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NON-LINEAR BEAM-BEAM EFFECT COMPUTER SIMULATION

A. Renieri

Laboratori Nazionali di Frascati del CNEN

Frascati, Italy

Abstract.

The motion of a single-particle colliding with a strong beam in a storage ring is studied by computer simulation, under the assumptions that there is no radiation noise and no damping and that the strong beam has a gaussian transverse charge distribution. The phase space regions of stochastic motion and the corresponding minimum ΔQ values are determined. There is evidence that the motion stochasticity is due to overlapping of neighbouring resonances (Chirikov criterion).

1. - Introduction.

The stochastic phase space behaviour of many one-dimensional non-linear conservative dynamic systems has been extensively studied by I. Guminski and C. Mira^{1,2,3}, E. Keil^{4,5}, using the resonance overlapping Zaslavskii-Chirikov⁶ criterion, evaluated the transverse amplitude and ΔQ limit for beam-beam non linear space charge effect.

In this paper we investigate the one-dimensional transverse phase space motion of a single particle colliding with a strong beam in a storage ring. The ΔQ threshold for the stochastic motion and the stochastic phase space regions are found using the properties of the phase space periodic points (cycles). The investigation is performed under the following conditions,

- head-on collision,
- strong beam shorter than β value at crossing point,
- the space charge forces from the strong beam are simulated by δ -function-like kicks given to the weak beam particle (point-like strong beam),
- no radiation noise and no damping,
- off energy closed orbit $\eta = 0$ at interaction point,
- no chromaticity,
- gaussian charge distribution function of the strong beam

$$f(x, z) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+z^2)}{2\sigma^2}} \quad (1)$$

- only one transverse mode excited.

We will use the following notations,

- β = betatron β -value at crossing point,
- μ = betatron phase difference between two neighbouring crossing points,
- x = transverse displacement,
- x' = dx/dS , where S is the machine longitudinal coordinate,
- $\theta(x)$ = transverse kick experienced by the particle with x transverse displacement.

For the strong beam distribution function (1), $\theta(x)$ is given by the eq.

$$\theta(x) = -\frac{r_0 N}{\gamma \sigma^2} \left\{ x \left(\frac{1 - e^{-\frac{x^2}{2\sigma^2}}}{\frac{x^2}{2\sigma^2}} \right) \right\} \quad (2)$$

Where r_0 is the particle classical radius, γ is the ratio between the total and the rest energy, N is the number of particles of the strong beam. Note that all computations have been carried out with at least 14 digits.

2. - Motion equations.

We write the motion equations as recurrences between the middle of one crossing and the middle of the next one. We assume that the machine is perfect, and that all crossing points are identical. We have,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{n+1} = \begin{pmatrix} \cos \mu & \beta \sin \mu \\ -\frac{1}{\beta} \sin \mu & \cos \mu \end{pmatrix} \begin{pmatrix} x \\ x' + \frac{1}{2} \theta(x) \end{pmatrix}_n + \frac{1}{2} \begin{pmatrix} 0 \\ \theta(x) \end{pmatrix}_{n+1} \quad (3)$$

where n labels the crossing.

If we put,

$$\xi = \frac{r_0 \beta N}{4\pi\gamma\sigma^2}, \quad F(y) = \frac{1 - e^{-y}}{y}$$

equations (3) become (see eq. (2)),

$$\begin{aligned} x_{n+1} &= x_n \cos \mu + \beta x'_n \sin \mu - 2\pi\xi F\left(\frac{x_n^2}{2\sigma^2}\right) \sin \mu x_n \\ x'_{n+1} &= -\frac{x_n}{\beta} \sin \mu + x'_n \cos \mu - \\ &\quad - 2\pi\xi F\left(\frac{x_n^2}{2\sigma^2}\right) \cos \mu x_n - 2\pi\xi x_{n+1} F\left(\frac{x_{n+1}^2}{2\sigma^2}\right) \end{aligned} \quad (4)$$

ξ is related to the linear ΔQ shift by the eq.,

$$\cos(\mu + 2\pi\Delta Q) = \cos \mu - 2\pi\xi \sin \mu$$

3. - Phase space variables.

The recurrence equations (4) can be derived

from the Hamiltonian,

$$H(q, p|S) = \frac{1}{2} \left(p^2 + \left(\frac{\mu}{L} \right)^2 q^2 \right) + \frac{4\pi\xi\mu\sigma^2}{L} G\left(\frac{q}{2\sigma^2}\right) + \sum_{n=-\infty}^{\infty} \delta(s-nL) \quad (5)$$

Where we have put,

$$q = x, \quad p = \frac{\mu \beta x'}{L}$$

L = longitudinal distance between two neighbouring crossing points,

$\delta(s)$ = Dirac function

$$G(x) = \int_0^x F(y) dy$$

In order to obtain variables more manageable than (q, p) , let us define,

$$H_0(q, p) = \frac{1}{L} \int_0^L H(q, p|S) dS = \frac{1}{2} p^2 + \frac{\mu^2 \sigma^2}{L^2} V\left(\frac{q}{2\sigma^2}\right) \quad (6)$$

$$(V(x) = x + \frac{2\xi}{\mu} G(x)).$$

From this Hamiltonian we derive two new variables (T, W) that will be used in describing the motion derived from the complete time-dependent Hamiltonian (5). We define,

$$T(q, p) = \frac{1}{\mu} \int_{-\lambda_M(q, p)}^{\lambda_M(q, p)} \frac{d\lambda}{\sqrt{V(\lambda_M^2(q, p)) - V(\lambda^2)}} \quad (7)$$

$$W(q, p) = \frac{|p|}{\mu} \frac{2\pi}{T(q, p)} \int_{q/\sqrt{2}\sigma}^{\lambda_M(q, p)} \frac{d\lambda}{q/\sqrt{2}\sigma \sqrt{V(\lambda_M^2(q, p)) - V(\lambda^2)}} \quad (8)$$

where $\lambda_M(q, p)$ is the solution of the equation,

$$V(\lambda_M^2(q, p)) = \left(\frac{L}{\mu\sigma}\right) H_0(q, p). \quad (9)$$

The physical meaning of T , λ_M and W is obvious. T is the motion period, in units dL , $\lambda_M = A/\sqrt{2}\sigma$ where A is the maximum q value, W is the phase conjugate variable. The solutions of the dynamic equations derived from the time-independent Hamiltonian H_0 are,

$$T(S) = T(0) \quad (10)$$

$$W(S) = W(0) - \frac{2\pi}{T(0)} \left(\frac{S}{L}\right) \pmod{2\pi}$$

In the following we restrict our investigation to the phase space strip enclosed between $W=0$ and $W=\pi/2$.

This is allowed by the symmetry of recurrence equations (4) with respect to the x and x' axis, that is, in terms of (T, W) , with respect to $W=0, \pi/2, \pi, 2\pi$.

4. - Cycles and separatrices.

With the new (T, W) variables, the recurrence equations (4) become,

$$T_{n+1} = A(T_n, W_n) \quad (11)$$

$$W_{n+1} = B(T_n, W_n).$$

Let us define the functions,

$$A_k(T_n, W_n) = T_{n+k} \quad (12)$$

$$B_k(T_n, W_n) = W_{n+k}$$

We call cycles of order N , the points (T_n, W_n) satisfying the equations,

$$T_n = A_N(T_n, W_n) \quad n=1, 2, \dots, N \quad (13)$$

$$W_n = B_N(T_n, W_n)$$

A further property of a cycle, beside N , is the number of turns (rotational number) made around the origin ($x=x'=0$) by the phase space representative point. In the following we use the notation "cycle N/r " for a cycle of order N and rotational number r . The characteristic matrix of a cycle N/r is,

$$\hat{C} = \begin{pmatrix} \frac{\partial A_N(T, W)}{\partial T} & \frac{\partial A_N(T, W)}{\partial W} \\ \frac{\partial B_N(T, W)}{\partial T} & \frac{\partial B_N(T, W)}{\partial W} \end{pmatrix} \quad (14)$$

The determinant of \hat{C} is always 1, because our recurrence is conservative. If the absolute value of the trace of \hat{C} is lower (greater) than 2 the cycle is stable (unstable) and is called centre (saddle). Separatrices start from saddles. They divide the phase space into stable and unstable phase regions. As an example we have, in Fig. 1, cycle $6/1$ and $5/1$ with their separatrices. Inside the $6/1$ separatrices we have stable motion (particle E), while outside the phase motion is unstable (particle A and H). In the computer graphs we use the notations, x = saddle; 0 = centre; N = separatrix of cycle $N/1$; $*$ = overlap between two or more separatrices. The vertical scale on the right (A/σ) is related to T by eq. (7) ($A = \sqrt{2} \lambda_M \sigma$). Roughly speaking A is the "instantaneous motion amplitude". From the physical meaning of T and W , we have, for a cycle N/r ,

$$T_1 \approx T_2 \approx \dots \approx T_N \approx \frac{N}{r}$$

$$W_n - W_{n-1} \approx \frac{2\pi r}{N}$$

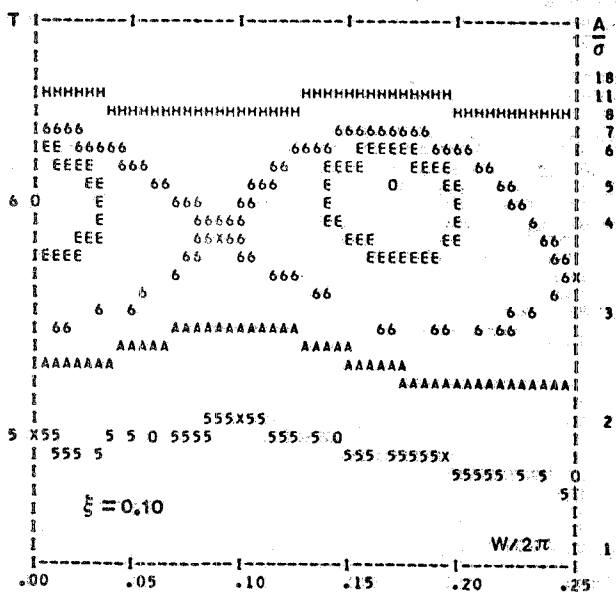


FIG. 1

5. - Stochastic threshold.

The phase space pattern of $N/1$ cycles, for $\mu/2\pi = 0.15$, is given in Figs. 2, 3 and 4.

Fig. 2, $\xi = 0.10$ ($\Delta Q = 0.087$); we have cycles $6/1$ and $5/1$,
 Fig. 3, $\xi = 0.16$ ($\Delta Q = 0.136$); there is an overlap region between cycles $6/1$ and $5/1$; a new cycle ($4/1$) appears,

Fig. 4, $\xi = 0.20$ ($\Delta Q = 0.171$); the whole phase space region between cycles $6/1$ and $4/1$ is stochastic, with the exception of the stable phase islands around the centres.

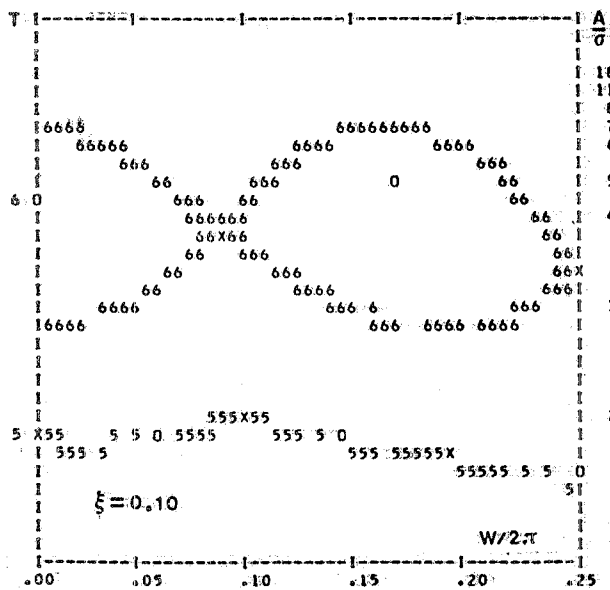


FIG. 2

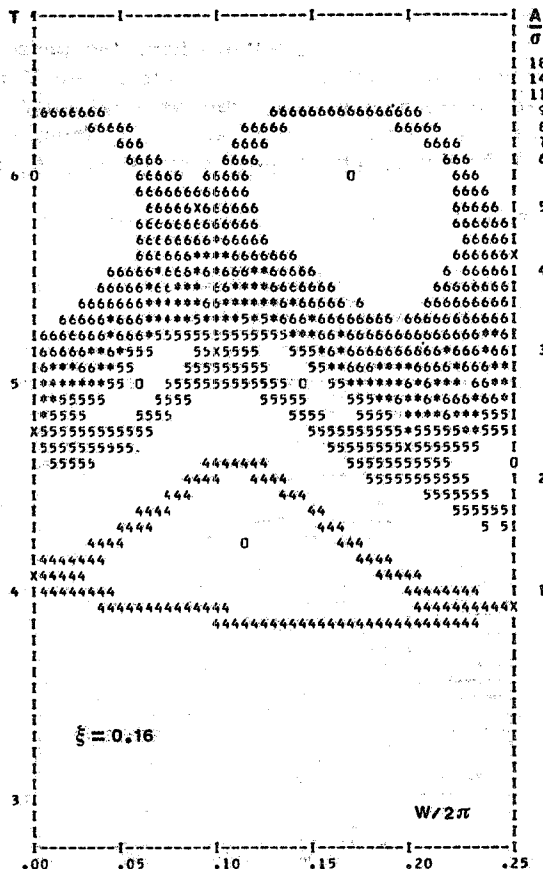


FIG. 3

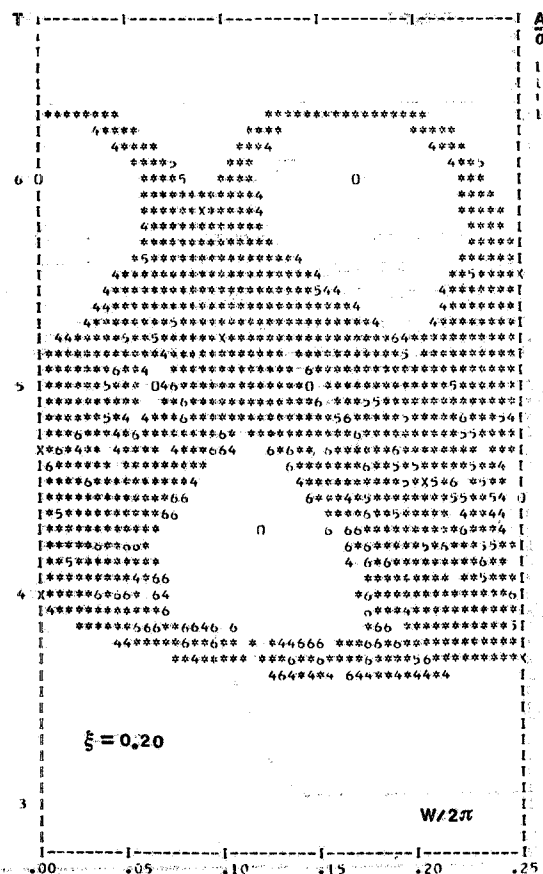


FIG. 4

In Fig. 5 ($\xi=0.20$) we have the phase space trajectories of particles A, B, H outside and W inside the stochastic region. From the vertical right scale we see that A/σ ranges, for particle W, from 1 to 10, so that this stochastic region may be very dangerous.

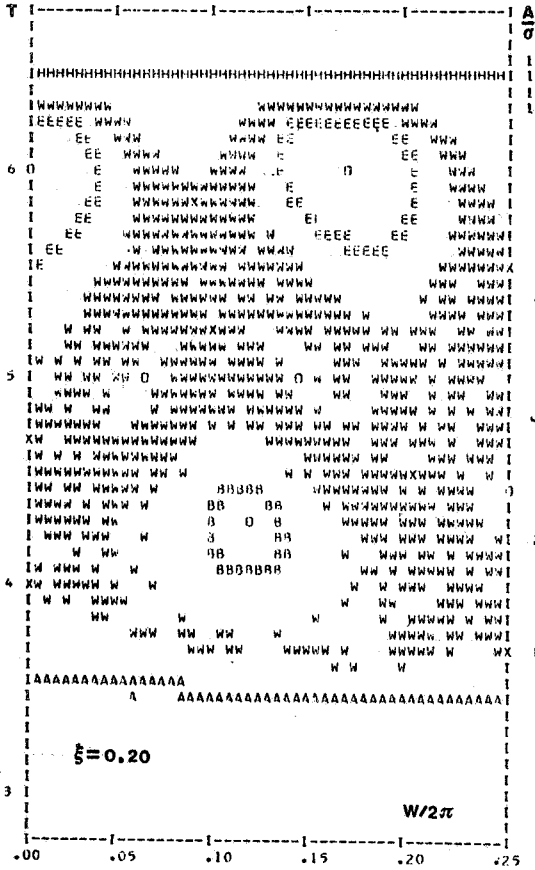


FIG. 5

If we add, in somehow artificial way, radiation noise and damping to our model, we obtain that the r.m.s. transverse dimension of the weak beam does not change until stochastic regions appear (See Fig. 6). When the phase space between 6/1 and 4/1

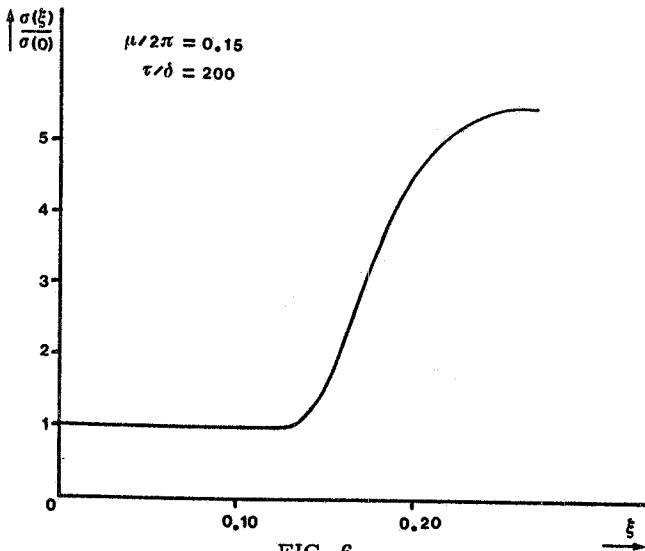


FIG. 6

is fully stochastic, the weak beam enlarges by a factor of five. This behaviour does not depend on the damping time value (τ), in the range

$$10^2 \lesssim \frac{\tau}{\delta} \lesssim 10^4$$

where δ is the time distance between two neighbouring crossing points. From this observation we may derive the energy independence of ξ in the weak beam-strong beam incoherent limit. However, we must remember that our model is very rough. The minimum requirement, for taking into account experimental data, is to have a bi-dimensional model (four-dimensional phase space).

6. - Conclusions.

According to Zaslavskii and Chirikov⁶, we may say that the stochastic region in Fig. 3 and 4 is due to the overlapping of the stochastic layers of neighbouring resonances. On the other hand, as showed by Gumoski^{2,3}, when a centre bifurcates into a saddle, the region in the neighbourhood of such a cycle becomes stochastic. We have indeed that the phase space between cycles 5/1 and 4/1 becomes stochastic just when centres 19/4 and 17/4 bifurcate into saddles. Equally we have that the bifurcation of centre 23/4 (21/4) makes the region between 6/1 and 11/2 (11/2 and 5/1) stochastic. Fig. 7 shows the plot of ξ bifurcation values versus μ of the centres 19/4 and 17/4. We have tested this behaviour for

$$0.05 \leq \mu/2\pi \leq 0.15;$$

for this range of μ we may write the following empirical rule.

The phase space region between cycles $N/1$ and $(2N+1)/2$ becomes stochastic when the centre $(4N+1)/4$ bifurcates into a saddle.

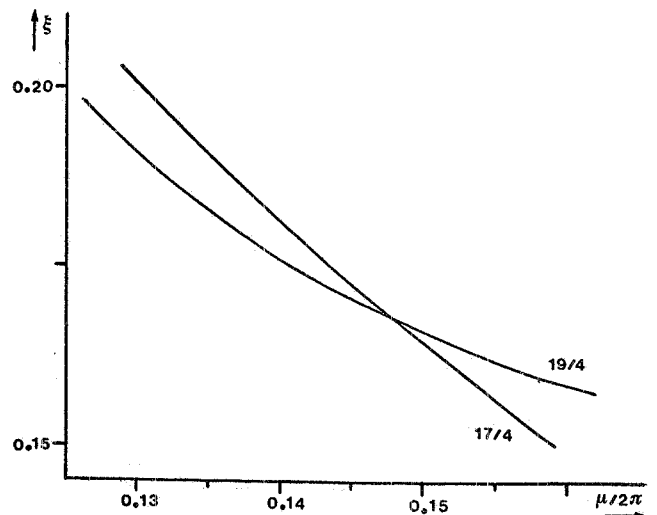


FIG. 7

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