D. Ruelle, Statistical Mechanics - Rigorous Results (Benjamin, New York, 1969); F. Calogero, Yu. A. Simonov and E. L. Surkov, Phys. Rev. 5, 1493 (1972). Conditions that are necessary for saturation are also given in Ruelle's book; see also the review paper by F. Calogero and Yu. A. Simonov in The Nuclear Many-Body Problem (F. Calogero and C. Ciofi degli Atti, eds.) (Editrice Compositori, Bologna, 1974), vol. I, p. 51; and the literature quoted there.

(2) - We are therefore assuming the potential $v(r)$ to be repulsive at short range but attractive at longer range, as it is actually the case in most problems of physical interest.

(3) - There might be breaks in the rising part of the curve, connected with phase transitions between different spatial arrangements; but these phenomena are outside of the scope of our discussion.

(4) - The standard definition of a regular potential $v(r)$ requires that it be finite-valued for $r > 0$, less singular than r^{-2} at the origin ($\lim_{r \to 0} r^{2-\varepsilon} v(r) = 0$, $\varepsilon > 0$) and asymptotically integrable ($\lim_{r \to \infty} r^{3+\varepsilon} v(r) = 0$, $\varepsilon > 0$). Hereafter we limit our consideration to such potentials.

(5) - In principle, of course, this is no limitation, since in any case we are primarily interested in the function $\bar{\varepsilon}(g)$ for $g \approx g_o$, where g_o is the physical value of the coupling constant; and eventually in the analytic continuation of this function to other values of g, including $g \approx 0$, even if for such values of g it looses any physical meaning.

(6) - Note that this occurs, in 3-dimensional space, even if the particles are identical fermions; see ref. (1).

(7) - G. A. Baker, Jr., Rev. Mod. Phys. 43, 479 (1971).

(8) - Actually the result that is proved in Ref. (7) is considerably less stringent, corresponding to eq. (10) but with $(2n)!$ in place of $n!$ (see eq. (3. 38), and the sentence following it, in Ref. (7)). The conjecture that the strongest condition (10) hold is hower made plausible (see eq. (3. 45), and the discussion leading to it and following it, in Ref. (7)).

(9) - Let us however mention that an example that is less artificial, being related to a (one-dimensional) mathematical model that displays some of the typical features of an N-body system, does reproduce the basic properties of the example given here : F. Calogero and A. Degasperis, J. Math. Phys. (submitted to).

(10) - The function $E(N; \rho, g)$ of eq. (11) is clearly meromorphic in ρ and g; its poles in the g plane occur at $g = \pm g_n$ and at $g = \rho + g_n$, and those in the ρ plane occur at $\rho = g - g_n$, with $g_n = N^{-1} + i\pi(2n+1)N^{-2}$, $n = 0, \pm 1, \pm 2, \ldots$.