To be submitted to
Physics Letters

COMITATO NAZIONALE PER L'ENERGIA NUCLEARE
Laboratori Nazionali di Frascati

LNF-74/25(P)
10 Maggio 1974

G. C. Caiazza and E. Etim: HARMONIC INVERSION AND RECIPROCITY RELATIONS OF DEEP INELASTIC STRUCTURE FUNCTIONS.

Servizio Documentazione
dei Laboratori Nazionali di Frascati del CNEN
Casella Postale 70 - Frascati (Roma)

Laboratori Nazionali di Frascati del CNEN
Servizio Documentazione

LNF-74/25(P)
10 Maggio 1974

G.C. Caiazza(o) and E. Etim(*): HARMONIC INVERSION AND RECIPROCITY RELATIONS OF DEEP INELASTIC STRUCTURE FUNCTIONS.

ABSTRACT. -

We give a new derivation of the reciprocity relations between deep inelastic structure functions based on a representation of the transformation by reciprocal radii as an integral transform in the space of harmonic functions. A simple picture, consistent with our intuition of short-distance operator product expansions, is suggested.

The reciprocity relations[1]

$$F_1(q^2, \frac{1}{\omega}) = \pm \omega \bar{F}_1(q^2, \omega)$$
$$F_2(q^2, \frac{1}{\omega}) = \pm \omega^3 \bar{F}_2(q^2, \omega) \qquad (1)$$

connecting scattering (F) and annihilation ($\bar{F}$) structure functions, in their physical regions, are not well understood. This is due in large

---

(o) - Istituto di Fisica dell'Università di Napoli
(*) - Work supported by the INFN

measure to the fact that their derivation involves a good amount of theo retical sophistication[1, 2, 3]. The proofs[4, 5] based on symmetry arguments, however, are quite simple. They also suggest a convincing intuitive picture which makes the reciprocity relations less mysterious. There exist two such arguments. The first[4] makes use of conformal symmetry while the second[5] invokes more familiar concepts, such as SO(3, 1) invariance, energy-momentum sum rules and spin-analyticity of the moments of the structure functions. It is the purpose of this letter to establish the equivalence of these two approaches. The exercise, it is hoped, will throw some light on the physical principles underlying the reciprocity relations. It will emerge, in fact, that the classical kin of the reciprocity relations is the Kelvin transformation in potential theory[6, 7].

Our main idea is that the equality of the energy-momentum sum rules and of all higher moments of the structure functions is guaranteed by a process of analytic continuation of the forward Compton amplitude in the ω-plane which correctly implements the symmetry under the conformal group. The continuation is by inversion or the transformation by reciprocal radii R, relative to the unit circle $|\omega| = 1$. The conformal symmetry constraint is that the analytic functions so related coincide on this circle. This establishes an equality between their Taylor coefficients

$$h_n(q^2) = \bar{h}_{n-1}(q^2); \quad n = 2, \ldots \tag{2}$$

As shown in ref. (5) this is the statement of the reciprocity relations in terms of the moments $h_n(q^2)$ and $\bar{h}_n(q^2)$ of the scattering and annihilation structure functions respectively. The functional equation

$$F_2(q^2, \frac{1}{\omega}) = \pm\, \omega^3 \bar{F}_2(q^2, \omega) \tag{3}$$

is obtained by taking inverse Mellin transform. The analyticity of the moments in n (spin) follows from the convergence of the Mellin transform integrals for $n = 2$ (energy-momentum conservation).

All this has a familiar classical ring[6] based on the formal analogy[7] between the light-cone expansion of the current product

$$J_\mu(x)\,J_\nu(0) = (\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\sum_n H^{(1)}_{\alpha_1..\alpha_n}(x)\,O_{\alpha_1..\alpha_n}(0) +$$

$$+ (g_{\mu\alpha_1}\partial_\nu\partial_{\alpha_2} + g_{\nu\alpha_2}\partial_\mu\partial_{\alpha_2} - g_{\mu\alpha_1}g_{\nu\alpha_2}\Box - g_{\mu\nu}\partial_{\alpha_1}\partial_{\alpha_2}) \cdot \tag{4}$$

$$\cdot \sum_n H^{(2)}_{\alpha_3..\alpha_n}(x)\,O_{\alpha_1..\alpha_n}(0)$$

and the expansion of the potential of a given charge distribution in terms of multipoles. In fact projecting (4) on to a spherical basis yields, in the limit of large $|q^2| \to \infty$ and fixed ω in $|\omega| \leq 1$, the asymptotic expansion

$$\frac{1}{2} m\,\omega^2\,T_2(q^2,\omega) = \sum_n h_n^{(-)}(q^2)\,C_n^{1}\!\left(\frac{\omega}{\omega_o}\right); \quad \omega_o = \frac{2m}{\sqrt{|q^2|}} \tag{5}$$

where $T_2(q^2,\omega)$ is the second invariant[*] of the forward Compton amplitude

$$T_{\mu\nu}(p,q) = \int d^4x\, e^{iqx} \langle p|T(J_\mu(x)\,J_\nu(0))|p\rangle \tag{6}$$

m the mass of the target and $C_n^{1}(z)$ the Gegenbauer polynomial. Eq. (5) is clearly the analogue of the potential generated at internal points of the unit circle by a plane charge distribution.

The role of conformal symmetry emerges from the observation that the light-cone representation of $T_2(q^2,\omega)$ in the region $|\omega| > 1$ can be obtained from eq. (5) by making use of the transformation by reciprocal radii R. On harmonic functions in d-dimensional space R acts as a

---

(*) - $T_1(q^2,\omega)$ could be treated analogously. We consider only one of them for simplicity.

finite integral transform[8, 9]

$$(\omega^2-1)^{\frac{1}{2}(\lambda-\frac{1}{2})} e^{-i\pi(\lambda-\frac{1}{2})} Q^{\lambda-\frac{1}{2}}_{n+\lambda-\frac{1}{2}}(\omega) = \frac{2^{\lambda-\frac{3}{2}}\Gamma(\lambda)}{\sqrt{\pi}} \int_{-1}^{+1} \frac{dz(1-z^2)^{\lambda-\frac{1}{2}} C^{\lambda}_n(z)}{\omega - z} \tag{7}$$

where $\lambda = (d-2)/2$. By expressing the regular part of $C^{\lambda}_n(z)$ in terms of Legendre function of the second kind[10] and making use of the duplication formula[11] $\Gamma(2\lambda) = \pi^{-1/2}\, 2^{2\lambda-3/2}\Gamma(\lambda)\Gamma(\lambda+\frac{1}{2})$, eq. (7) becomes

$$(\omega^2-1)^{\frac{1}{2}(\lambda-\frac{1}{2})} e^{-i\pi(\lambda-\frac{1}{2})} Q^{\lambda-\frac{1}{2}}_{n+\lambda-\frac{1}{2}}(\omega) = \frac{1}{\pi}\int_{-1}^{+1} \frac{dz(1-z^2)^{\frac{1}{2}(\lambda-\frac{1}{2})} Q^{\lambda-\frac{1}{2}}_{-n-\lambda-\frac{1}{2}}(z)}{\omega - z} \tag{8}$$

which is manifestly a reciprocal relation. Making use of this, and symmetrising[12] in ω to respect crossing symmetry, one gets the complete asymptotic expansion of $T_2(q^2, \omega)$ in the limit $|q^2| \to \infty$

$$\frac{1}{2}m\omega^2 T_2(q^2,\omega) = \begin{cases} \dfrac{e^{-i\pi/2}}{\sqrt{(2\pi)}}\left(\dfrac{\omega^2}{\omega_o^2}-1\right)^{-\frac{1}{4}} \displaystyle\sum_{n=2}^{\infty} h_n^{(-)}(q^2)\, Q^{\frac{1}{2}}_{-n-\frac{3}{2}}\left(\frac{\omega}{\omega_o}\right); \quad \omega \leq 1 & \text{(9a)} \\[2ex] \dfrac{e^{i\pi/2}}{\sqrt{(2\pi)}}\left(\dfrac{\omega^2}{\omega_o^2}-1\right)^{-\frac{1}{4}} \displaystyle\sum_{n=1}^{\infty} h_n^{(+)}(q^2)\, Q^{\frac{1}{2}*}_{n+\frac{1}{2}}\left(\frac{\omega}{\omega_o}\right); \quad \omega \geq 1 & \text{(9b)} \end{cases}$$

where only even n terms contribute in (9a) and odd n in (9b).

Notice the appearance of the complex conjugate of $Q^{\frac{1}{2}}_{n+\frac{1}{2}}(z)$

in (9b) required to keep the argument of ω in (9a) and (9b) the same[(*)].

We have here a remarkable property of the transformation by reciprocal radii, that is, whether acting on quantum fields or on analytic functions, it preserves harmonicity. In the one case $\phi(x)$ and $\phi^R(x) = R\,\phi(x)\,R$ both satisfy the massless Klein-Gordon equation[(9)] and in the other both $C_n^{\lambda}(z)$ and $Q^{\lambda-1/2}_{n+\lambda-\frac{1}{2}}(z)$ satisfy the same (harmonic) differential equation.

The series in eq. (9) can also be regarded from another point of view, namely that their domain of convergence is delimited by the circle $|\omega| = \omega_o$; this circle is very small and contracts to a point as $|q^2| \to \infty$. For this reason (9a) is said to be an asymptotic expansion of $T_2(q^2, \omega)$ in the dilatation limit[(13)] $\omega \to 0$, from which one recovers the scattering amplitude by analytic continuation. By the same token (9b) defines the dilatation limit $\omega \to \infty$ of the corresponding annihilation amplitude. The description in terms of dilatation limits corresponds to the fact that, due to physical channel singularities, the asymptotic expansion, at large $|q^2| \to \infty$, of the forward Compton amplitude, as gotten from the light-cone expansion of the currents is not uniform in ω.

In discussing the relationship between (9a) and (9b) it is useful, however, to decouple ω from ω_o so that the singularities of $T_2(q^2, \omega)$ remain unaffected by the limit $|q^2| \to \infty$. The series in eqs. (9a) and (9b) therefore define analytic functions with branch cuts in $|\mathrm{Re}\,\omega| \geq 1$ and $|\mathrm{Re}\,\omega| \leq 1$ respectively. The presence of ω_o in the spherical functions is important nevertheless[(5)]. It exhibits the canonical q^2-dependence of the moments of the structure functions as is easily seen by using the contours in Fig. 1 to invert eq. (9). This gives the Mellin transforms

---

(*) - The transformation by reciprocal radii is different from the simple inverse transformation $\omega \to 1/\omega$; the two coincide only on the real axis.

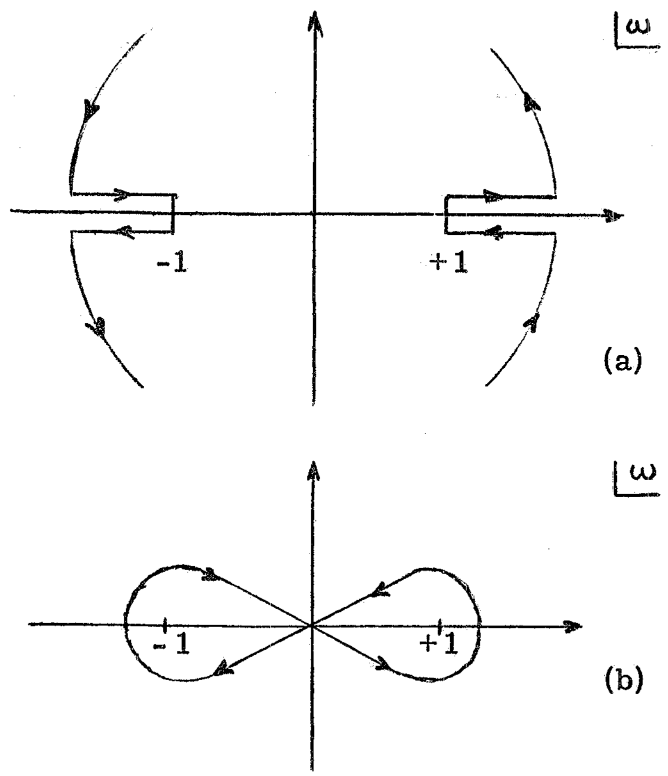

FIG. 1 - The contours for inverting eqs. (9a) and (9b) respectively.

$$h_n(q^2) = \left(\frac{m^2}{q^2}\right)^{-\frac{n+2}{2}} h_n^{(-)}(q^2) = \frac{2}{\pi}\int_1^\infty d\omega\, \omega^{-n} F_2(q^2, \omega) \qquad (10a)$$

$$\bar{h}_n(q^2) = \left(\frac{m^2}{q^2}\right)^{\frac{n}{2}} h_n^{(+)}(q^2) = \frac{2}{\pi}\int_0^1 d\omega\, \omega^{n+2}\, \bar{F}_2(q^2, \omega) \qquad (10b)$$

The connection between (10a) and (10b) is now obtained by setting $\omega = \exp(i\varphi)$ in eq. (9) and comparing the two series term-wise. The result is eq. (2) as promised. Inserting from here into eq. (10) and taking inverse Mellin transform yields eq. (3).

In d- dimensional space eq. (2) reads

$$h_n(q^2) = \bar{h}_{n-\lambda}(q^2) \qquad (11)$$

which, inserted in the inverse Mellin transforms

$$F_2(q^2, \frac{1}{\omega}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dn\; h(n; q^2)\, \omega^{-(n-\lambda)} \qquad (12a)$$

$$F_2(q^2,\omega) = \frac{1}{2\pi i}\int_{\bar{c}-i\infty}^{\bar{c}+i\infty} dn\, \bar{h}(n;q^2)\,\omega^{-(n+2\lambda+1)} \tag{12b}$$

yields

$$F_2(q^2,\tfrac{1}{\omega}) = \omega^{2\lambda+1}\,\bar{F}_2(q^2,\omega) \tag{13}$$

in agreement with the result[4] obtained by invoking invariance of the conformal Casimir $C_n = 2(n-1)(n+2\lambda)$ of the spin n irreducible tensors of canonical dimension $l_n = n+2\lambda$ in the light-cone expansion of eq. (4) under the SO(d-1) operation $n \rightarrow -n-2\lambda+1$. This dependence of the reciprocity relations on the dimension of the underlying space is an important characteristic of the transformation by reciprocal radii. It suggests a simple intuitive picture whereby the reciprocity relations between the discontinuities of $T_2(q^2,\omega)$ in different physical regions is similar to an analogous relationship[6] between systems of charges located at inverse points with respect to a sphere and generating the same potential at all points of this surface. This point of view is strongly supported by analyticity and unitarity[14] and is consistent with our intuition of short-distance operator product expansions[7].

REFERENCES. -

(1) - V.N. Gribov and L.N. Lipatov, Phys. Letters 37B, 78 (1971); Soviet J. Nuclear Phys. 15, 438, 675 (1972).

(2) - P.M. Fishbane and J.D. Sullivan, Phys. Rev. 6D, 3568 (1972).

(3) - N. Christ, B. Hasslacher and A. Mueller, Phys. Rev. 6D, 1453 (1972).

(4) - S. Ferrara, R. Gatto and G. Parisi, Phys. Letters 44B, 381 (1973).

(5) - E. Etim, Phys. Letters 45B, 478 (1973).

(6) - L. L. Helms, Introduction to potential theory (J. Wiley and Sons, Inc., New York, 1969), p. 36; T.M. MacRobert, Spherical harmonics (Pergamon Press, Oxford, 1967), p. 150.

(7) - K. Wilson, Proc. 1971 Intern. Conf. on Electron and Photon Interactions at High Energies, Cornell, Ithaca (N.Y.), p. 116.

(8) - Tables of integral transforms, vol. II, p. 281.

(9) - R acts as a Hankel transform on radial functions. H.A. Kastrup, Phys. Rev. 140B, 183 (1965); Phys. Rev. 142, 1060 (1966).

(10) - Higher transcendental functions, Bateman manuscripts, vol. I, p. 140.

(11) - Cfr. ref. (10), p. 5.

(12) - The symmetrisation involves the replacements

$$e^{\pm i\pi\mu/2} Q^{\mu}_{\nu}(z) \rightarrow \frac{1}{2}(e^{\pm i\pi\mu/2} Q^{\mu}_{\nu}(z) + e^{\mp i\pi\mu/2} Q^{\mu}_{\nu}(-z))$$

$$e^{\mp i\pi\mu/2} Q^{\mu}_{-\nu}(z) \rightarrow \frac{1}{2}(e^{\mp i\pi\mu/2} Q^{\mu}_{-\nu}(z) + e^{\pm i\pi\mu/2} Q^{\mu}_{-\nu}(-z),$$

cfr. ref. (10), p. 143.

(13) - G. Mack, Nuclear Phys. B35, 592 (1971).

(14) - G.F. Chew, S-matrix theory of strong interactions (W.A. Benjamin Inc., New York, 1961).