G. Parisi :
DEEP INELASTIC SCATTERING IN A FIELD THEORY WITH COMPUTABLE LARGE-MOMENTA BEHAVIOUR

Deep Inelastic Scattering in a Field Theory with Computable Large-Momenta Behaviour.

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In a recent paper [1] SYMANZIK has shown that in a theory characterized by a $g\varphi^4$ interaction with a negative coupling constant the behaviour of the Green's functions in the deep Euclidean region [2] can be directly computed in a perturbative fashion: the dimensions of all the operators are canonical; the corrections to the leading terms are a factor $1/\log q^2$ smaller.

The aim of this note is to prove that in this model also the deep inelastic structure functions can be directly computed in the Bjorken limit [3]. Some general statements can be done: Bjorken scaling [3] and the parton model [4,5] are canonically realized [6], *e.g.* the total cross-section for e^+e^+ annihilation into hadrons is the same as in the free field theory [7].

As a simple application we compute the structure function $F_2(\omega)$ for deep inelastic scattering at the first order in the coupling constant g. It is amusing to note that in standard perturbation theory no diagram contributes to $F_2(\omega)$ at this order.

We briefly resume the results of ref. [1].

In a charged $g\varphi^4$ field theory, if $0 > g > g_{\min}$, the asymptotically stable fixed point [8] is located at $g = 0$ and the effective coupling constant $g(\lambda)$ [2] which appears in the solution of the Callan-Symanzik equation [9,10], is asymptotic of order $-1/\log \lambda$. The renormalized Green's functions of the fundamental field φ and of the other local operators are asymptotical the same as in the free field theory, apart from a multiplicative factor.

(1) K. SYMANZIK: *A field theory with computable large-momenta behaviour*, DESY preprint 72/68 (1972).
(2) K. SYMANZIK: *Comm. Math. Phys.*, **23**, 49 (1971).
(3) J. D. BJORKEN: *Phys. Rev.*, **179**, 1547 (1969).
(4) S. D. DRELL, D. J. LEVY and TUNG MOW YAN: *Phys. Rev. Lett.*, **22**, 744 (1969); *Phys. Rev.*, **187**, 2159 (1969).
(5) N. CABIBBO, G. PARISI, M. TESTA and A. VERGANELAKIS: *Lett. Nuovo Cimento*, **4**, 569 (1970).
(6) G. PARISI: *Phys. Lett.*, **42** B, 114 (1972).
(7) N. CABIBBO, G. PARISI and M. TESTA: *Lett. Nuovo Cimento*, **4**, 35 (1970).
(8) K. G. WILSON and J. KOGUT: *The renormalization group and the ε expansion*, COO 2220-2 (1972); *Phys. Rep.*, to be published.
(9) C. G. CALLAN jr.: *Phys. Rev. D*, **2**, 1541 (1970).
(10) K. SYMANZIK: *Comm. Math. Phys.*, **18**, 48 (1970).

This last factor can be easily obtained from the knowledge of the various $\gamma_0(g)$ functions which appear in the Callan-Symanzik equations for the Green's functions of the various operators [11,12].

In the case of the renormalized field φ_R one finds

$$(1) \qquad Z = \lim_{p^2 \to \infty} [p^2 G_R(p^2)] = \exp\left[2 \int_g^0 \frac{\gamma(g')}{\beta(g')} \, dg' \right] .$$

The current propagator is asymptotically the same as in the free theory: the relevant $\gamma_J(g)$ function is identically zero as a consequence of the Ward identities [10]. The total cross-section $\sigma_T(E)$ for e^+e^- annihilations into hadrons satisfies asymptotically the parton model predictions [7].

The vertex of the electromagnetic current and of two renormalized fields is equal to Z^{-1} for large enough current mass and for arbitrary but fixed masses of the two fields [13,14].

In e^+e^- collisions to two-particle final state gives a contribution to the total cross-section equal to $Z^{-2}\sigma_T(E)$ [7]. It is a simple task to show that the parton model diagram

$$(2)$$

gives a contribution equal to the total cross-section $\sigma_T(E)$ and therefore all other diagrams do not contribute to the total cross-section.

If we consistently assume that also production amplitudes are dominated by diagram (2), we find that the produced hadrons cluster in two jets; the mass distribution of each jet is proportional to the Källén-Lehmann spectral function as discussed in ref. [7].

We recall that the spectral functions $\varrho(S)$ of the unrenormalized field $\varphi_B = Z^{-\frac{1}{2}}\varphi_R$ is for large values of the mass independent of the coupling constant and equal to [1]

$$(3) \qquad \varrho(S) \sim \frac{1}{25 S \log^2 (S/m^2)} .$$

Parton model predictions are clearly satisfied [7].

Let us now study the Wilson expansion of two currents [15]. For simplicity we first consider a scalar current, namely φ^2.

We can write the following light-cone expansion:

$$(4) \qquad \begin{cases} \varphi^2(x)\,\varphi^2(0) \xrightarrow[x^2\to 0]{} \sum_N E_N(x^2, m^2, g)\, O_{\mu_1 \dots \mu_N} x^{\mu_1} \dots x^{\mu_N} , \\[2ex] \varphi_R(x)\,\varphi_R(0) \xrightarrow[x^2\to 0]{} \sum_N F_N(x^2, m^2, g)\, O_{\mu_1 \dots \mu_N} x^{\mu_1} \dots x^{\mu_N} . \end{cases}$$

[11] N. CHRIST, B. HASSLACHER and A. MULLER: Columbia University preprint CO-3067(2)-9 (1972).
[12] C. G. CALLAN jr. and D. GROSS: Broken conformal invariance, Princeton Report (1972).
[13] M. GELL-MANN and F. ZACHARIASEN: Phys. Rev., 123, 1065 (1961).
[14] K. SYMANZIK: private communication.
[15] K. WILSON: Phys. Rev., 179, 1499 (1969); R. A. BRANDT and G. PREPARATA: Nucl. Phys., 27 B, 541 (1971).

The functions $E_N(x^2, m^2, g)$ and $F_N(x^2, m^2, g)$ satisfy the Callan-Symanzik equation [11]

$$(5) \qquad \begin{cases} \left[x^2 \dfrac{\delta}{\delta x^2} + \beta(g) \dfrac{\delta}{\delta g} + 2\gamma_{\varphi^2}(g) + \gamma_N(g) + 1 \right] E_N(x^2, m^2, g) = 0 \,, \\[2mm] \left[x^2 \dfrac{\delta}{\delta x^2} + \beta(g) \dfrac{\delta}{\delta g} + 2\gamma_\varphi(g) + \gamma_N(g) \right] F_N(x^2, m^2, g) = 0 \,, \end{cases}$$

where β and γ are C^∞ functions in the coupling constant and can be computed in perturbation theory.

In the small-x^2 limit the solution can be found to be

$$(6) \qquad \begin{cases} E_N(x^2, m^2, g) \to \dfrac{1}{x^2} Z_{\varphi^2} Z_N^{\frac{1}{2}} \,, \\[2mm] F_N(x^2, m^2, g) \to Z Z_N^{\frac{1}{2}} \,, \end{cases}$$

where

$$(7) \qquad Z_{\varphi^2} = \exp\left[2 \int_g^0 \frac{\gamma_{\varphi^2}(g')}{\beta(g')} \, dg' \right], \qquad Z_N = \exp\left[2 \int_g^0 \frac{\gamma_N(g')}{\beta(g')} \, dg' \right].$$

From eqs. (4)-(6) it follows that

$$(8) \qquad \varphi^2(x)\,\varphi^2(0) \to Z^{-1} Z_{\varphi^2} \frac{\varphi_R(x)\,\varphi_R(0)}{x^2} = Z_{\varphi^2} \frac{\varphi_B(x)\,\varphi_B(0)}{x^2} \,,$$

which is the parton model result [16].

One finds a similar result for the Wilson expansion of two currents: the only difference is that now $Z_J = 1$. The cross-section is longitudinal and the transverse cross-section goes to zero like $1/\log p^2$.

The only contribution to the total cross-section is given by the parton model diagrams [5]

$$(9)$$
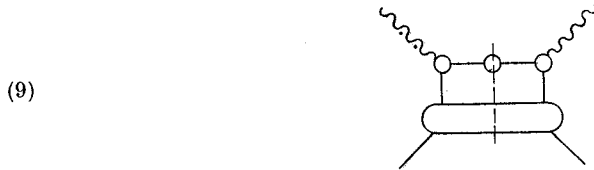
The function $F_2(\omega) = \lim\limits_{\substack{q^2 \to -\infty \\ 2\nu/q^2 = -\omega}} \nu W_2(q^2, \omega)$ can be computed using the sum rule

$$(10) \qquad \int_1^\infty F_2(\omega)\, \omega^{-N} d\omega = Z_N^{\frac{1}{2}} \,.$$

[16] H. FRIZSCH and M. GELL-MANN: *Proc. 1971 Coral Gable Conference* (1971).

We have normalized the operators O_{μ_1, μ_N} in such a way that

(11)
$$\langle P | O_{\mu_1, \mu_N} | P \rangle = P_{\mu_1} \dots P_{\mu_N} - \text{traces}.$$

An interesting consistency check can be done: the first diagrams of (9) gives a contribution to $F_2(\omega)$ equal to $Z\delta(\omega - 1)$. On the other side eq. (10) implies that the coefficient of $\delta(\omega - 1)$ is equal to $Z_\infty^{\frac{1}{2}} = \lim_{N \to \infty} Z_N^{\frac{1}{2}}$.

It follows that

(12)
$$Z_\infty^{\frac{1}{2}} = Z \Rightarrow \exp\left[\frac{1}{2} \int_g^0 \frac{\gamma_\infty(g') \, dg'}{\beta(g')}\right] = \exp\left[\int_g^0 \frac{\gamma(g') \, dg'}{\beta(g')}\right].$$

Equation (12) must be valid for arbitrary g in the range $0 > g > g_{\min}$ and this is possible only if

(13)
$$2\gamma(g) = \gamma_\infty(g) .$$

It is interesting to note that relation (13) can be proven using the same methods as in ref. [17].

From the knowledge of the functions β and γ_N at order g^{N+1}, the function $F_2(\omega)$ can be computed up to the order g^N.

Using the known results [2,18]

(14)
$$\gamma_N(g) = \frac{1}{384\pi^4} \left[\frac{1}{6} - \frac{1}{N(N+1)}\right] g^2 + O(g^3), \qquad \beta(g) = \frac{5}{48} g^2 + O(g^3);$$

one finds

(15)
$$\int_1^\infty F_2(\omega) \omega^{-N} d\omega = 1 + \frac{g}{40\pi^2} \left[\frac{1}{6} - \frac{1}{N(N+1)}\right].$$

Solving the simple momentum problem the function $F_2(\omega)$ is obtained at order g:

(16)
$$F_2(\omega) = \delta(\omega - 1) - \frac{g}{40\pi^2} \left[(\omega - 1)\omega^{-2} - \frac{1}{6} \delta(\omega - 1)\right].$$

We note that:

1) The function $F_2(\omega)$ is analytic in the whole complex plane; we assume that its analytic continuation is equal to the structure function for deep inelastic annihilation [19].

2) If the analyticity continuation ansatz is true, the particle multiplicity for e^+e^- annihilation into hadrons is finite (the result is not trivial as far as the first order in g corresponds to the sum of an infinite number of diagrams).

[17] G. PARISI: *How to measure the dimension of the parton field*, Frascati report LNF-72/105 (1972).
[18] K. WILSON: *Quantum field theory in less than 4-dimensions*, CLNS 198 (1972).
[19] R. GATTO and G. PREPARATA: *Nucl. Phys.*, **47** B, 313 (1972) and references therein.

3) The total annihilation cross-section obtained from the energy conservation sum rule ([4,20]) is equal to the right one.

4) The Gribov-Lipatov ([21]) reciprocity relation $F_2(\omega) = -\omega^3 F_2(1/\omega)$ is satisfied.

The $g\varphi^4$ theory with a negative coupling constant is therefore an example of a renormalizable theory where all predictions of the parton model are true. However these predictions should fail for positive coupling constant.

It may be interesting to note that in a theory like the gluon model the situation described in this note can be realized only at the price of taking an imaginary coupling constant, giving up the positivity of the Hilbert space of the states.

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([20]) E. ETIM and Y. SRIWASTAVA: *Phys. Lett.*, **41** B, 59 (1972).
([21]) V. N. GRIBOV and L. N. LIPATOV: *Phys. Lett.*, **37** B, 78 (1971).