Laboratori Nazionali di Frascati

LNF-73/76

A. F. Grillo:
CONFORMAL INVARIANCE IN QUANTUM FIELD THEORY.

Rivista Nuovo Cimento 3, 146 (1973).

RIVISTA DEL NUOVO CIMENTO VOL. 3, N. 2 Aprile-Giugno 1973

Conformal Invariance in Quantum Field Theory.

A. F. GRILLO

Laboratori Nazionali di Frascati del CNEN - Frascati

(ricevuto il 7 Aprile 1973)

1. – Introduction.

Since the discovery of scaling behaviour in deep inelastic electroproduction [1], quite a lot of theoretical investigations have been devoted to the study of the origin of this phenomenon.

Various models, such as the parton model, have been invented, that give a partially satisfactory (or unsatisfactory) explanation of experimental results, but the most important achievement which emerges is the emphasis that has been put on fundamental properties of field theory such as dilatation [2-6] and conformal [6-9] invariance.

In the deep inelastic experiments, physics is tested at very high energies, *i.e.* at short distances, where masses become negligible, so the limiting theory we are interested in is a zero-mass theory.

As a consequence, when masses become less and less important, one is free to rescale the energy units in the matrix elements: this is the content of the dilatation invariance [2, 3]. In a physical (massive) field theory things are not so simple, and one has to invent quite sophisticated methods to study the consequences of broken dilatations [4, 5].

A crucial concept related to dilatations is that of scale dimensions of a field, a number that only for a free field is the same as the naive mass dimension; in general, whenever it can be defined, it specifies the transformation of the field under dilatations (see the Appendix for mathematical details). Here it is worth noting that commutation relations, being nonlinear relations between operators, can fix the scale dimension; in particular canonical commutation relations put the (canonical) dimension equal to the free one.

It is an important idea, due to WILSON [3], that the renormalization procedure of any sensible field theory could eventually give an anomalous part to the dimension of fields: this comes from the infinite strength renormalization, and is a parameter which is determined by the interaction and in some sense characterizes the dynamics.

In a massless theory, apart from pathologies and with the cautions we will discuss in the next Section, there are other transformations that can be symmetries for the theory, namely the conformal transformations [6-9]. Again, for an exact mathematical definition we refer to the Appendix; it is however interesting to note that dilatations and conformal transformations, together with Poincaré transformations, form the largest group that leaves the light-cone invariant.

The full conformal group puts more restrictions on physical quantities, such as matrix elements and so on, than only dilatations, so one can hope to derive much more information on the structure of the theory we are considering. And in fact this is the case: vacuum expectation values are severely constrained [10], and in some cases uniquely determined, in terms of parameters such as anomalous dimensions and normalization constants.

These quantities are in principle fixed by the dynamics, so their computation is in general impossible; but conformal invariance allows us to make statements also on these.

In fact it has been noted that the Bethe-Salpeter equations that contain the dynamics become purely algebraic equations (at least in some cases) in these parameters. As they are nonlinear, one can hope that they have at most a discrete set of solutions: this is the aim of the so-called bootstrap approach [11-16], where the name means that self-consistency of the two sides of B.S. equations fixes the value of physical constants.

We have however to pay a very heavy price for this, for, as soon as we allow dimensions to be anomalous, we find that the LSZ limit no longer exists: in fact we lose the particle description [17], and, since scattering states are not present, also the S-matrix cannot be defined.

In particular there is no hope of obtaining information on the matrix element

$$\langle p|J_\mu(x)\, J_\nu(0)|p\rangle$$

which appears in deep inelastic scattering.

One way to circumvent this difficulty is that of applying conformal invariance to matrix elements of off-shell operators, such as the electromagnetic currents in $\pi^0 \to \gamma\gamma$ [18] or $e^+e^- \to e^+e^- +$anything [19].

The other way out is to consider conformal invariance not for the whole matrix elements, but only for products of operators and expand the products in sums of local operators [3, 20-26] to obtain conformal covariant operator-product expansions which exhibit many interesting properties.

All we have said for conformal invariance is implicitly assumed to hold in a zero-mass theory, however the relevance for the physical massive theory can be examined at the same level as for dilatations [8, 9].

In the following Sections we will consider more in details these aspects.

2. – Conformal invariance as an asymptotic symmetry in quantum field theory.

In the following Sections **3** and **4** a number of results will be derived by assuming the validity of conformal invariance right from the beginning.

It is then necessary to justify the assumption, since it is easy to check that dilatation (and also conformal) invariance cannot be valid in the presence of masses: in fact, since (see Appendix)

$$[P^2, D] = 2iP^2 , \tag{2.1}$$

no mass eigenstates can exist in a dilatation-invariant theory.

The problem then arises of understanding in what sense results concerning an unrealistic invariant theory can reflect themselves in the (massive) real-world physics. We want to present here only the relevant results, without giving any derivation.

Consider a definite theory, *e.g.* the self-interacting $g\varphi^4$ scalar theory. Formal manipulations, namely the use of canonical commutation relations for the interacting field, led to the proof that it is possible, at least in not too complicated cases, to redefine the energy-momentum tensor $\theta_{\mu\nu}$ in such a way that [27]

$$D = \int D_0(x)\, \mathrm{d}^3x , \qquad K_\mu = \int K_{0\mu}(x)\, \mathrm{d}^3x \tag{2.2}$$

and

$$D_\mu = x^\lambda \theta_{\lambda\mu} , \qquad K_{\mu\nu} = (2x_\mu x^\lambda - g^{\mu\lambda}x^2)\,\theta_{\nu\lambda} . \tag{2.3}$$

In order to generate a symmetry these currents have to be conserved:

$$\partial^\nu K_{\mu\nu} = 2x_\mu \theta_\lambda^{\ \lambda} = 2x_\mu(\partial^\lambda D_\lambda) = 0 , \tag{2.4}$$

so that in a conformally invariant theory the energy-momentum tensor has to be traceless.

At the same level one can show that

$$\theta_\lambda{}^\lambda(x) \sim m^2\varphi^2\,, \tag{2.5}$$

so that in the naive Ward identities following from eqs. (2.2) and (2.3) the Weinberg power-counting argument enables us to neglect the « soft » breaking term (2.5) at very short distances in the Euclidean region. The behaviour of Green's functions is then fixed, in the same limit, to be

$$G_n(\lambda^{-1}x_1, \ldots, \lambda^{-1}x_n) = \langle 0|T[\varphi(\lambda^{-1}x_1) \ldots \varphi(\lambda^{-1}x_n)]|0\rangle \underset{\lambda\to\infty}{\sim} \lambda^{-n} G_n(x_1, \ldots, x_n)\,, \tag{2.6}$$

the dimensions of fields having their canonical value and all things behaving canonically.

Equation (2.6) should be valid order by order in perturbation theory: it is however a well-known fact that, on the contrary, logarithmic terms appear that invalidate eq. (2.6) in *every order* of perturbation theory [3-6, 28].

The reason for this clash is quite evident: the formal manipulations used in deriving eq. (2.6) are valid only in view of the fact that the theory has been cut off to eliminate divergences. Now, a cut-off is a large mass, that largely breaks dilatation invariance.

So we do not expect that the renormalization procedure will maintain the canonical dimensions of fields or that the naive Ward identities continue to be valid.

For dilatational invariance the new Ward identities have been found (*) by CALLAN [14] and SYMANZIK [4, 5]:

$$\sum_{i=1}^{n}\left[x_\mu^i \frac{\partial}{\partial x_\mu^i} - (1+\gamma(g))\right]\langle 0|T[\varphi(x_1) \ldots \varphi(x_n)]|0\rangle = \tag{2.7}$$

$$= \int \mathrm{d}^4x \langle 0|T[\varphi(x_1) \ldots \varphi(x_n)\,\theta(x)]|0\rangle\,,$$

where $\theta(x)$, which is a sort of *effective* breaking term

$$\theta(x) = -\eta(g)m^2\varphi^2(x) - \beta(g)\varphi^4(x)\,, \tag{2.8}$$

depends, through the functions $\gamma(g)$, $\beta(g)$, $\eta(g)$, on the detailed dynamics of the theory.

(*) Obviously this is the form of the naive W.I. when $\beta(g) = \gamma(g) = 0$, $\eta(g) = -1$ which would have been obtained by using formal manipulations.

If the r.h.s. of eq. (2.7) had been negligible at short distances, then the anomalous dimension of φ could have been defined as

$$d = 1 + \gamma(g) . \tag{2.9}$$

In general this is not the case, since $\theta(x)$ is quadrilinear in φ and then the Weinberg argument no longer holds.

If however the physical value of the coupling constant g were at a zero of $\beta(g)$ (perhaps for some self-consistency reason, as proposed by WILSON [29]), then the theory would asymptotically be dilatation invariant with anomalous dimensions given by eq. (2.9). This hypothesis turns out to be too restrictive: we do not need to know if the real world is actually sitting on a zero of β but simply that such a zero does exist [4, 5], even if it is different from the actual value.

From the explicit solution of the Callan-Symanzik equation (2.7) it is then possible to see that in this case the asymptotic form of G_n approaches the solution of the homogeneous equation, which scales with anomalous dimensions fixed by the position of the zero.

As a remark, we note that it is possible to accommodate in this scheme also « dilatation multiplets » [23, 30-32] as defined in the Appendix: their Callan-Symanzik equation becomes a matrix equation and, interestingly enough, allows for logarithmic terms in the solution, without destroying the underlying dilatation invariance. The dynamical requirements in order to have operators transforming in this way are, however, rather obscure.

If we come back to the Callan-Symanzik equation, it has been noted [4, 5] that in the presence of a zero of β its asymptotic form is identical to the equation derived by GELL-MANN and LOW for the renormalization group [33].

This in turn implies that the asymptotic deep Euclidean limit is equivalent to the zero-mass Gell-Mann–Low limit (*i.e.* a zero-mass limit performed in the renormalization procedure in such a way that infra-red divergences are not introduced).

The way is now open to give meaning to asymptotic conformal invariance: the reason why the formal argument leading from dilatation to conformal symmetry can in principle not be valid is that it is not possible to exclude *a priori* that the relation (2.4) is modified by nonsoft terms, *i.e.* terms containing the fourth power of the field.

This happens to be exactly the case: but it has been proved by SCHROER [8] and later by PARISI [9], that one can identify $\theta(x)$ with θ_λ^λ, without any extra term, and that the « true » Ward identities read

$$\sum_{i=1}^{n}\left[x_{i\mu}\left(x_i^\lambda \frac{\partial}{\partial x_i^\lambda}\right) - x_i^2 \frac{\partial}{\partial x^{i\mu}} - 2(1+\gamma(g))\right] \langle 0|T[\varphi(x_1) \dots \varphi(x_n)]|0\rangle = \tag{2.10}$$

$$= 2\int \mathrm{d}^4x\, x_\mu \langle 0|T[\varphi(x_1) \dots \varphi(x_n)\theta(x)]|0\rangle .$$

So, asymptotic conformal invariance is regained in much the same way as dilatation invariance is: namely only if a zero of $\beta(g)$ exists, the zero-mass limit of the theory can be conformal invariant. In the following Sections we will always make the more restrictive hypothesis that we are studying a theory exactly at the zero of β.

3. – Some results from exact conformal invariance: Wightman functions and operator products.

Consider now a zero-mass (Gell-Mann–Low) field theory in which conformal invariance is a « true » symmetry (*i.e.* $\beta = 0$).

The transformation laws of fields are then determined, and this puts many restrictions on the associated Green's functions [10].

An instructive example comes from the study of two-point functions (*e.g.* Wightman functions)

$$G_2(x, y) = \langle 0|\varphi(x)\varphi(y)|0\rangle = G_2(x-y)\,. \tag{3.1}$$

Simple algebra, using dilatation invariance, shows that

$$G_2(x-y) = c[-(x-y)^2 + i\varepsilon(x_0-y_0)]^{-l} \tag{3.2}$$

with c an unspecified constant and l the dimension of φ.

When $l=1$, the two-point function is the free one, so the field itself is free [33, 34]. We are thus interested in examining the case in which $l \neq 1$. The spectrum condition requires $l \geqslant 1$, so, in all interesting cases, the matrix elements of fields decrease asymptotically (in the LSZ sense) faster than the particle factors in the reduction formulae, and consequently particle interpretation is lost (infra-particle theory [17] in the language of SCHROER).

Taking the discontinuity of G_2 we have the V.E.V. of the commutator, which, if dimensions are not integer, has the form

$$G_c(x-y) = c[\theta(x-y)^2]\,\varepsilon(x_0-y_0)[(x-y)^2]^{-l}\,, \tag{3.3}$$

while it is proportional to derivatives of delta-functions when the dimensions are integer.

This fact exhibits the most serious difficulty coming from conformal invariance: in fact it is obvious that, if there exists a unitary operator $U(g)$ that represents conformal transformations g (which can mix spacelike with timelike intervals) we can have, in the commutator

$$U(g)[\varphi(x), \varphi(y)]\,U^{-1}(g) = S(g, x)S(g, y)[\varphi(g^{-1}x), \varphi(g^{-1}y)]\,, \tag{3.4}$$

a violation of causality, if $l \neq$ integer (*i.e.* the commutator has support also *inside* the light-cone).

The conclusion is that no unitary $U(g)$ exists at least in a strict sense; a more careful examination reveals that what is not left invariant is the boundary (*is*) prescription [35], while the functional form of vacuum expectation values is left invariant.

The concept of « weak conformal invariance » as proposed by HORTAÇSU, SCHROER and SEILER has its basis on this circumstance. They propose not to consider conformal transformations as a symmetry for operators, but rather for Green's functions continued into the complex domain of co-ordinates (*e.g.* by Wick rotation) where no causality problem arises.

The quantities defined in the pseudo-Euclidean space-time are then obtained as different boundary values of the above-defined functions.

In the case of operator quantities one has to prove that the application of conformal invariance does not lead to wrong conclusions, by taking the appropriate matrix elements; in this way one can prove, as we shall see later on, that in fact the kind of invariance relevant for operator products is the weak one [36].

With this in mind, we can proceed in the study of the two-point function. Since its functional form is completely determined by dilatation invariance, the requirement of being also conformal covariant is either identically verified, or it implies that $c = 0$.

A selection rule in fact can be shown to hold, namely that the (conformal invariant) two-point function of every pair of operators is zero, if they do not belong to the same representation of the conformal group [37, 38].

Translated for Lorentz tensors (*i.e.* symmetric and traceless) this means that

$$\langle 0|O_{\alpha_1\ldots\alpha_n}(x)O_{\beta_1\ldots\beta_m}(0)|0\rangle = 0 \tag{3.5}$$

if $n \neq m$ or $l_n \neq l_m$.

The properties of two-point functions are then completely specified by conformal invariance, so also current conservation must be implicit, whenever it is required by the theory. It is a simple matter to prove that, in fact, the generalized conservation laws [36]

$$\partial_x^{\alpha_1}\langle 0|O_{\alpha_1\ldots\alpha_n}(x)O_{\beta_1\ldots\beta_n}(y)|0\rangle = 0 \tag{3.6}$$

hold, if and only if the dimensions of the tensor operators are « canonical », *i.e.*

$$l_n = 2 + n\,, \tag{3.7}$$

in other words the same dimensions as they have in a free-field theory.

Conservation in the operator form then follows [20]

$$\partial^{\alpha_1} O_{\alpha_1 \ldots \alpha_n}(x) = 0 \tag{3.8}$$

if the metric of the Hilbert space of the theory is positive definite; in the following we will see the implications of this result.

The above relations are a particular case of a more general statement that can be phrased in the following way: any n-point Green's function that transforms covariantly under the conformal group depends on an arbitrary function [10] of $n(n-3)/2$ (if $n \leqslant 6$) or $4n-15$ variables of the form

$$\omega_i = \frac{(x_i - x_{i+1})^2 (x_{i+2} - x_{i+3})^2}{(x_i - x_{i+2})^2 (x_{i+1} - x_{i+3})^2} . \tag{3.9}$$

The afore-mentioned selection rule is a particular case of this result: in fact, for $n = 2$, we have $n(n-3)/2 = -1$, *i.e.* the function is over-determined.

Analogous, and more stringent, conclusions can be stated for fields belonging to « dilatation multiplets » from the study of their two-point functions. Even the definition of such fields in fact loses sense, if one insists that the following conditions be verified: $D|0\rangle = 0$ and $K_\mu|0\rangle = 0$ *and* positivity in the Hilbert space. So one is led to the conclusion that in this case conformal invariance is (possibly) realized in a spontaneously broken way.

When $n = 3$ $(n(n-3)/2 = 0)$ the Green's function is completely determined, apart from an arbitrary multiplicative constant: for scalar fields we have, for example,

$$\langle 0|\varphi(x_1)\varphi(x_2)\varphi(x_3)|0\rangle = c'[(x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2]^{-l/2} . \tag{3.10}$$

The four-point function $(n = 4,\ n(n-3)/2 = 2)$ is fixed up to an arbitrary function of two « harmonic ratios »:

$$\langle 0|\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle = [(x_1 - x_3)^2 (x_2 - x_4)^2]^{-l} F(\eta, \varrho) ,$$

where the variables η, ϱ are of the kind described above.

It is not possible to put further restrictions on F, unless one makes some *dynamical* assumption, *i.e.* one specifies better the model one is considering. As an example we can consider the model proposed by MIGDAL [38], in which the field $\varphi(x)$ is composed of n free fields:

$$\varphi(x) = \psi_1(x) \ldots \psi_n(x) , \tag{3.11}$$

and analytical continuation is performed in n; this model, which is interesting for other reasons, is trivial from this point of view, in that the model is solvable and all the n-point functions are determined.

Another possible approach will be pointed out in the next Section: we emphasize that a conformally invariant four-point function has no clear-cut relation with quantities such as for example the structure functions in deep inelastic scattering, due to the lack of an LSZ particle interpretation in all the physically interesting cases.

A bridge to this kind of applications is provided by the well-known expansion [9, 39, 40] for the product of two or more local operators

$$A(x)B(0) \underset{x^2 \to 0}{\sim} \sum_n c_n(x^2) x^{\alpha_1} \dots x^{\alpha_n} O_{\alpha_1 \dots \alpha_n}(0) , \tag{3.12}$$

where the c-number functions $c_n(x^2)$ have singularities at short and lightlike distances governed by the dimensions of the fields:

$$c_n(x^2) = a_n(-x^2 + i\varepsilon x_0)^{-\frac{1}{2}(l_A + l_B - l_n + n)} . \tag{3.13}$$

Conformal invariance can then be required to hold on both sides of eq. (3.12), so giving relations inside « towers » of operators forming an irreducible representation of the conformal group [41, 42], *i.e.* an operator $O_{\alpha_1 \dots \alpha_n}(x)$ transforming as in eq. (3.12) and its derivatives: at the leading order in x^2 these relations take the form (*) (**)

$$\begin{cases} A(x)B(0) \sim \sum_n c_n(x^2) x^{\alpha_1} \dots x^{\alpha_n} \int_0^1 \mathrm{d}\lambda \, \lambda^{a-1}(1-\lambda)^{b-1} O_{\alpha_1 \dots \alpha_n}(x) , \\ a = \frac{1}{2}(l_A - l_B + l_n + n) , \qquad b = \frac{1}{2}(l_B - l_A + l_n + n) . \end{cases} \tag{3.14}$$

The expansion (3.14) automatically solves some problems left open by eq. (3.12): for instance that of causality, in the sense that

$$[A(x)B(0), C(z)] = 0, \qquad \text{if } (z-x)^2 < 0, \ z^2 < 0 , \tag{3.15}$$

on both sides of the Wilson expansion: this is a dynamical requirement for eq. (3.12), while it is automatically satisfied for every n in the conformal invariant expression.

(*) The validity of the Wilson expansion has been proved also in the Callan-Symanzik framework, if a zero of the $\beta(g)$ exists. Also the equation (3.13) can be verified in this case, as a relation between leading and next-to-leading terms (in x) when the short-distance limit is approached in the Euclidean region [24, 25].

(**) The $c_n(x^2)$ have the form (3.13); note that the constants a_n are *not* specified by conformal invariance.

Operator-product expansions can also be derived that account for conformal invariant relations between leading and nonleading terms in x^2: their derivation is very difficult, since one cannot use only the conformal algebra, but rather the action of finite transformations [43, 44] or the 6-dimensional formalism [45]. The two approaches have been proved to lead to equivalent results, at least in the Euclidean region [46]; a lot of care must be taken in order to translate these results into the ordinary Minkowski space [47].

The use one can make of these « whole space » expansions (*) is mainly related to the study of the structure of the fourth-order [37] (and higher-order [48]) functions: the insertion of operator-product expansions in V.E.V.'s in fact allows one to reduce a higher-order function to a sum of smaller-order ones. However the causality requirements are not easy to verify for a « whole space » expansion, so the best one can hope to prove is that, *e.g.*, the four-point function *is* causal [49].

An important feature one can easily check is that the invariance of the operator-product expansions is equivalent to the invariance of the two respective three-point functions *plus* the selection rule [45, 50]; in this sense Wilson expansions are « weakly » conformal invariant.

As a consequence normalization factors are connected each other: for example, from eqs. (3.2), (3.10) and (3.13) we obtain

$$c' = ca_0 .$$

This fact is important for phenomenological applications, as these normalization factors can be possibly measured in experiments.

Let us illustrate this possibility by briefly describing the result derived, in a slightly different framework, by CREWTHER [18]. The use of Wilson expansions for e.m. and axial currents

$$(3.16)\qquad \begin{cases} T\big(J_\mu(x)J_\nu(0)\big) = R(g_{\mu\nu}x^2 - 2x_\mu x_\nu)\, I(\pi x^2)^{-4} + K\varepsilon_{\mu\nu\varrho\sigma}x^\varrho A^\sigma(0)(3\pi^2 x^4)^{-1}, \\ T\big(A_\mu(0)A_\nu(y)\big) = R' I(g_{\mu\nu}y^2 - 2y_\mu y_\nu)(\pi y^2)^{-4} \quad (I \text{ is the identity operator}), \end{cases}$$

and conformal invariance for the 3-point function allows one to write the anomalous Adler's constant (which is fixed by PCAC and $\pi^0 \to \gamma\gamma$ decay: $S \sim 0.5$)

$$(3.17)\qquad S = -\frac{1}{12}\pi^2 \varepsilon^{\mu\nu\alpha\beta}\int \mathrm{d}^4x\, \mathrm{d}^4y\, x_\alpha y_\beta \langle 0|J_\mu(x)J_\nu(0)\,\partial^\lambda A_\lambda(y)|0\rangle ,$$

(*) For the validity of these expansions we have to require that either the physical world is at a zero of $\beta(g)$ or that $\beta(g) \equiv 0$, as happens to be the case in the two-dimensional Thirring model.

as

$$3S = KR' , \tag{3.18}$$

where K can be measured in polarized deep inelastic scattering and R' is connected through asymptotic chiral invariance to the electron-positron total cross-section.

Similar relations can also be found for other off-mass-shell processes [19], like $e^+e^- \to e^+e^- +$ anything.

Wilson's expansion can also be used to transfer properties coming from conformal invariance to on-mass-shell matrix elements such as the pion form factor [12, 51]; unfortunately application to the most interesting case, namely the deep inelastic electroproduction, is hopeless, since in the matrix element between equal-momentum states all the derivatives of operators disappear and an infinite sum over *inequivalent* representations of the conformal group remains:

$$\langle p|J_\mu(x)\,J_\nu(0)|p\rangle \underset{x^2\to 0}{=} \sum_n c_n(x^2)\langle p|O_{\alpha_1\ldots\alpha_n}(0)|p\rangle x^{\alpha_1}\ldots x^{\alpha_n} . \tag{3.19}$$

The observed scaling in deep inelastic experiments implies that an infinite number of local tensors contribute with « canonical dimension » $l_n = 2 + n$, so that

$$c_n = a_n(-x^2 + i\varepsilon x_0)^{-\frac{1}{2}(l_A + l_B - 2)} . \tag{3.20}$$

Conformal invariance in turn implies that the operators are conserved. We have then this situation: dilatation invariance at short distances implies conformal invariance—scaling plus conformal invariance constrain the limiting (Gell-Mann–Low) theory to possess an infinite number of conserved tensor quantities. It has been proved [52] that the infinite conservation laws in simple models (such as the $g\varphi^4$) mean that the theory, though formally interacting, is free.

However this is not always the case in more complicated theories, such as the above-mentioned model proposed by MIGDAL [38]; there the conservation laws are the consequences of the conservation of the « type » of fundamental constituents $\psi_n(x)$.

It is however necessary to note that we should perhaps have put a question mark on the statement of the « observed scaling »: in fact recent semi-empirical [53, 54] analyses have shown that the observed data are also consistent with anomalous dimensions of the type

$$l_n = 2 + n + a\left[1 - \frac{12}{(n+1)(n+2)}\right] , \tag{3.21}$$

which have been suggested also from theoretical arguments [55-57].

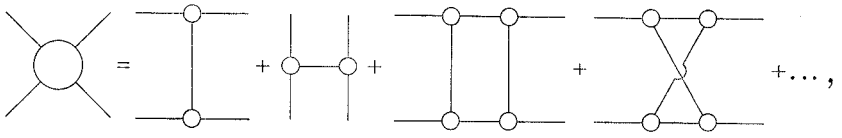
4. – The bootstrap approach.

The considerations developed in the preceding Sections are in some sense the «kinematics» of conformal invariance: quantities such as anomalous dimensions or normalizations are not specified and not constrained by the theory inself.

An interesting tool for studying dynamical properties of the theory is provided for by the bootstrap approach, first proposed by MIGDAL [13] and POLYAKOV [11] and by PARISI and PELITI [14]. We can consider this approach as complementary to the study of «kinematical» properties of a conformal invariant field theory in the sense that it starts from the knowledge of the structure of the Wightman (specifically 2- and 3-point) functions.

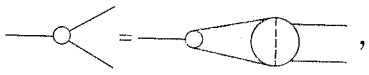
Let us take as an example the case of the 3-point bootstrap equations, closely following the original treatment of MIGDAL [13].

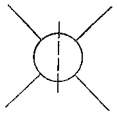
Consider a theory with a conformal invariant trilinear interaction, such as the $g\varphi^3$ in 6-dimensional space-time (or the *ps-n* interaction in four dimensions): two- and three-point functions are functionally determined and we can make the hypothesis that all n-point functions have a skeleton expansion [15, 58]

(4.1) 

that is an expansion in Feynman diagrams without self-energy parts and vertex corrections, in which propagators and vertices are the «true» conformal invariant ones.

This kind of expansion can then be used also for the Bethe-Salpeter kernel that appears in the renormalized Schwinger-Dyson equation for the three-point function (inhomogeneous terms are absent since they would violate dilatation invariance)

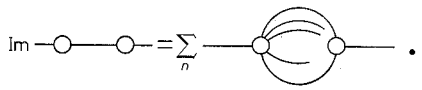
(4.2) 

where  means that the B.S. kernel is two-particle irreducible.

We know that conformal invariance fixes the form of propagators and vertex functions appearing in the expansion (4.1) up to normalization factors. It can be shown that all the normalization factors can be reabsorbed in the nor-

malization of the vertex function (the « coupling constant »). So we are left with an integral equation that, whenever the integrals involved in eq. (4.2) do not introduce divergences, thus spoiling its meaning, becomes an algebraic equation in the unknown coupling constant and anomalous dimensions [16, 58] which are the only parameters left: since the equation is nonlinear one can hope for at most a discrete set of solutions.

As for the propagator, since unitarity is not satisfied term by term in the skeleton expansion, this requirement can be adopted as a basis for bootstrap equations (note that it is sufficient to know the imaginary part of the propagator) [14]:

(4.3) 

Also a Bethe-Sapeter approach can be used [16].

The main point for the validity of the previous statements deeply relies on the absence of divergences: this has been proved by MACK and TODOROV [15] for some range of the value of the dimensions. Up to now there is no solution of the bootstrap equations in 4 dimensions, whereas solutions have been found in the framework of calculations of critical indices [16], and, through analytical continuation in the dimension of the space-time, in $6+\varepsilon$ dimensions [56].

The other hard problem in this approach is the verification of the Ward identities when the vertex functions contain currents or the energy-momentum tensor: a detailed treatment of this point can be found in the paper by MACK and SYMANZIK [16].

Finally, as has been pointed out by MACK [53], the bootstrap approach is also important because it can be thought of as a constructive way to the Gell-Mann–Low limiting zero-mass theory: it is clear in fact that conformal invariance is maintained in every step of the procedure thus yielding at the end a *dynamical* and self-consistent invariant theory.

5. – Two-dimensional models.

Two-dimensional field theory has often been used as a laboratory for the study of quantum field theory: as an example the Thirring model [59, 60], being solvable in its zero-mass limit, allows one to study many formal porperties of quantum field theory.

From the point of view of conformal invariance, such models stem from a twofold interest: the structure of conformal transformations is rather peculiar in two dimensions [60] and, secondly, in the Thirring model it can be proved that the Callan-Symanzik function is identically zero [61], so the theory is *exactly* conformal invariant in the Gell-Mann–Low limit.

As for the peculiarity of the conformal group in two dimensions we can see, from the Appendix, that in the defining equations $D = 2$ is an exceptional case (see eq. (A.3*b*)): in fact the generating function $\lambda(x)$ is no longer constrained to be linear in x, but has rather to verify the equation

$$\Box \lambda(x) = 0 . \tag{5.1}$$

The full conformal group becomes then infinite dimensional, *i.e.* every harmonic function can generate conformal transformations. Of course this algebra contains a subalgebra which is the restriction of that in 4 dimensions ($O_{4,2}$), that is $O_{2,2}$: this algebra plays a special role, as we shall see later.

The structure of the generators, the transformation laws of the fields and so on can be easily derived in general, but are better illustrated in the framework of the Thirring model.

Here, as is well known, we have a self-interacting « spinor » field with equations of motion of the form

$$\partial\!\!\!/ \psi(x) = g : (\gamma \cdot J)\psi : (x) \qquad (J_\mu = : \bar{\psi}(x)\gamma_\mu \psi(x) :) . \tag{5.2}$$

Here the canonical approach, starting from the commutation relations of the fields, can be substituted with some advantage by an alternative one that uses the currents

$$J_\mu(x) \quad \text{and} \quad J^5_\mu(x) = \varepsilon_\mu{}^\nu J_\nu(x) = : \psi\gamma_5\gamma_\mu\psi : (x) , \qquad \varepsilon_{01} = -\varepsilon_{10} = -1, \tag{5.3}$$

as dynamical quantities.

The relation between the axial and vector currents, which is a consequence of the fact that the structure of the Dirac matrices is poorer than in 4 dimensions, is an important feature of the model. In fact, together with the conservation of both currents (which can be proven in the zero-mass limit), it implies that the currents fulfil a *free* Klein-Gordon equation, which makes formal manipulations very easy to handle, and, more significantly, allows us to prove that the Callan-Symanzik function is zero for every value of the coupling constant [61]. This is a very crucial point: it means in fact that the short-distance limiting theory is *exactly* dilatation and conformal invariant. Of course the conformal group here implied is the « restricted » ($O_{2,2}$) one, as we shall better see later.

Coming back to the « extended » algebra, simple computations allow us to write its generators in terms of the currents [60], via the energy-momentum tensor

$$L(f) = \int \mathrm{d}x^1 \theta_{0\nu}(x) f^\nu(x) . \tag{5.4}$$

Here $f^\nu(x)$ is the generating function of infinitesimal conformal transformations, connected to $\lambda(x)$ through eq. (A. 3a), and

$$\theta_{\mu\nu}(x) = :J_\mu(x)\, J_\nu(x): - \tfrac{1}{2} g_{\mu\nu} :J_\alpha(x) J^\alpha(x): . \tag{5.5}$$

It is easy to see that, if we use the standard light-cone variables

$$u = x^0 + x^1 , \qquad v = x^0 - x^1 , \tag{5.6}$$

the whole algebra splits into the direct product of the u- and v-part (in the same way as the $O_{2,2}$ subalgebra splits into $O_{2,1} \otimes O_{2,1}$) which in a particular basis can be expressed as

$$\begin{cases} L_n^u = \int \mathrm{d}u \theta_+(u)\, u^{1-n} & (\theta_+ = \theta_{00} + \theta_{01}) , \\ L_n^v = \int \mathrm{d}v\, \theta_-(v)\, v^{1-n} & (\theta_- = \theta_{00} - \theta_{01}) , \end{cases} \tag{5.7}$$

with commutation relations

$$[L_n^u, L_m^u] = (m-n) L_{n+m}^u + c\text{-number} , \qquad [L_n^u, L_m^v] = 0 . \tag{5.8}$$

Amusingly enough, this is the same « gauge » algebra that appears in the dual models [62, 63].

We have anticipated that only $O_{2,2}$ is the invariance group: in fact, while the equations of motion are formally invariant under the whole algebra, Wightman functions are invariant only under $O_{2,2}$, thus resembling a situation in which the symmetry is spontaneously broken.

Finally, the fact that $\beta(g) \equiv 0$, *i.e.* the exact invariance of the zero-mass theory, allows us to write a « whole space » operator-product expansion; it turns out that the expansion originally proposed by DELL'ANTONIO, FRISHMAN and ZWANZIGER [64] is invariant not only under $O_{2,2}$ transformations, but also under the whole algebra.

6. – Conclusions.

In the preceding Sections we have tried to illustrate some of the most relevant results that follow from the assumed conformal invariance of field theory: they are encouraging, mainly in the direction of giving a self-contained theory that can be thought of as a limit of realistic theories. However, the relevance to the real world is not clear and normally requires additional hypoteses, such as the operator-product expansion.

There are of course many other ways in which this argument can be studied: as an example we can quote the work presented by NOBILI to this Conference [65],

in which broken dilatation invariance (*i.e.* partial conservation of the dilatation current) is used to derive constraints on Green's functions.

Beyond these applications, there is a number of purely theoretical problems left open, and the most important one is perhaps that of the formulation of quantum electrodynamics in a conformal invariant way.

In fact, although conformal invariance found its first physical application as the symmetry group of the Maxwell equations, its extension to the quantized theory is very difficult due to the fact that conformal and gauge transformations are strictly connected.

An important achievement in this direction has been found by ADLER [66], with the formulation of a conformal quantum electrodynamics for massless particles, not taking into account photon self-energy parts.

APPENDIX

Some general results about the conformal group.

Conformal transformations are defined as the space-time transformations that leave the metric tensor (in the flat space) invariant up to multiplication by a scalar function of the co-ordinates:

$$x_\mu \to x'_\mu: \qquad g'_{\mu\nu}(x) = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\varrho}{\partial x'^\nu} g_{\lambda\varrho} = \Lambda(x) g_{\mu\nu} . \tag{A.1}$$

Their name comes from the fact that they leave the cosines of infinitesimal angles invariant.

In the following, unless otherwise specified, we will consider a D-dimensional space-time: in it infinitesimal transformations

$$x'_\mu \simeq x_\mu + \delta x_\mu(x) , \qquad \Lambda(x) \simeq 1 - \lambda(x) , \tag{A.2}$$

are constrained to obey the equations

$$\partial_\mu \delta x_\nu + \partial_\nu \delta x_\mu = g_{\mu\nu}\lambda(x) , \tag{A.3a}$$

$$[(D-2)\partial_\lambda\partial_\varrho + g_{\lambda\varrho}\Box]\lambda(x) = 0 , \tag{A.3b}$$

which, for $D \neq 2$, fix their form to be

$$\delta x_\mu = \delta a_\mu + \delta\omega_{\mu\nu}x^\nu + x_\mu \delta\varrho + (x^2\delta c_\mu - 2(x\cdot\delta c)x_\mu) . \tag{A.4}$$

As $\delta\omega_{\mu\nu}$ is antisymmetric, δx depends on $(D+1)(D+2)/2$ parameters, as is appropriate for an $O_{D,2}$ group: in the case usually considered $D = 4$ and the group becomes $O_{4,2}$. An interesting feature is that $O_{4,2}$ does not act linearly in the Minkowski space, but rather in a 6-dimensional space [6*a*, *b*], thus conformal invariance can be induced through a set of linear relations and the results mapped in a well-defined way into the physical space-time.

Going back to eq. (A.4) we can easily identify the first two terms as infinitesimal Poincaré transformations, while the other two correspond to dilatation and special conformal transformations, that in the finite form read

$$\text{(A.5)} \qquad \begin{cases} x'_\mu = \varrho x_\mu: & \text{dilatations}, \\ x'_\mu = \dfrac{x_\mu - c_\mu x^2}{1 - 2c\cdot x + c^2 x^2}: & \text{special conformal transformation}. \end{cases}$$

Note that all these transformations leave the light-cone ($x^2 = 0$) invariant.

Action of transformation on fields is related to the existence of unitary operators such that

$$\text{(A.6)} \qquad U(g)\varphi(x)\,U^\dagger(g) = S(g, x)\varphi(x)$$

(g is a member of the group) with multiplication law

$$S(gg', x) = S(g, x)\,S(g', g^{-1}x)\,.$$

Here $\varphi(x)$ stands for an arbitrary collection of fields.

It turns out to be more convenient to deal with infinitesimal transformations, *i.e.* representations of the algebra: they can be found with the help of the standard induced-representation method.

For our purposes we need to consider only dilatations and special conformal transformations.

For dilatations we have

$$\text{(A.7)} \qquad [\varphi(x), D] = i(x^\lambda \partial_\lambda + L)\varphi(x)\,,$$

where the matrix L can be put in the block-diagonal form

$$\text{(A.8)} \qquad L = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$

and the matrices λ have the Jordan form

$$\text{(A.9)} \qquad \lambda_j = \begin{bmatrix} l_j & & 0 \\ 1 & \ddots & \\ 0 & & 1\ l_j \end{bmatrix}.$$

If the rank of the j-th block is 1, *i.e.* $\lambda_j = l_j$, then the j-th field is said to have « scale dimension » l_j; in the other case the fields belonging to the block are said to form a « dilatation multiplet » and l_j is called « diagonal dimension » [30-32].

For conformal transformations we have

$$\text{(A.10)} \qquad [\varphi(x), K_\mu] = i\left(2x_\mu x^\lambda \partial_\lambda - x^2 \partial_\mu - 2x^\lambda [g_{\mu\lambda} L + i\Sigma_{\mu\lambda}]\right)\varphi(x) + \tilde{K}_\mu \varphi(x)$$

($\Sigma_{\mu\lambda}$ is the spin part of the Lorentz generator $M_{\mu\nu}$), where the form of $\tilde{K}_\mu$ is roughly similar to that of L; however all the physically interesting cases are those for which $\tilde{K}_\mu = 0$.

Finally we list the commutation relations of the conformal algebra

$$\text{(A.11)} \qquad \begin{cases} [K_\mu, M_{\lambda\nu}] = i(g_{\mu\lambda}K_\nu - g_{\mu\nu}K_\lambda)\,, & [D, M_{\lambda\mu}] = 0\,, \\ [K_\mu, P_\nu] = -2i(g_{\mu\nu}D + M_{\mu\nu})\,, & [D, P_\mu] = iP_\mu\,, \\ [K_\mu, K_\nu] = 0\,, & [K_\lambda, D] = iK_\lambda\,, \end{cases}$$

plus that of the Poincaré algebra.

REFERENCES

[1] See, for example, G. MILLER *et al.*: *XV International Conference on High-Energy Physics* (Kiev, 1970), and references therein.

[2] K. WILSON: *On short-distance behaviour in quantum field theory*, Cornell Report, unpublished.

[3] K. WILSON: *Phys. Rev.*, **179**, 1499 (1969).

[4] C. G. CALLAN jr.: *Phys. Rev. D*, **2**, 1451 (1970).

[5] K. SYMANZIK: *Comm. Math. Phys.*, **18**, 227 (1970); **23**, 71 (1971).

[6] There are many reviews on the argument; see for example: *a*) G. MACK and A. SALAM: *Ann. of Phys.*, **53**, 174 (1969); *b*) S. FERRARA, R. GATTO and A. F. GRILLO: *Springer Tracts in Modern Physics*, Vol. **67**; *c*) B. SCHROER: talk given at the *IV Simposio Brasileiro de Fisica Teorica* (Rio de Janeiro, 1972); *d*) G. MACK: *Lecture Notes in Physics*, Vol. **17** (Berlin, 1972), p. 300.

[7] D. G. BOULWARE, L. S. BROWN and R. D. PECCEI: *Phys. Rev. D*, **2**, 293 (1970).

[8] B. SCHROER: *Lett. Nuovo Cimento*, **2**, 627 (1971).

[9] G. PARISI: *Phys. Lett.*, **39** B, 643 (1972).

[10] E. G. SCHREIER: *Phys. Rev. D*, **3**, 980 (1971); and also ref. [6*b*] and [11, 12] below.

[11] A. M. POLYAKOV: *JETP Lett.*, **12**, 381 (1970).

[12] A. A. MIGDAL: *Phys. Lett.*, **37** B, 98 (1971).

[13] A. A. MIGDAL: *Phys. Lett.*, **37** B, 396 (1971).

[14] G. PARISI and L. PELITI: *Lett. Nuovo Cimento*, **2**, 627 (1971).

[15] G. MACK and I. T. TODOROV: *Phys. Rev.*, to be published.

[16] G. MACK and K. SYMANZIK: *Comm. Math. Phys.*, **27**, 247 (1972).

[17] B. SCHROER: *Fortschr. Phys.*, **11**, 1 (1963).

[18] R. J. CREWTHER: *Phys. Rev. Lett.*, **28**, 1421 (1972).

[19] S. FERRARA, A. F. GRILLO and G. PARISI: Rome preprint No. 428 (1972), to be published.

[20] S. FERRARA, R. GATTO, A. F. GRILLO and G. PARISI: *Phys. Lett.*, **38** B, 332 (1972).

[21] W. ZIMMERMANN: in *1970 Brandeis Lectures*, Vol. **1** (Cambridge, Mass., 1970).

[22] K. G. WILSON and W. ZIMMERMANN: *Comm. Math. Phys.*, **24**, 87 (1972).

[23] P. OTTERSON and W. ZIMMERMANN: *Comm. Math. Phys.*, **24**, 107 (1972).

[24] C. G. CALLAN and D. GROSS: Princeton report (1972).

[25] S. FERRARA, A. F. GRILLO and G. PARISI: Frascati Report LNF-72/103 (1972), to be published.
[26] G. PARISI: *Phys. Lett.*, **39** B, 643 (1972).
[27] C. G. CALLAN jr., S. COLEMAN and R. JACKIW: *Ann. of Phys.*, **59**, 42 (1970).
[28] S. COLEMAN and R. JACKIW: *Ann. of Phys.*, **67**, 552 (1972).
[29] K. WILSON: *Phys. Rev. D*, **2**, 1473 (1970).
[30] G. F. DELL'ANTONIO: New York University preprint (1972).
[31] S. FERRARA, R. GATTO and A. F. GRILLO: *Phys. Lett.*, **42** B, 264 (1972).
[32] B. SCHROER: *Lecture Notes in Physics*, Vol. **17** (Berlin, 1972), p. 364.
[33] M. GELL-MANN and F. LOW: *Phys. Rev.*, **95**, 1300 (1954).
[34] K. POHLMEYER: *Comm. Math. Phys.*, **12**, 204 (1969).
[35] M. HORTAÇSU, B. SCHROER and R. SEILER: *Phys. Rev. D*, **5**, 1319 (1972).
[36] See ref. [6*a*]; S. FERRARA, R. GATTO, A. F. GRILLO and G. PARISI: *Lett. Nuovo Cimento*, **4**, 115 (1972).
[37] See ref. [6*a*]; S. FERRARA, R. GATTO, A. F. GRILLO and G. PARISI: *Nucl. Phys.*, **49** B, 77 (1972).
[38] A. A. MIGDAL: Chernogolovka preprint (1972).
[39] Y. FRISHMAN: *Phys. Rev. Lett.*, **25**, 960 (1970).
[40] R. BRANDT and G. PREPARATA: *Nucl. Phys.*, **27** B, 541 (1971).
[41] S. FERRARA, R. GATTO and A. F. GRILLO: *Nucl. Phys.*, **34** B, 349 (1971).
[42] S. FERRARA, R. GATTO and A. F. GRILLO: *Phys. Lett.*, **36** B, 124 (1971).
[43] L. BONORA, G. SARTORI and M. TONIN: *Nuovo Cimento*, **10** A, 667 (1972).
[44] L. BONORA, S. CICCARIELLI, G. SARTORI and M. TONIN: talk presented at the *Informal Meeting on Conformal Invariance, Frascati, 1972.*
[45] S. FERRARA, R. GATTO and A. F. GRILLO: *Phys. Rev. D*, **5**, 3102 (1972).
[46] L. BONORA: *Lett. Nuovo Cimento*, **3**, 548 (1972).
[47] S. FERRARA, A. F. GRILLO and G. PARISI: *Lett. Nuovo Cimento*, **5**, 147 (1972).
[48] S. FERRARA and G. PARISI: *Nucl. Phys.*, **42** B, 281 (1972).
[49] S. FERRARA, R. GATTO, A. F. GRILLO and G. PARISI: to be published.
[50] See ref. [6*a*].
[51] S. FERRARA, A. F. GRILLO and G. PARISI: *Nuovo Cimento*, **12**, 952 (1972).
[52] G. PARISI: *Phys. Lett.*, **42** B, 114 (1972).
[53] G. MACK: *Lecture Notes in Physics*, Vol. **17** (Berlin, 1972), p. 300.
[54] G. PARISI: Frascati Report LNF-72/93 (1972), to be published.
[55] N. CHRIST, B. HASSLACHER and A. MUELLER: *Light-cone and perturbation theory*, Report CO-3067(2)-9 (1972).
[56] G. MACK: *Expansions around canonical dimensions in conformal invariant quantum field theory*, Berna preprint (1972).
[57] I. T. TODOROV: *Lecture Notes in Physics*, Vol. **17** (Berlin, 1972), p. 270.
[58] W. THIRRING: *Ann. of Phys.*, **3**, 91 (1958).
[59] K. JOHNSON: *Nuovo Cimento*, **20**, 773 (1961).
[60] S. FERRARA, R. GATTO and A. F. GRILLO: *Nuovo Cimento*, **12**, 959 (1972).
[61] M. GOMES and J. H. LOWENSTEIN: University of Pittsburg preprint, NYO-3829-90 (1972).
[62] See, for example, E. DEL GIUDICE, P. DI VECCHIA and S. FUBINI: *Ann. of Phys.*, **70**, 378 (1972) and references therein.
[63] S. FUBINI, H. HANSON and R. JACKIW: to be published.
[64] G. F. DELL'ANTONIO, Y. FRISHMAN and D. ZWANZIGER: *Phys. Rev. D*, **6**, 988 (1972).
[65] R. NOBILI: presented at this Conference.
[66] S. L. ADLER: *Phys. Rev. D*, **6**, 3445 (1972).