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CONFORMAL-COVARIANT WILSON EXPANSION IN PERTURBATION THEORY

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Abstract: The Wilson expansion at light-like distances is investigated in perturbation theory. Ward identities for broken conformal symmetry are used; it is found that, for any value of the physical coupling constant, the Wilson expansion is asymptotically conformal covariant provided there exists a non-trivial scale-invariant skeleton theory.

1. Introduction

In recent time conformal symmetry [1] has been recognized as useful to understand asymptotic properties of quantum field theory [2]. In particular the hypothesis of exact conformal invariance yields to powerful constraints on the Wilson operator-product expansions [3].

As a consequence of the invariance infinite towers of operators, namely $O_{\alpha_1 \ldots \alpha_n}(x) = \partial_{\alpha_1} \ldots \partial_{\alpha_n} O(x)$ can be summed explicitly in such a way to preserve automatically the locality and the spectrum condition [4].

However the real world does not possess such an invariance so the question arises if conformal symmetry could be at least a good asymptotic symmetry. The true zero-momentum Ward identities for this symmetry have been derived and they differ from the canonical ones [5] in a non-trivial interacting theory [6–8]. The divergences of the dilation and conformal currents contain non-soft terms like $\beta(g)\phi^4(x)$ where $\beta(g)$ is the Callan-Symanzik function (we definitely consider the $g\phi^4$ theory), which is not asymptotically negligible. Nevertheless it can be shown [6] that, if $\beta(g)$ vanishes for some $g_\infty > g$, where g is the physical coupling constant, conformal invariance [7], which is violated at each other in perturbation theory, becomes true after summation of all orders. In this paper we study the constraints which broken conformal symmetry puts on the Wilson expansion at short and lightlike distances. In particular we find that, if scale symmetry is asymptotically true, the Wilson expansion is also asymptotically conformal covariant. However the preasymptotic terms of the zero-mass theory no longer preserve such a covariance.

2. Ward identities for broken conformal symmetry

In this section we recall some known results on Ward identities of conformal algebra.

The true Ward identity for scale [6] symmetry is the Callan-Symanzik equation:

$$\left(-\frac{1}{2}\sum_{i=1}^{n-1} k_i \frac{\partial}{\partial k_i} + \beta(g)\frac{\partial}{\partial g} - n(1-\gamma(g))\right) G_n(k_1 \dots k_{n-1}, m^2, g)$$

$$= \Delta G_n(k_1 \dots k_{n-1}, m^2, g)\,, \tag{1}$$

where $\Delta G_n = m^2 \partial/\partial m_0^2 G_n$.

Following Symanzik [9] one can define

$$G_n^{\mathrm{AS}}(k_1 \dots k_{n-1}, m^2, g) = G_n(k_1 \dots k_{n-1}, m^2, g) - \int_1^\infty 2\frac{\mathrm{d}\lambda}{\lambda}$$

$$\times \Delta G_n\left(k_1 \dots k_{n-1}, \frac{m^2}{\lambda^2}, \bar{g}(g, \lambda^{-1})\right) \exp\left\{ n\int_{\bar{g}(g,\lambda^{-1})}^{g} \mathrm{d}g' \,\frac{1-\gamma(g')}{\beta(g')}\right\}, \tag{2}$$

where

$$\bar{g}(g, \lambda) = \rho^{-1}(\lg \lambda^2 + \rho(g))\,, \qquad \rho(g) = \int^g \frac{\mathrm{d}g'}{\beta(g')}$$

and ρ^{-1} is the functional inverse of ρ.

The second term in the right-hand side becomes negligible in the deep-Euclidean region, for non-exceptional momenta, so that G_n^{AS} is asymptotically equal to G_n and satisfies the homogenous Callan-Symanzik equation whose solution is

$$G_n^{\mathrm{AS}}(\lambda k_1 \dots \lambda k_{n-1}, m^2, g) = G_n^{\mathrm{AS}}(k_1 \dots k_{n-1}, m^2, \bar{g}(g, \lambda))$$

$$\times \exp\left\{-n\int_g^{\bar{g}(g,\lambda)} \mathrm{d}g' \,\frac{[1-\gamma(g')]}{\beta(g')}\right\}. \tag{3}$$

As a consequence of eq. (3), if $\beta(g)$ vanishes at some $g_\infty > g$ then

$$G_n^{\mathrm{AS}}(\lambda k_1 \dots \lambda k_{n-1}, m^2, g) \to \lambda^{-n[1-\gamma(g_\infty)]}\, Z^n(g, \lambda)$$

$$\times G_n^{\mathrm{AS}}(k_1 \dots k_{n-1}, m^2, g_\infty) + \dots\,, \tag{4}$$

where

$$Z(g, \lambda) = \exp\left\{-\int_{g}^{\bar{g}(g,\lambda)} \mathrm{d}g' \frac{[\gamma(g') - \gamma(g_\infty)]}{\beta(g')}\right\}$$

is at most a logarithmically divergent function of λ. Dots stand for non-leading asymptotic terms.

Then n-point Green functions turn out to be scale invariant at short distances with anomalous dimensions. Similar Ward identities can be obtained for the composite operators of the theory, in particular for the bilinear operator ϕ^2. However pathologies associated to reducible scale invariance [10] can be present for higher-dimensionality objects. Along the same lines it is possible to derive of dilatation and conformal currents are related by the canonical formula [1, 5]

$$\partial^\mu \kappa_{\mu\nu}(x) = 2x_\nu \, \partial^\mu \, \mathcal{D}_\mu(x) = -4x_\nu(\beta(g) N_4(\phi^4(x)) + m^2\eta(g) N_2(\phi^2(x)) = 2x_\nu \, \theta(x) \,. \tag{5}$$

For our purposes it is better to work in configuration space. The true Ward identities for scale and special conformal transformations become*

$$\left[\sum_i x_i^\mu \frac{\partial}{\partial x_i^\mu} - n(1+\gamma(g)) + y^\mu \frac{\partial}{\partial y^\mu} - \delta(g)\right] \langle 0 | \mathrm{T}(\phi(x_1) \ldots \phi(x_n) O^{\{\alpha\}}(y)) | 0 \rangle = \int \mathrm{d}^4 z \, \langle 0 | \mathrm{T}(\phi(x_1) \ldots \phi(x_n) \, O^{\{\alpha\}}(y) \theta'(z)) | 0 \rangle \,, \tag{6}$$

$$\left\{\left[\sum_i 2x_{i\lambda} \, x_i^\rho \frac{\partial}{\partial x_i^\rho} - x_i^2 \frac{\partial}{\partial x_{i\lambda}} + 2\,(1+\gamma(g))\, x_{i\lambda}\right] \delta^{\{\alpha\}}_{\{\beta\}} + \left[2y_\lambda \, y^\rho \frac{\partial}{\partial y^\rho} - y^2 \frac{\partial}{\partial y_\lambda} + 2\delta_0(g) y_\lambda\right] \delta^{\{\alpha\}}_{\{\beta\}} - 2iy_\rho \, \Sigma^{\rho\{\alpha\}}_{\lambda\{\beta\}}\right\} \times \langle 0 | \mathrm{T}(\phi(x_1) \ldots \phi(x_n) O^{\{\beta\}}(y) | 0 \rangle = 2\int \mathrm{d}^4 z z_\lambda \langle 0 | \mathrm{T}(\phi(x_1) \ldots \phi(x_n) \, O^{\{\alpha\}}(y) \, \theta'(z) | 0 \rangle \,, \tag{7}$$

where this Ward identity holds for those fields which in free field theory satisfy

$$[O^{\{\alpha\}}(0), K_\lambda] = 0 \,, \tag{8}$$

* Here $\theta'(x)$ differs from $\theta(x)$ by contact terms, as it will be evident in the following.

and $\Sigma^{\{\alpha\}}_{\rho\lambda\{\beta\}}$, $\delta_0(g)$ are their spin matrix and dimension respectively.

Operators with $K_\lambda \neq 0$ at $x = 0$ are of the form [3]

$$\partial_{\alpha_1} \dots \partial_{\alpha_m} O^{\{\alpha\}}(x) ; \tag{9}$$

their Ward identity can be easily derived using the Jacobi identity and the cummutation relation

$$[K_\lambda, P_\rho] = -2i(g_{\lambda\rho} D + M_{\lambda\rho}) , \tag{10}$$

where K_λ, P_ρ, D, $M_{\lambda\rho}$ are the 15 generators of conformal algebra.

The Ward identities (7) imply that for $g = g_\infty$, G_N^{AS} is exactly conformal invariant so the n-point Green functions for $g \neq g_\infty$ still manifest such a symmetry at short distances as a consequence of eq. (4).

3. The Wilson expansion

It has been shown [3] that strict conformal invariance permits one to group the operators in towers of the form $(O^{\{\alpha\}}, \partial_\alpha O^{\{\alpha\}}, \partial_\alpha\partial_\beta O^{\{\alpha\}}, \dots)$ when $O^{\{\alpha\}}(x)$ is a Lorentz irreducible tensor with conformal properties given by eq. (8). The contribution C_n of each tower to the Wilson expansion [11] is determined completely by the coefficient C_0 of the lowest member of the tower. Because conformal symmetry is an asymptotic symmetry we should expect useful constraints on the C_n to follow. In fact the contrary is true; one gets equations for the C_n from which anomalies have not disappeared unless $g = g_\infty$. However we can derive also Callan-Symanzik equations for all C_n which teach us that the short distance behaviour of C_n is controlled, at all g, by their form at $g = g_\infty$. This implies that the leading term of the Wilson expansions is conformal covariant. Nothing can be said on preleading corrections to the asymptotic terms. For simplicity, without any loss of generality, we can confine ourselves to a tower of operators which starts with the Lorentz scalar $\phi^2(x)$, so we are interested in the contribution of such a tower to the operator product $\phi(x)\phi(0)$ at short distances.

We assume that the contribution of different towers (inequivalent representations) are independent so the sum over each tower must satisfy by itself the constraints from broken conformal covariance.

The Wilson expansion reads as

$$\langle 0 | \mathrm{T}(\phi(x)\phi(y)\phi(t_1)\dots\phi(t_n)) | 0 \rangle = \sum_m C_m((x-y)^2, g)\,(x-y)^{\mu_1} \dots$$

$$\times\,(x-y)^{\mu_m} \langle 0 | \mathrm{T}(\partial_{\mu_1} \dots \partial_{\mu_m} \phi^2(y)\phi(t_1)\dots\phi(t_n)) | 0 \rangle . \tag{11}$$

If the ϕ^4 operator is involved in the Green function some contact terms must be introduced and expansion (11) is modified as follows [8]:

$$\langle 0|\mathrm{T}(\phi(x)\,\phi(y)\,\phi^4(z)\,\phi(t_1)\dots\phi(t_n))|0\rangle \sim \sum_m C_m((x-y)^2, g)\,(x-y)^{\mu_1}\dots$$

$$\times\,(x-y)^{\mu_m}\langle 0|\mathrm{T}(\partial_{\mu_1}\dots\partial_{\mu_m}\phi^2(y)\phi^4(z)\phi(t_1)\dots\phi(t_n))|0\rangle$$

$$+\sum_m \bar{C}_m((x-y)^2, g)\,\delta^4(x-z)\,(x-y)^{\mu_1}\dots(x-y)^{\mu_m}$$

$$\times\,\langle 0|\mathrm{T}(\partial_{\mu_1}\dots\partial_{\mu_m}\phi^2(y)\phi(t_1)\dots\phi(t_n))|0\rangle$$

$$+\sum_m R_m((x-y)^2, g)\,\delta_{\mu_1}(x-z)\,(x-y)^{\mu_2}\dots(x-y)^{\mu_m}$$

$$\times\,\langle 0|\mathrm{T}(\partial_{\mu_1}\dots\partial_{\mu_m}\phi^2(y)\phi(t_1)\dots\phi(t_n)|0\rangle + \dots\,, \tag{12}$$

where the dots stand for terms with higher derivatives of the δ-function which can be neglected, as it will be clear below.

As it is well known [12],

$$\bar{C}_m((x-y)^2, g) = \frac{\partial}{\partial g} C_m((x-y)^2, g)\,, \tag{13}$$

and $R_m((x-y)^2, g)$ are preleading corrections to the coefficients $\bar{C}_m((x-y)^2, g)$.

We remark that leading singularities at short distances (and light-like distances) are mass independent so if the Wilson expansion is true we can consistently neglect the mass term in eq. (5).

As a consequence we are interested in a local operator with the same (minimum) twist. However conformal algebra connects in general also operators with different twists so for our purposes only the subalgebra which preserves the twist is relevant.

This corresponds to consider only collinear conformal transformations on the light cone.

Broken symmetry Ward identities for such transformations read as

$$2x_+^2\frac{\partial}{\partial x_+} - \gamma(g)x_+ + \sum_i\left(2t_{i_+}t_i^\rho\frac{\partial}{\partial t_i^\rho} - t_i^2\frac{\partial}{\partial t_{i_+}} + 2(1+\gamma(g))\,t_{i_+}\right)$$

$$\times\,\langle 0|\mathrm{T}(\phi(x)\phi(0)\phi(t_1)\dots\phi(t_n))|0\rangle$$

$$= 2\int \mathrm{d}^4 z z_+\,\langle 0|\mathrm{T}(\phi(x)\phi(0)\,\phi(t_1)\dots\phi(t_n)\,\theta(z))|0\rangle\,, \tag{14}$$

where we made a Poincaré transformation in such a way that

$x_\mu = (\frac{1}{2}(x_+ + x_-), \frac{1}{2}(x_+ - x_-), 0, 0)$.

The corresponding Ward identity for operators belonging to the tower are

$$\Sigma \left(2t_{i_+} t_i^\rho \frac{\partial}{\partial t_i^\rho} - t_i^2 \frac{\partial}{\partial t_{i_+}} + 2(1+\gamma(g)) t_{i_+}\right) \langle 0 | \mathrm{T}(\partial_+^m \phi^2(0)\, \phi(t_1) \ldots$$

$$\times\, \phi(t_n)) | 0 \rangle + m(\delta(g) + m - 1) \langle 0 | \mathrm{T}(\partial_+^{m-1} \phi^2(0)\, \phi(t_1) \ldots \phi(t_n) | 0 \rangle$$

$$= 2 \int \mathrm{d}^4 z z_+ \langle 0 | \mathrm{T}(\partial_+^m \phi^2(0)\, \phi(t_1) \ldots \phi(t_n)) | 0 \rangle . \tag{15}$$

Inserting the Wilson expansion (11) and (12) in the left- and right-hand side of eq. (14) respectively and comparing with (15) the following set of equations must be satisfied:

$$x^2 \frac{\mathrm{d}}{\mathrm{d}x^2} C_n(x^2, g) + (n + \gamma(g) + 1)\, C_n(x^2, g)$$

$$= (n+1)(\delta(g) + n)\, C_{n+1}(x^2, g) - 2\beta(g) R_n((x-y)^2, g) ; \tag{16}$$

however, according to ref. [12] the coefficients $C_n(x^2, g)$ must be consistent with the Ward identities for broken scale symmetry i.e. the Callan-Symanzik equations:

$$\left(x^2 \frac{\mathrm{d}}{\mathrm{d}x^2} + \beta(g) \frac{\partial}{\partial g} + 1 + \gamma(g) - \tfrac{1}{2}\delta(g)\right) C_n(x^2, g) = 0 . \tag{17}$$

We study eq. (16) under the assumption that $g_\infty > g$ exists. For $g = g_\infty$ one has

$$\left(x^2 \frac{\mathrm{d}}{\mathrm{d}x^2} + 1 + \gamma(g_\infty) - \tfrac{1}{2}\delta(g_\infty)\right) C_n(x^2, g_\infty) = 0 , \tag{18}$$

$$x^2 \frac{\mathrm{d}}{\mathrm{d}x^2} C_n(x^2, g_\infty) + (n + 1 + \gamma(g_\infty))\, C_n(x^2, g_\infty)$$

$$= (n+1)(\delta(g_\infty) + n)\, C_{n+1}(x^2, g_\infty) , \tag{19}$$

whose solution is

$$C_n(x^2, g_\infty) = \left(\frac{1}{x^2}\right)^{1+\gamma(g_\infty) - \frac{1}{2}\delta(g_\infty)} C_0(g_\infty) \frac{\Gamma(\frac{1}{2}\delta(g_\infty) + n)}{\Gamma(\delta(g_\infty) + n)} . \tag{20}$$

This is the solution which was obtained using the constraints of exact conformal symmetry on the operator-product expansion [3].

For $g \neq g_\infty$ we know that

$$C_n(x^2, g) \to C_n(x^2, g_\infty) Z(x^2, g) , \qquad x^2 \to 0 ,$$

where Z is a function less singular than any power of x^2 (ref. [9]).

4. Conclusions

We have applied the constraints that broken conformal symmetry implies on the short-distance Wilson expansion.

We have found that, unless the dynamical assumption of the existence of a zero $g = g_\infty$ of the β-function is made, the Wilson expansion cannot be conformal covariant; the terms $\beta(g) R_n$ cannot be neglected on the right-hand side of eq. (16) without obtaining a contradiction with the Callan-Symanzik equation (17).

However if such a zero does exist conformal symmetry as well as scale symmetry is a good asymptotic symmetry both for Green functions and the Wilson expansion.

These types of results are in contradiction with the apparent conclusions of ref. [8].

In particular the contribution of a whole conformal tower of operators, which gives rise to the same singularity on the light cone, turns out to be

$$\phi(x)\phi(0) \sim Z(x^2, g) \sum_n C_n(x^2, g) x^{\alpha_1} \dots x^{\alpha_n} \partial_{\alpha_1} \dots \partial_{\alpha_n} \phi^2(0)$$

$$\sim \left(\frac{1}{x^2}\right)^{1+\gamma(g_\infty)-\frac{1}{2}\delta(g_\infty)} C_0 Z(x^2, g) \int_0^1 \mathrm{d}\sigma \, [\sigma(1-\sigma)]^{\frac{1}{2}\delta(g_\infty)-1} \phi^2(\sigma x) .$$

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