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E. Etim : INVERSION TRANSFORMATION IN QUANTUM FIELD THEORY AND RECIPROCITY CONSTRAINTS. -

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ABSTRACT. -

The transformation by reciprocal radii, as an isomorphism between the two collinear SU(1, 1) subgroups of the conformal group in the scaling limit, is shown to induce the reciprocity relations between scattering and annihilation structure functions.

It is difficult to appreciate fully the importance of the reciprocity relations between scattering and annihilation structure functions[(1)] because of the lack of a simple intuition for them, even in the parton model[(2)]. This is due in large measure to the fact that derivations of the reciprocity relations have been highly sophisticated[(1, 2, 3, 4)]. There is however an intimate connection between asymptotic scale invariance of quantum field theory and the reciprocity relations, so that if the latter are well understood the pattern of the breaking of the former would be also.

From the point of view of S-matrix theory, the reciprocity relations

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$$F_1(\omega, q^2) = \pm \frac{1}{\omega} \bar{F}_1(\frac{1}{\omega}, q^2)$$

$$F_2(\omega, q^2) = \pm \frac{1}{\omega^3} \bar{F}_2(\frac{1}{\omega}, q^2) \qquad (1)$$

between scattering (F) and annihilation ($\bar{F}$) structure functions, are not unexpected, because such interconnections between discontinuities of the S-matrix in different physical regions, are implied by unitarity and analyticity. To render this fact intuitive appeal is often made to the formal analogy between the analytic S-matrix and the Coulomb potential[5]. In deep inelastic scattering this analogy is implicit in the comparison of the expansion of the product of two operators in a basis of local ones in the short distance limit to the long distance expansion of the potential of a given charge distribution in terms of multipoles[6]. The reciprocity relations can also be understood in the classical setting of this formal analogy. In fact reciprocity relations, known generally as Kelvin transformations, form an integral part of classical potential theory[7]. They are generated by a non-linear transformation R of space, called inversion or transformation by reciprocal radii, and carry an inherent dependence on the dimension of the underlying space manifold. This is true also of eq. (1), which in a space-time of dimension d would read[3]

$$F_2(\omega, q^2) = \pm \frac{1}{\omega^{d-1}} \bar{F}_2(\frac{1}{\omega}, q^2) \qquad (1')$$

In quantum field theory the transformation by reciprocal radii leads beyond Poincare invariance to conformal symmetry. From its operation on space-time coordinates, given by

$$R\, x_\mu = \bar{x}_\mu = x_\mu / x^2 \qquad (2)$$

one gets its action on the generators of the conformal group[8]

$$R\,J_{\mu\nu}\,R = J_{\mu\nu} \qquad R\,P_\mu\,R = K_\mu$$
$$R\,K_\mu\,R = P_\mu \qquad R\,D\,R = -D \tag{3}$$

where $J_{\mu\nu}$ and P_μ are the Poincare generators and K_μ, D those of special conformal transformation and dilatation respectively. The conformal group is thus completely generated if one adjoins R and D to the Poincare group.

The action of R on quantum fields, on the other hand is given by[9]

$$\phi^R_{\alpha_1..\alpha_n}(x) = R\,\phi_{\alpha_1..\alpha_n}(x)\,R = \left(\frac{1}{x^2}\right)^{l_n} R_{\alpha_1\beta_1}(x)\ldots R_{\alpha_n\beta_n}(x)\,\phi_{\beta_1..\beta_n}(x)$$
$$\psi^R_\alpha(x) = R\,\psi_\alpha(x)\,R = \left(\frac{1}{x^2}\right)^{1} S_{\alpha\beta}(x)\,\psi_\beta(x) \tag{4}$$

where

$$R_{\alpha\beta}(x) = g_{\alpha\beta} - 2x_\alpha x_\beta/x^2$$
$$S_{\alpha\beta}(x) = (x^\mu\gamma_\mu)_{\alpha\beta}/(x^2)^{1/2} \tag{5}$$

and $\phi_{\alpha_1..\alpha_n}(x)$ is a conformal tensor field of spin n and scale dimension l_n and $\psi(x)$ a spin 1/2 field of scale dimension 1.

In the scaling limit the conformal group reduces to two SU(1, 1) subgroups with algebras $L^{(+)}$ and $L^{(-)}$, given by

$$L^{(+)}_+ = iP_+ = \frac{i}{2}(P_o + P_3)$$
$$L^{(+)}_o = \frac{i}{2}(J_{o3} + D) \tag{6}$$
$$L^{(+)}_- = iK_- = \frac{i}{2}(K_o - K_3)$$

and

$$L^{(-)}_+ = iK_+ = \frac{i}{2}(K_o + K_3)$$
$$L^{(-)}_o = \frac{i}{2}(J_{o3} - D) \tag{7}$$
$$L^{(-)}_- = iP_- = \frac{i}{2}(P_o - P_3)$$

which leave the light-like rays of the vectors $p^{(+)} \equiv (p_+, 0, 0, 0)$ and $p^{(-)} \equiv (0, p_-, 0, 0)$ respectively, invariant.

The connection between the transformation by reciprocal radii and the reciprocity relations now follows from the observation that, according to eq. (3), R is an isomorphism between $L^{(+)}$ and $L^{(-)}$, so that one can compute the structure functions either in a basis of the eigenvalues p_μ of the momentum generators P_μ or in a basis of the eigenvalues $\bar{p}_\mu = Rp_\mu$ of K_μ. This equivalence is the substance of SU(1, 1) symmetry. This was assumed in a recent note(10), where it was shown that the forward Compton amplitude

$$T_{\mu\nu}(p, q) = \int d^4z e^{iqz} \langle p | T(J_\mu(x) J_\nu(y)) | p \rangle \; ; \quad z = x - y \tag{8}$$

upon diagonalisation with respect to representations of SU(2, 2), in which the target and the intermediate states belong to massless and continuous mass representations respectively, becomes a generalized Born term whose form is completely fixed by the symmetry requirements. Now the inversion operator R, is all-embracing; it affects not only field operators and the SU(2, 2) generators but it also transforms the test function space of the translation eigenfunctions exp $(ip \cdot x)$ (regarded as distributions) into the test function space of the eigenfunctions $\exp(ik\bar{x})$ of K_μ(8). By reason of this fact, the R isomorphism between the two SU(1, 1) subgroups translates into the fundamental equality

$$T_{\mu\nu}(p, q) = T^R_{\mu\nu}(\bar{p}, \bar{q}) \tag{9}$$

to be understood in the sense of analytic continuation in the invariants, where

$$T^R_{\mu\nu}(\bar{p}, \bar{q}) = \int d^4\bar{z}\, e^{i\bar{q}\bar{z}} (\bar{p} | T(J^R_\mu(x) J^R_\nu(y)) | \bar{p}) \; ; \quad \bar{z} = \bar{x} - \bar{y} \tag{10}$$

and $\qquad |\bar{p}) = R|p\rangle.$

Using standard reduction techniques, the discontinuities of $T_{\mu\nu}(p, q)$ and $T^R_{\mu\nu}(\bar{p}, \bar{q})$ for space-like q^2 $(q^2 < 0)$ and $q_0 > 0$ can be calculated from the SU(1, 1) light-cone expansions[10, 11] of the vertices $\langle p | J_\mu(x) | n \rangle$ and $(\bar{p} | J^R_\mu(x) | n)$ (where $| n\rangle = R | n)$ is a set of intermediate states) in the limit $x_- \to 0$ and $x^2 \to 0$, giving[10] (*)

$$\mathrm{Im} \int d^4x e^{iqx} \langle p | J_-(x_+, x_-, \vec{x}_\perp) \phi_-^{(n)}(0) | 0 \rangle =$$

$$= \mathrm{Im} \int dx_+ dx_- e^{\frac{i}{2}(q_+x_- + q_-x_+)} (x_-)^{-\frac{1}{2}(d_n+1)} (x_+)^{-\frac{1}{2}(\tau_n-1)} \cdot$$

$$\cdot \int_0^1 du\, u^{\frac{1}{2}(1-\tau_n)} (1-u)^{\frac{1}{2}(\tau_n-3)} e^{\frac{i}{2}up_-x_+} = \qquad (11a)$$

$$= (-q^2)^{\frac{1}{2}(\tau_n-3)} (q_+)^{1+\tau_n} \omega^{\frac{1}{2}(1-\tau_n)} (\omega-1)^{\tau_n-2} \cdot$$

$$\cdot {}_2F_1\left(\frac{\tau_n-1}{2}, \frac{\tau_n-1}{2}; \tau_n-1; 1-\frac{1}{\omega}\right) \theta(\omega-1)$$

$$\mathrm{Im} \int d^4x e^{i\bar{q}\bar{x}} (p | J_-^R(x_+, x_-, \vec{x}_\perp) \phi_-^{(n)R}(0) | 0) =$$

$$= \mathrm{Im} \int d\bar{x}_+ d\bar{x}_- e^{\frac{i}{2}(\bar{q}_+\bar{x}_- + \bar{q}_-\bar{x}_+)} (\bar{x}_+)^{-\frac{1}{2}(d_n+1)} (\bar{x}_-)^{-\frac{1}{2}(\tau_n-1)} \cdot$$

$$\cdot \int_0^1 du\, u^{\frac{1}{2}(1-\tau_n)} (1-u)^{\frac{1}{2}(\tau_n-3)} e^{\frac{i}{2}u\bar{p}_+\bar{x}_-} = \qquad (11b)$$

$$= (-\bar{q}^2)^{\frac{1}{2}(\tau_n-3)} (\bar{q}_-)^{1+\tau_n} \bar{\omega}^{\frac{1}{2}(1-\tau_n)} (\bar{\omega}-1)^{\tau_n-2} \cdot$$

$$\cdot {}_2F_1\left(\frac{\tau_n-1}{2}, \frac{\tau_n-1}{2}; \tau_n-1; 1-\frac{1}{\bar{\omega}}\right) \theta(\bar{\omega}-1) .$$

where $\phi^{(n)}(x)$ is a conformal field of spin $n = \frac{1}{2}(d_n - \tau_n)$ and scale dimension $l_n = \frac{1}{2}(d_n + \tau_n)$ and $\omega = -p_-/q_-$ and $\bar{\omega} = -p_+/q_+ =$

(*) - eq. (11b) involves a double limit in which the argument of $\phi_-^{(n)R}(y)$ is set equal to zero only after the operator product expansion.

$= - q_-/p_- = 1/\omega$.

In configuration space the (leading) most singular distribution $(x_-)^{-\alpha}$ $(\alpha > 0)$ on the light-cone $(x_- \to 0,\ x^2 \to 0)$ transforms under R into the least singular distribution $(\bar{x}_+)^{-\alpha}$ as $\bar{x}_+ = 1/x_- \to \infty$. This is a very general result[8] which depends on the way the conformal group affects the structure of singularities in quantum field theory. For instance the canonical light-cone singularity $\delta(x^2)$ in x-space transforms into the identically zero distribution in $\bar{x}$-space.

Taking the discontinuities of bothsides of (9) and making use of eq. (11) gives, for each conformal quantum number $\lambda = (\tau_n - 3)/2$[10]

$$F_{1R}^{(\lambda)}(\bar{\omega} = \frac{1}{\omega}) = F_1^{(\lambda)}(\frac{1}{\omega}) = (-)^{1+2\lambda}\, \omega F_1^{(\lambda)}(\omega) \tag{12a}$$

$$F_{2R}^{(\lambda)}(\bar{\omega} = \frac{1}{\omega}) = F_2^{(\lambda)}(\frac{1}{\omega}) = (-)^{1+2\lambda}\, \omega^3 F_2^{(\lambda)}(\omega) \tag{12b}$$

where the scaling of the structure functions in the Bjorken limit is quaranteed by the assignement of the target to a massless representation of the conformal group. For time-like q^2 $(q^2 > 0)$ there is an overall sign change in eq. (11); therefore for the annihilation structure functions one finds, for each λ

$$\bar{F}_{1,2R}^{(\lambda)}(\frac{1}{\bar{\omega}} = \omega) = \bar{F}_{1,2}^{(\lambda)}(\omega) = (-)^{1+2\lambda}\, F_{1,2}^{(\lambda)}(\omega) \tag{13}$$

which, together with (12), yields(*)

$$\begin{aligned} F_1^{(\lambda)}(\frac{1}{\omega}) &= \omega F_1^{(\lambda)}(\omega) \\ F_2^{(\lambda)}(\frac{1}{\omega}) &= \omega^3 F_2^{(\lambda)}(\omega) \end{aligned} \tag{14}$$

independently of λ and hence, upon summation over λ, one gets the reciprocity relations of eq. (1) for scalar targets (positive sign) and

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(*) - The sign difference between eq. (14) and eq. (11) of ref. (9) comes from the fact that here we define the annihilation structure functions to be both positive.

scaling structure functions.

On the basis of the Coulomb potential analogy, the above manipulations admit of a very simple physical reading: $T_{\mu\nu}(p, q)$ and $T^R_{\mu\nu}(\bar{p}, \bar{q})$ are the analytic solutions of the light-cone operator product expansion representation for the forward Compton amplitude in the regions $|\omega| > 1$ and $|\omega| < 1$ of the complex ω-plane respectively, which coincide on the unit circle $|\omega| = 1$, and hence constitute analytic continuations of each other.

This circumstance alone imposes the reciprocity constraint between their discontinuties, in much the same way that charges located at inverse points with respect to a sphere[*], and generating the same potential at all points on this surface, are reciprocally related[12].

This point of views is developed further in our next paper[13] where the transformation by reciprocal radii is implemented directly on analytic functions and not on quantum fields.

Discussion with S. Ferrara is gratefully acknowledged.

(*) - Note that in the complex plane the transformation by reciprocal radii with respect to the unit circle is different from the simple reciprocal transformation $\omega \to 1/\omega$; the two coincide only on the real axis.

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