COMITATO NAZIONALE PER L'ENERGIA NUCLEARE
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LNF-73/64
7 Novembre 1973

A. Renieri: INCOHERENT BEAM-BEAM EFFECT: A COMPUTER SIMULATION.

(Performed during the PEP Summer Study, Stanford-Berkeley, August 1973).

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A. Renieri: INCOHERENT BEAM-BEAM EFFECT: A COMPUTER SIMULATION.(x)

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1. - INTRODUCTION. -

In this paper we investigate, by means of a computer simulation, the incoherent beam-beam effect under the following conditions:

a) Weak beam against strong beam (i.e. single particle against strong beam);

b) Head-on collision;

c) Strong beam shorter than β value at crossing point;

d) No noise and no damping.

From b) and c) conditions it follows that only the (x, z) transverse modes are affected by beam-beam interaction.

In order to have a charge distribution function of the strong beam, which can be easily handled in the analytical computation of the space-charge forces, we have chosen a distribution function of the form:

$$f(x, z) = \frac{\sigma_x \sigma_z}{\pi^2} \frac{1}{x^2 + \sigma_x^2} \frac{1}{z^2 + \sigma_z^2} \qquad (1)$$

which is indeed unphysical, because it has very long tails.

The computer program simulates the space charge forces from the strong beam as a δ-function-like kick given to the weak beam particle, this is allowed by c) condition (point-like strong beam). So we can put the motion equations of the weak beam particle in the form of turn by turn recurrence equations.

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2. - MOTION EQUATIONS. -

We assume that the machine is perfect, so that the transverse modes recurrence equations are :

$$\begin{pmatrix} x \\ x' \\ z \\ z' \end{pmatrix}_{n+1} = \begin{pmatrix} \cos\mu_x , & \beta_x \sin\mu_x , & 0 , & 0 \\ -\frac{1}{\beta_x}\sin\mu_x , & \cos\mu_x , & 0 , & 0 \\ 0 , & 0 , & \cos\mu_z , & \beta_z \sin\mu_z \\ 0 , & 0 , & -\frac{1}{\beta_z}\sin\mu_z , & \cos\mu_z \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} x \\ x' \\ z \\ z' \end{pmatrix}_n + \begin{pmatrix} \beta_x \Theta_x(x,z) \operatorname{sen}\mu_x \\ \Theta_x(x,z)\cos\mu_x \\ \beta_z \Theta_z(x,z) \operatorname{sen}\mu_z \\ \Theta_z(x,z)\cos\mu_z \end{pmatrix}_n , \qquad (2)$$

where n labels the turn, β_x and β_z are the β values at the crossing point, μ_x and μ_z are the betatron phase differences between a crossing and the next one. $\Theta_x(x,z)$ and $\Theta_z(x,z)$ are the transverse kicks experienced by the (x, z) particle. $\Theta_x(x,z)$ and $\Theta_z(x,z)$ are given in App. A (eq. A.2).

If we put

$$\beta_x \Theta_x(x,z) = -4\pi\, \xi_x F_x(x,z) , \qquad \beta_z \Theta_z(x,z) = -4\pi\, \xi_z F_z(x,z) ,$$

where ξ_x and ξ_z are defined by the equations :

$$\xi_x = -\frac{1}{4\pi} \beta_x \frac{\partial}{\partial x} \Theta_x(x,z) \Big|_{x=z=0}$$

$$\xi_z = -\frac{1}{4\pi} \beta_z \frac{\partial}{\partial z} \Theta_z(x,z) \Big|_{x=z=0} \qquad (3)$$

the equations (2) become :

$$\begin{pmatrix} x \\ x' \\ z \\ z' \end{pmatrix}_{n+1} = \begin{pmatrix} \cos\mu_x & , \beta_x \sin\mu_x , & 0 & , 0 \\ -\frac{1}{\beta_x}\sin\mu_x , & \cos\mu_x , & 0 & , 0 \\ 0 & , 0 & , \cos\mu_z & , \beta_z \sin\mu_z \\ 0 & , 0 & , -\frac{1}{\beta_z}\sin\mu_z , & \cos\mu_z \end{pmatrix} .$$

$$\cdot \begin{pmatrix} x \\ x' \\ z \\ z' \end{pmatrix}_n - 4\pi \begin{pmatrix} \xi_x F_x(x,z)\sin\mu_x \\ \frac{\xi_x}{\beta_x} F_x(x,z)\cos\mu_x \\ \xi_z F_z(x,z)\sin\mu_z \\ \frac{\xi_z}{\beta_z} F_z(x,z)\cos\mu_z \end{pmatrix}_n \qquad (4)$$

From eq.(3) and eq.(A.2) (see App. A) we derive :

$$\xi_x = \frac{r_o N}{\pi^2 \gamma} \frac{\beta_x}{\frac{\sigma_x}{\sigma_z}(\sigma_x^2 + \sigma_z^2)} \left(1 + \frac{\pi}{2} \frac{\sigma_x(\sigma_x^2 - \sigma_z^2)}{\sigma_z(\sigma_x^2 + \sigma_z^2)}\right)$$

$$\xi_z = \frac{r_o N}{\pi^2 \gamma} \frac{\beta_z}{\frac{\sigma_z}{\sigma_x}(\sigma_x^2 + \sigma_z^2)} \left(1 + \frac{\pi}{2} \frac{\sigma_z(\sigma_z^2 - \sigma_x^2)}{\sigma_x(\sigma_x^2 + \sigma_z^2)}\right) , \qquad (5)$$

where r_o is the particle classical radius, γ is the ratio between the total and the rest energy, N is the number of particles of the strong beam.

ξ_x and ξ_z are related to the linear ΔQ_x, ΔQ_z shifts by the equations :

$$\cos(\mu_x + 2\pi\, \Delta Q_x) = \cos \mu_x - 2\pi\, \xi_x \sin \mu_x$$

$$\cos(\mu_z + 2\pi\, \Delta Q_z) = \cos \mu_z - 2\pi\, \xi_z \sin \mu_z \ .$$

In what follows we will use the action-phase dynamic variables,

$$I^x,\ \phi^x;\ I^z,\ \phi^z$$

defined by the equations :

$$x = \sqrt{I^x} \cos \phi^x , \qquad z = \sqrt{I^z} \cos \phi^z ,$$

$$x' = -\frac{\sqrt{I^x}}{\beta_x} \sin \phi^x , \qquad z' = -\frac{\sqrt{I^z}}{\beta_z} \sin \phi^z \ .$$

Then the eq. (4) become :

$$\begin{pmatrix} I^x \\ \phi^x \\ I^z \\ \phi^z \end{pmatrix}_{n+1} = \begin{pmatrix} I^x \\ \phi^x \\ I^z \\ \phi^z \end{pmatrix}_n + \begin{pmatrix} 0 \\ \mu_x \\ 0 \\ \mu_z \end{pmatrix} + \tag{6}$$

$$+ \begin{pmatrix} 8\pi\, \xi_x F_x (2\pi\, \xi_x F_x + \sqrt{I^x} \sin \phi^x) \\ \mathrm{arctg}\left(\dfrac{4\pi\, \xi_x F_x \cos \phi^x}{\sqrt{I^x} + 4\pi\, \xi_x F_x \sin \phi^x} \right) \\ 8\pi\, \xi_z F_z (2\pi\, \xi_z F_z + \sqrt{I^z} \sin \phi^z) \\ \mathrm{arctg}\left(\dfrac{4\pi\, \xi_z F_z \cos \phi^z}{\sqrt{I^z} + 4\pi\, \xi_z F_z \sin \phi^z} \right) \end{pmatrix}_n$$

where we have put :

$$F_x = F_x(\sqrt{I^x}\cos\phi^x, \sqrt{I^z}\cos\phi^z)$$

$$F_z = F_z(\sqrt{I^x}\cos\phi^x, \sqrt{I^z}\cos\phi^z) .$$

In this paper we will investigate the particular case :

$$\begin{aligned} \beta_x &= \beta_z = \beta \\ \mu_x &= \mu_z = \mu \qquad \left(\frac{\mu}{2\pi} = 0.05\right) \\ \sigma_x &= \sigma_z = \sigma \qquad \text{(circular beam)} \end{aligned} \tag{7}$$

In this conditions we have, from eq. (5) :

$$\xi = \xi_x = \xi_z = \frac{r_0 N \beta}{2\pi^2 \gamma \sigma^2}$$

Note : All computations have been carried out with at least 14 digits.

3. - CLOSED ORBITS. -

Firstly we compute the closed orbits that arise from eq.(6), under the conditions (7). For closed orbit we mean the orbit which satisfies the eq.

$$\begin{aligned} I^x_{n+N} &= I^x_n \qquad n = 1, 2, \ldots, N \\ \phi^x_{n+N} &= \phi^x_n \\ I^z_{n+N} &= I^z_n \\ \phi^z_{n+N} &= \phi^z_n , \end{aligned} \tag{8}$$

this is a closed orbit of order N. The problem is symmetric in x and z, and so, for small N, we have :

$$I^x_n = I^z_n , \qquad \phi^x_n = \phi^z_n .$$

The closed orbits corresponding to

$$\xi = 0.035, \quad 0.051, \quad 0.15, \quad 0.25$$

are plotted in Fig. 1, 2, 3, 4. We can see that if we increase ξ, the number and the "density" of the closed orbits increase.

4.- (I^x, I^z) PHASE PLANE REPRESENTATION.-

If we start from some initial point

$$(I^x_o, \ \phi^x_o, \ I^z_o, \ \phi^z_o),$$

which does not belong to a closed orbit, and if we look at the (I^x, I^z) phase plane, we find that the representative point fills a strip-like region (see Fig. 5). The area of the region covered by the representative point does not increase after the first $\sim 10^4$ turns. All the calculations are runned up to 5×10^4 turns.

If we look at the area of this strip, we find that it blows up when the starting point is near to a closed orbit.

In Fig. 6, 7, 8 and 9 we have plotted that area versus the starting invariant

$$I = I^x_o = I^z_o \qquad (\phi^x_o = \pi \ , \ \phi^z_o = \pi + 0.02)$$

Fig. 6 , $\xi = 0.035$ ($\Delta Q = 0.028$).

We observe two peaks corresponding to closed orbits 16 and 18. In this case it seems that there is no contribution from odd closed orbits.

Fig. 7 , $\xi = 0.051$ ($\Delta Q = 0.035$).

The 16 and 18 peaks increase and a new closed orbit (N = 14) appears.

Fig. 8 , $\xi = 0.15$ ($\Delta Q = 0.085$).

There are many resonances (odd too), which begin to overlap.

Fig. 9 , $\xi = 0.25$ $(\Delta Q = 0.12)$.

There is no more the resonant structure, but merely a plateau after an I^* threshold value (in this case $I^* = 3.5\ \sigma^2$).

If we increase ξ, I^* becomes smaller and the plateau increases too, e.g. for $\xi = 0.35$ $(\Delta Q = 0.15)$ we obtain $I^* \approx 1.5\ \sigma^2$.

5.- CONCLUSIONS.-

My personal feeling is that, for $\xi \cong 0.25$, we are in a sort of stochastic region, due to the overlap of the resonances. But this is a very preliminary conclusion, because we need to investigate inside the complete phase space $(I^x, \phi^x, I^z, \phi^z)$. Only after this investigation we can be sure that the behaviour, we are talking about, is really stochastic. Furthermore I want to extend this kind of calculation to more general cases ($\sigma_x \neq \sigma_z$, $\mu_x \neq \mu_z$, crossing at angle and so on), and I want to use a gaussian strong beam charge distribution function, that is a more physical distribution. Indeed I have now the CTETA subroutine (from J. Augustin) for a fast and accurate calculation of the kick given by a gaussian beam to the colliding weak beam particle.

APPENDIX A.-

The kick given by the strong beam to the weak one is :

$$\theta_x(x,z) + i\,\theta_z(x,z) = \frac{2r_oN}{\gamma}\;\frac{\sigma_x\sigma_z}{\pi^2}\int_{-\infty}^{+\infty} dx_o \int_{-\infty}^{+\infty} dz_o \;\cdot \tag{A.1}$$

$$\cdot\;\frac{1}{(x_o^2+\sigma_x^2)(z_o^2+\sigma_z^2)}\;\frac{1}{x-x_o-i(z-z_o)}\;.$$

If we put :

$$S_\pm = \sigma_x \pm z\;, \qquad R_\pm = \sigma_z^2 - x^2 + (\sigma_x \pm z)^2\;,$$

$$V_\pm = \sigma_z^2 + x^2 + (\sigma_x \pm z)^2\;, \qquad D_\pm = R_\pm^2 + 4x^2 S_\pm\;,$$

$$A_1 + i B_1 = \frac{\sigma_z}{\pi}\left(\frac{1}{2xS_+ - iR_+} + \frac{1}{2xS_- - iR_-}\right),$$

$$A_2 + i B_2 = \frac{2\sigma_z}{\pi}\left(\frac{1}{R_+ + 2ixS_+} + \frac{1}{R_- - 2ixS_-}\right),$$

$$A_3 + i B_3 = \frac{2}{\pi}\left(\frac{iS_+V_+ - xR_+}{D_+} + \frac{iS_-V_- + xR_-}{D_-}\right),$$

$$A_4 + i B_4 = \frac{iS_+V_+ - xR_+}{D_+} - \frac{iS_-V_- + xR_-}{D_-}\;,$$

and if we perform the integration, eq. (A.1) becomes :

$$\theta_x(x,z) + i\,\theta_z(x,z) = \frac{r_oN}{\gamma}\left\{(A_1+iB_1)\ln\left(\frac{z^2+\sigma_z^2}{x^2+\sigma_x^2}\right) + \right. \tag{A.2}$$

$$\left. + (A_2+iB_2)\,\mathrm{arctg}\left(\frac{x}{\sigma_x}\right) + (A_3+iB_3)\,\mathrm{arctg}\left(\frac{z}{\sigma_z}\right) + A_4 + iB_4\right\}$$

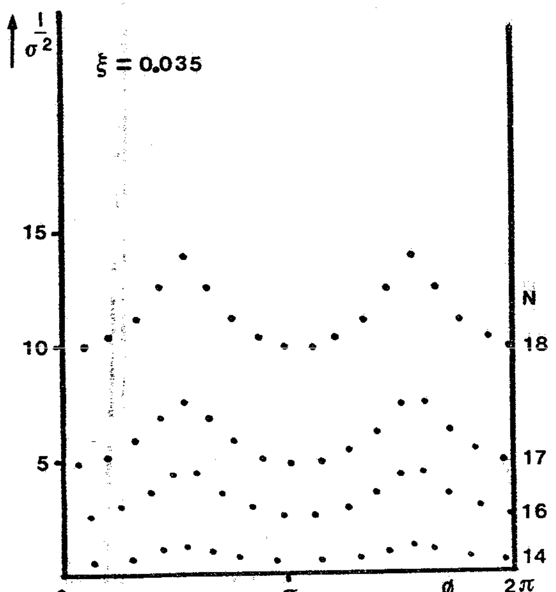

FIG. 1

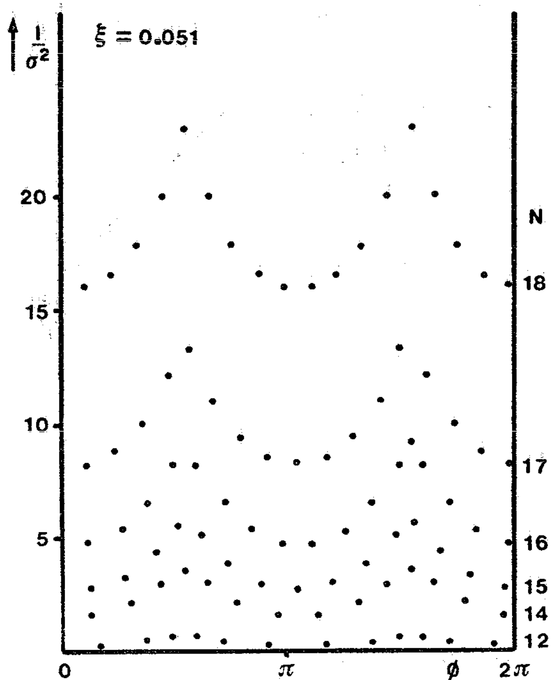

FIG. 2

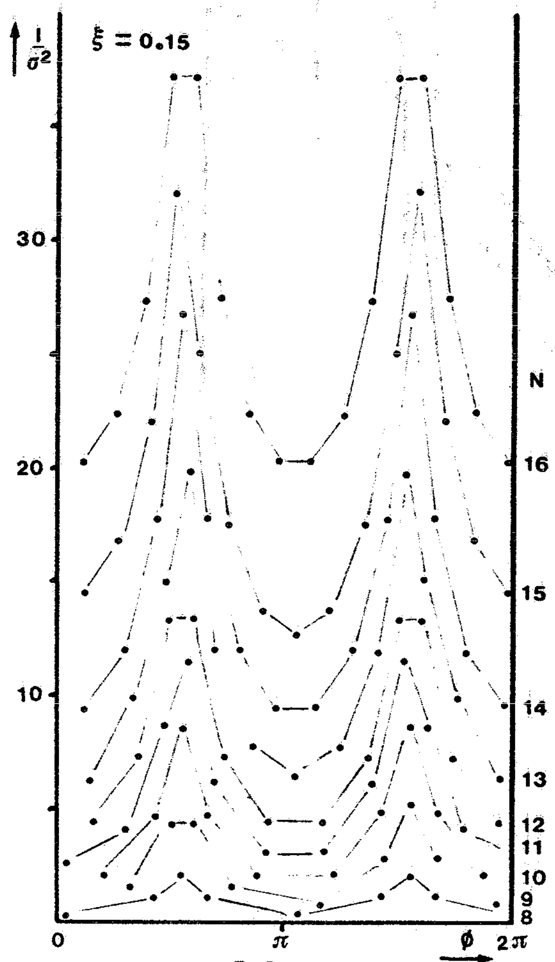

FIG. 3

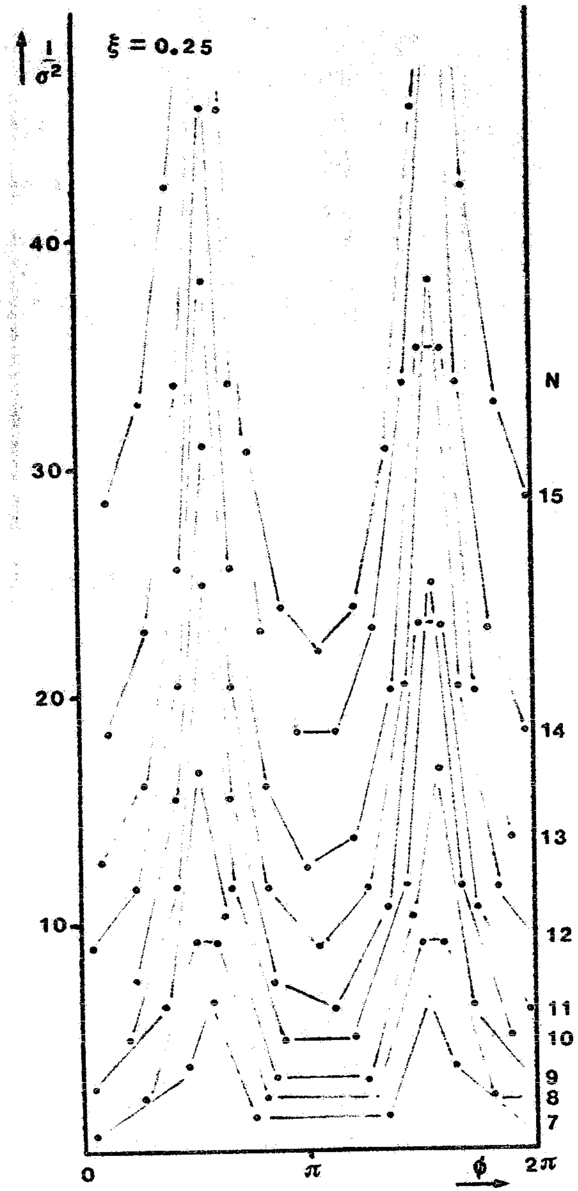

FIG. 4

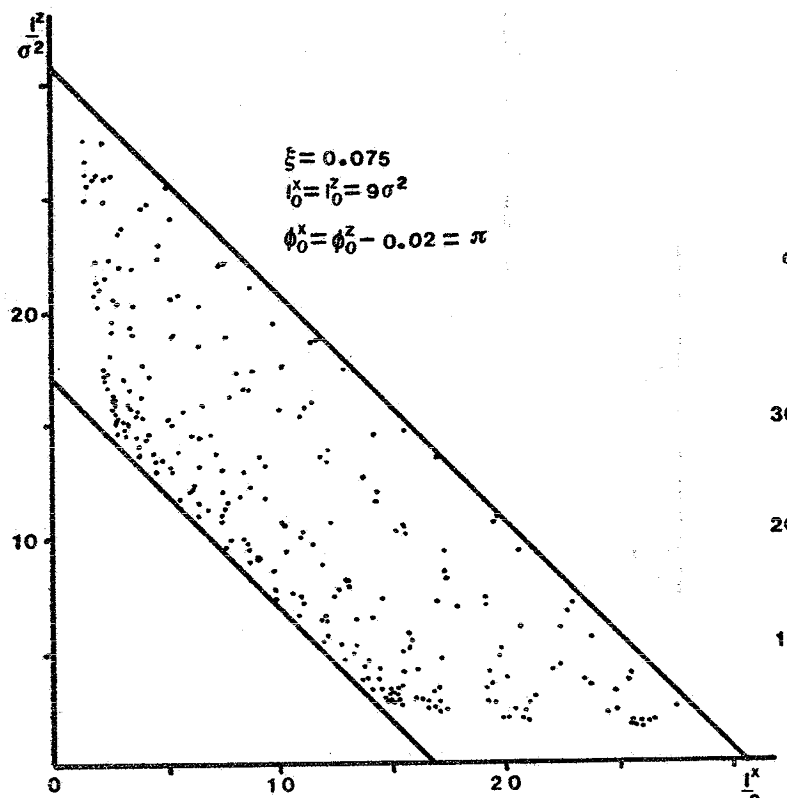

FIG. 5

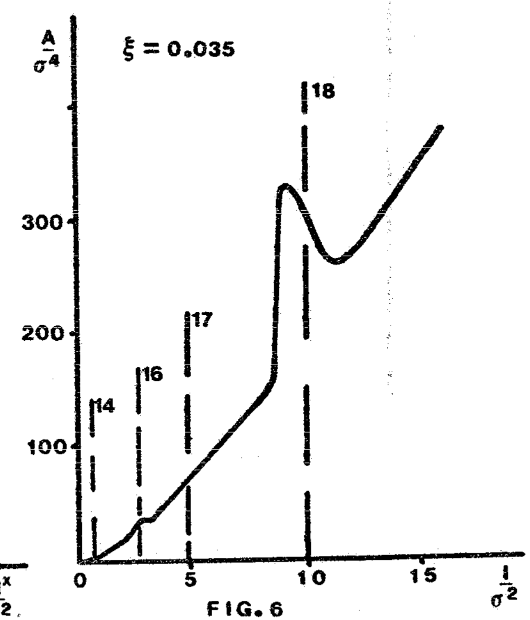

FIG. 6

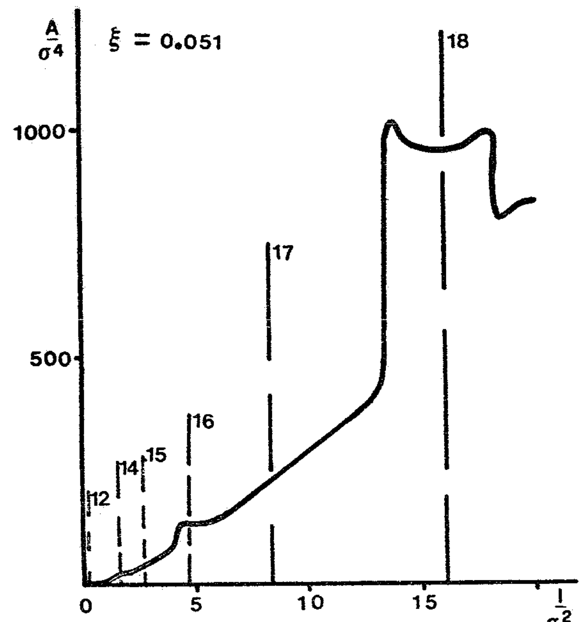

FIG. 7

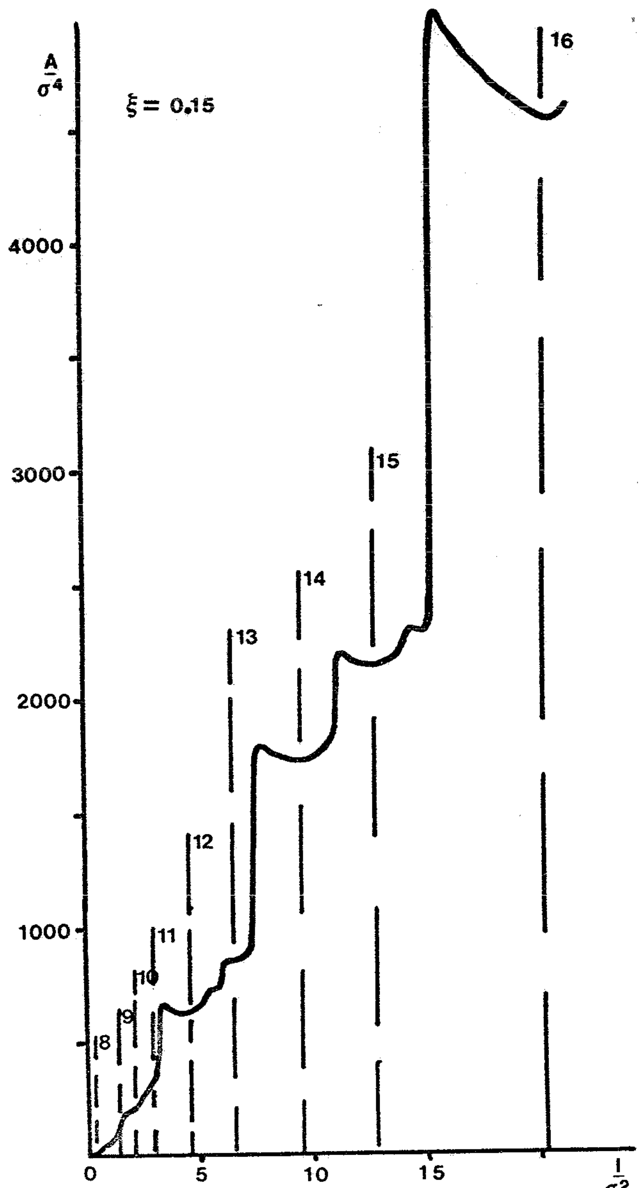

FIG. 8

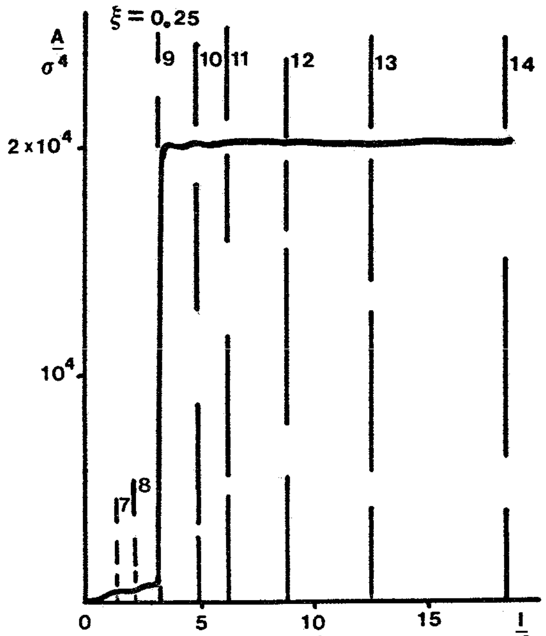

FIG. 9