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RECIPROCITY RELATION

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## A GROUP THEORETICAL DERIVATION OF THE GRIBOV–LIPATOV RECIPROCITY RELATION

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The Gribov–Lipatov reciprocity relation is shown to follow directly from group-theoretical considerations employing the conformal group.

Gribov and Lipatov have recently been able to calculate the deep inelastic electroproduction and annihilation structure functions in a leading logarithmic approximation in pseudoscalar and in vector theory [1, 2]. A remarkable result of their sophisticated calculation is the appearance of a “reciprocity relation” connecting the annihilation and the scattering structure functions. Christ, Hasslacher and Mueller [3] have recalculated the leading logarithmic terms by a simpler procedure making use of the powerful techniques by Callan [4] and Symanzik [5] in conjunction with operator product expansions [6]. The reciprocity relation also obtains [7] in a  $\lambda\varphi^4$  model with negative coupling constant [8]. In this note we show that the reciprocity relation can directly be obtained, in renormalizable theories, by a simple mathematical argument of conformal algebra – no special assumption of conformal invariance has to be added however.

The main point of our argument is the following. The reciprocity relation, for instance

$$F_2\left(\frac{1}{\omega}\right) = -\omega^3 F_2(\omega) \quad (1)$$

for the asymptotic structure function  $F_2(\omega, q^2) = \nu W_2(\omega, q^2)$  at large  $q^2$ , essentially equivalent to a property of reflection invariance of the Mellin transform, in  $\omega$ , of the structure function itself<sup>†</sup>. Such

symmetry simple follows from a corresponding symmetry over the relevant representations of the conformal Casimir operators – which applies up to the leading logarithmic approximation (but not in the full theory).

Specifically, let us call  $\gamma_N(g)$ , where  $g$  is the coupling constant, the “anomaly” in the dimension  $l_N$  of the relevant traceless tensors  $O_{\alpha_1 \dots \alpha_N}(x)$  appearing in the operator expansion for  $j_\mu(x)j_\nu(0)$ . For  $\gamma_N$  one has a perturbation expansion

$$\gamma_N(g) = g^2 \gamma_N^{(2)} + O(g^3) \quad (2)$$

and the mentioned reflection property of the Mellin-transformed structure function (in leading logarithmic approximation) requires

$$\gamma_N^{(2)} = \gamma_{-N-1}^{(2)} \quad (3)$$

To demonstrate now the group-theoretical origin of the “twin” symmetry in eq. (3): one of the various possibilities for computing  $\gamma_N^{(2)}$  in renormalizable field theory is to look at the coefficient of  $\log(q^2)$  in the vacuum expectation value of a product  $O_{\alpha_1 \dots \alpha_N}(0)\varphi(x)\varphi(y)$ , which is readily seen to be finite

<sup>†</sup> See ref. [2]. It is assumed that in the Mellin-transformed variable there are only isolated poles and/or essential singularities (as satisfied in all the above leading logarithmic calculations [1–3, 7]).

and conformal covariant in configuration space. The coefficient  $\gamma_N^{(2)}$  is thus the normalization of a conformally covariant quantity. It must depend on  $N$  through conformal invariant operators, namely the Casimir operators of the conformal group representation of  $O_{\alpha_1 \dots \alpha_N}(x)$ , i.e.

$$\gamma_N^{(2)} = \gamma^{(2)}(C_N^{(I)}) \tag{4}$$

where  $C_N^{(I)}$  is the value of the quadratic Casimir operator [9] of  $O(4, 2)$ . Indeed for such  $(\frac{1}{2}N, \frac{1}{2}N)$  Lorentz tensors of canonical dimension

$$l_N = 2 + N \tag{5}$$

(as they are for the purpose of calculating  $\gamma_N^{(2)}$ ) the remaining Casimir operators  $C_N^{(II)}$  and  $C_N^{(III)}$  are not independent. Covariantly one writes  $C^{(I)} = \frac{1}{2}J_{AB}J^{AB}$ , where  $J_{AB}$  is a six-dimensional angular momentum, and, on the  $(\frac{1}{2}N, \frac{1}{2}N)$  tensor  $O_{\alpha_1 \dots \alpha_N}$ ,

$$C_N^{(I)} = l_N(l_N - 4) + N(N + 2). \tag{6}$$

From eq. (5):  $C_N^{(I)} = C_{-N-1}^{(I)}$ ; q.e.d.

It is interesting to consider the generalization in a  $D$ -dimensional space-time. One has  $C_N^{(I)} = l_N(l_N - D) + N(N + D - 2)$  and for canonical dimensions,  $l_N = N + D - 2$ ,  $C_N^{(I)} = 2(N - 1)(N + D - 2)$ . Therefore  $\gamma_N^{(2)} = \gamma_{-N+3-D}^{(2)}$  and the generalized Gribov-Lipatov relation is  $F_2(1/\omega) = -\omega^{D-1}F_2(\omega)$ . Indeed: in the  $\lambda\varphi^3$  theory in six dimensions ( $D = 6$ ), which is conformal invariant,  $\gamma_N^{(2)}$  is proportional to [10]

$$[(N+1)(N+2)]^{-1} - \frac{1}{12}$$

and thus  $\gamma_N^{(2)} = \gamma_{-N-3}^{(2)}$ .

It is also to be pointed out that the above reflection property is precisely a symmetry property with respect to the space-rotation subgroup,  $O(D-1)$ , in  $D$ -dimensional space-time. In fact we recall that the symmetric-

traceless conformal tensors with  $\rho_N = D - 2 + N$  are conserved [9]. They thus carry spin  $N$  alone. But then  $C^{(I)}$  coincides (apart from a constant) with the quadratic Casimir of the rotation subgroup

$$\frac{1}{2}J_{AB}J^{AB} = 2J_{ab}J^{ab} + 2(2-D)$$

( $A, B = 1, \dots, D + 2$ ;  $a, b = 1, \dots, D - 1$ ). This is why we have used the term "twin symmetry".

Our discussion has been limited so far to renormalizable theories. In superrenormalizable theories the reciprocity relation does not seem to hold [11]. This is what our viewpoint leads us to expect, because of the appearance of ratios of dimensioned constants in the so-called scaling limit.

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