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Anomalous dimensions in one-dimensional quantum field theory*

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We study the short distance behavior of the Green's functions of two operators in a soluble one-dimensional model of quantum field theory with dimensionless coupling constant. Integer power behavior does not occur. The leading terms of the Wilson expansion of two operators at short distances are determined.

I. INTRODUCTION

The behavior of Green's functions at short distances is one of the most interesting and controversial problems in quantum field theory.[1]

The first fundamental step towards a deeper understanding of the problem was made by Wilson.[2] He suggested that the product of two operators satisfies the following asymptotic expansion at short distances:

$$\phi(x)\phi(0) \simeq \sum_n \xi_n(x)\, O_n(0), \tag{1}$$

where the functions $\xi_n(x)$ become singular when x goes to zero.

In free field theory these functions are integer powers; in perturbation theory this simple behavior is destroyed by the appearance of terms of the form $g \log(x^2)$.

Wilson suggested that the logarithms may sum to a power which may not be integral: This happens in the soluble two-dimensional Thirring model.[3] The exponent of the power is dependent on the renormalized coupling constant.

In this work we prove that similar pathologies arise also in a very simple one-dimensional model of quantum field theory.

We study the Lagrangian

$$\mathcal{L} = \int dt\,\{\tfrac{1}{2}[\dot\phi(t)]^2 - \tfrac{1}{2}m^2[\phi(t)]^2 - g[\phi(t)]^{-2}\}. \tag{2}$$

This is the only possible one with dimensionless coupling constant: the Lagrangian density must have dimension 1, the field ϕ, therefore, has dimension $-\frac{1}{2}$, so that the only possible scale invariant interaction is $g\phi^{-2}$.

This model can be solved using the equivalence between one-dimensional quantum field theory and nonrelativistic quantum mechanics. The analogous quantum mechanical problem is the quantal oscillator: a harmonic oscillator with centrifugal potential g/x^2; its solution is known from the early days of quantum mechanics.[4]

We find that for $g > -\frac{1}{8}$ the Wilson expansion for the product of two fields ϕ is

$$\phi(t)\phi(0) \simeq \phi^2(0) + t\dot\phi(0)\phi(0) + \tfrac{1}{2}t^2\ddot\phi(0)\phi(0) + Dt^{a+2}\phi(0)^{-2a-1}\delta[\phi(0)], \tag{3}$$

where $a = \frac{1}{2}(1+8g)^{1/2}$ and all the neglected terms in the Wilson expansion are of the type $t^{\beta_i} O_i(0)$, where β_i is equal to n or to $n+a+2$ with integer n.

We note that $\phi^{-2a-1}\delta[\phi]$ is not an operator but a nonbounded sesquilinear form, which is defined on any finite linear combination of the energy eigenvectors.

This result is interesting because it shows that anomalous dimensions in the short distance behavior are a very common phenomenon, which is present not only in relativistic quantum field theory, but also in the old nonrelativistic quantum mechanics.

In Sec. II we rederive the analogy between one-dimensional quantum field theory and nonrelativistic quantum mechanics. We write the solution of the quantal oscillator and use it to compute the Wightman functions for the quantum field problem.

In Sec. III we study the behavior of the two-point Wightman function at short distances and find anomalous dimensions. We extend our study to the two-point correlation function between two arbitrary energy eigenstates and finally arrive at the Wilson expansion (3).

In Sec. IV we briefly discuss our results and make an interesting but unproven conjecture.

II. SOLUTION OF THE MODEL

The Lagrangian of our one-dimensional problem is (2).

One can easily find the associated Euler–Lagrange equation

$$\ddot\phi(t) = m^2\phi(t) - 2g[\phi(t)]^{-3} \tag{4}$$

and the Hamiltonian

$$H = \tfrac{1}{2}[\pi(t)]^{2} + \tfrac{1}{2}m^2[\phi(t)]^2 + g[\phi(t)]^{-2}. \tag{5}$$

ϕ and π satisfy the canonical commutation relations

$$[\phi(t), \pi(t)] = -i. \tag{6}$$

In order to find the eigenvectors of the Hamiltonian we use the standard representation

$$\phi(0) \to x, \qquad \pi(0) \to i\,\frac{d}{dx}. \tag{7}$$

This trick reduces the problem to finding the eigensolutions of the following differential equation:

$$\left(-\frac{d^2}{2dx^2} + \tfrac{1}{2}m^2x^2 + \frac{g}{x^2}\right)\psi(x) = E\psi(x); \tag{8}$$

Equation (8) is the Schrödinger equation for the quantal oscillator. The eigenfunctions and eigenvectors of Eq. (8) are

$$E_n = m[2n + a + 1];$$

$$\psi_n(x) = (4m)^{1/4}\left(\frac{\Gamma(n+1)}{\Gamma(a+n+1)}\right)^{1/2}[mx^2]^{(2a+1)/4} \times \exp\left(-\frac{mx^2}{2}\right) L_n^a[mx^2], \tag{9}$$

where $a = \frac{1}{2}(1 + 8g)^{1/2}$, n is a nonnegative integer, and $L_n^a(x)$ are the Laguerre polynomials.[5]

If $g \lesssim -\frac{1}{8}$ the spectrum of the Hamiltonian is no longer bounded below; there exists no ground state and the physical meaning of the problem is lost.

The Wightman functions of the theory are

$$\langle 0|\phi(t_1)\cdots\phi(t_n)|0\rangle = \langle\psi_0|x(t_1)\cdots x(t_n)|\psi_0\rangle, \quad (10)$$

where ψ_0 is the ground state of (8) and $x(t)$ is the position operator at time t in the Heisenberg representation. We note that $x(t)$ satisfies the same equation of motion (4) as $\phi(t)$.

If we define

$$x_{nm} = \langle\psi_n|x|\psi_m\rangle, \quad (11)$$

we have from (10)

$$\langle 0|\phi(t)\phi(0)|0\rangle = \sum_n e^{itE_n}|x_{0n}|^2. \quad (12)$$

Similar expression can be easily derived for general N-point Wightman functions.

III. THE WILSON EXPANSION

In this section we compute the two-point Wightman function and study its behavior at small t.

The formula for x_{0n} is

$$x_{0n} = -[2(\pi m)^{-1/2}][\Gamma(a+n+1)\Gamma(a+1)\Gamma(n+1)]^{-1/2} \times \Gamma(a+\tfrac{3}{2})\Gamma(n-\tfrac{1}{2}). \quad (13)$$

In the limit $n \to \infty$ we find

$$x_{0n} \simeq [-\tfrac{1}{2}(\pi m)^{-1/2}]\,\Gamma(a+\tfrac{3}{2})\Gamma^{-1/2}(a+1)n^{-[(a/2)+(3/2)]}. \quad (14)$$

This asymptotic behavior of x_{0n} implies that in the small t region

$$\langle 0|\phi(t)\phi(0)|0\rangle = (1/m)[C_0 + C_1\cdot(mt) + C_2\cdot(mt)^2 + C_3\cdot(mt)^{a+2} + \cdots], \quad (15)$$

where C_0, C_1, C_2, C_3 are g-dependent constants and the neglected terms have higher power in t. It is interesting to observe that the power of the fourth term is not integral and is a continuous function of the coupling constant.

We now look for the two-point correlation function between two arbitrary energy eigenstates s and r.

We need only to compute the asymptotic behavior for large n of x_{sn}.

If we decompose

$$L_s^a(x^2m) = \sum_0^s {}_k b_k^s(x^2m)^k \quad (16)$$

we find, using Eq. (9), that

$$x_{sn} = \frac{1}{m^{1/2}}\left(\frac{\Gamma(s+1)\Gamma(n+1)}{\Gamma(a+s+1)\Gamma(n+1+a)}\right)^{1/2} \times \sum_0^s {}_k\left(b_k^s\frac{\Gamma(a+k+\frac{3}{2})\Gamma(n-k-\frac{1}{2})}{\Gamma(n+1)\Gamma(-k-\frac{1}{2})}\right)$$

$$\simeq \frac{1}{m^{1/2}}\Gamma^{1/2}(s+1)\Gamma^{-1/2}(a+s+1)\sum_0^s {}_k \times [\Gamma(s+k+\tfrac{3}{2})b_k^s n^{-(a/2)-(3/2)-k}]. \quad (17)$$

The terms proportional to b_k^s with $k \neq 0$ go faster to zero. The final result for small t is

$$\langle r|x(t)x(0)|s\rangle = \frac{1}{m}[D_0^{r,s} + D_1^{r,s}\cdot(mt) + D_2^{rs}\cdot(mt)^2 + D_3\left(\frac{\Gamma(s+1)\Gamma(r+1)}{\Gamma(a+s+1)\Gamma(a+r+1)}\right)^{1/2} b_0^s b_0^r(mt)^{2+a}] \quad (18)$$

where $D_0^{r,s}, D_1^{r,s}, D_2^{r,s}$ are constants dependent on g, r and s, but D_3 is a function of only g. b_0^s can also be defined as

$$b_0^s = \lim_{x\to 0}\frac{1}{(4m)^{1/4}}\frac{\Gamma(a+s+1)}{\Gamma(s+1)}(mx^2)^{-(2a+1)/4}\psi_s(x), \quad (19)$$

so that (18) is equivalent to

$$\langle r|x(t)x(0)|s\rangle \simeq \langle r|x^2(0)|s\rangle + t\,\langle r|\dot{x}(0)x(0)|s\rangle + \tfrac{1}{2}t^2\langle r|\ddot{x}(0)x(0)|s\rangle + t^{a+2}\tfrac{1}{2}D_3\int\psi_r^+(x)\psi_s(x)x^{-2a-1} \times \delta(x) + \cdots. \quad (20)$$

The coefficient of t^{a+2} can also be interpreted as the mean value of the sesquilinear form $x^{-2a-1}\delta(x)$ between the states.

Equation (20) can be rewritten in operational form, and in this way we find the Wilson expansion (3) for the product of two fields.

We note that the operational form of the short distance singularities is mass independent, and the index of the power depends only on the dimensionless coupling constant.

One can investigate the general form of the neglected terms, computing the exact two-point correlation function. This can be done by inserting in (18) the exact expression (17) for x_{rn}, and not its asymptotic expansion. One finds

$$\langle r|x(t)x(0)|s\rangle = \frac{e^{-ir\cdot tm}}{m}\left(\frac{\Gamma(s+1)\Gamma(r+1)}{\Gamma(a+s+1)\Gamma(a+r+1)}\right)^{1/2} \times \frac{1}{\Gamma(1+a)}\sum_0^r {}_k\sum_0^s {}_{k'} b_{k'}^s b_k^r\,\Gamma(a+k+\tfrac{3}{2})\Gamma(a+k'+\tfrac{3}{2}) \times F(-k-\tfrac{1}{2}, -k'-\tfrac{1}{2}; 1+a; e^{imt}). \quad (21)$$

Using the well-known decomposition of the hypergeometric function[5], we arrive at

$$\langle r|x(t)x(0)|s\rangle = \frac{e^{-irtm}}{m}\left(\frac{\Gamma(s+1)\Gamma(r+1)}{\Gamma(a+s+1)\Gamma(a+r+1)}\right) \times \sum_0^r {}_k\sum_0^s {}_{k'} b_{k'}^s b_k^r\Big[\Gamma(a+k+k'+2) \times F(-k-\tfrac{1}{2}, k'-\tfrac{1}{2}; -a-k-k'-1; 1-e^{imt}) + [1-e^{imt}]^{2+a+k+k'} \quad (22)$$

$$\times\frac{\Gamma(-a-k-k'-2)\Gamma(a+k+\frac{3}{2})\Gamma(a+k'+\frac{3}{2})}{\Gamma(-k-\frac{1}{2})\Gamma(-k'-\frac{1}{2})} \times F(a+k+\tfrac{3}{2}, a+k'+\tfrac{3}{2}; a+k+k'+3; 1-e^{imt})\Big].$$

The first term generates short-distance singularities with integer powers, and the second term contains only powers of the form $n + a + 2$. (The hypergeometric function is regular at the origin.)

IV. CONCLUSIONS

The results of our study show that in a one-dimensional model of quantum field theory with dimensionless coupling constant, the fundamental field does not change dimension; but operators with anomalous dimension appear in the Wilson expansion of the product of two fields. The dimension of these operators is coupling-constant dependent.

We also find a very simple expression for the leading anomalous term of the Wilson expansion.

The next step along this line of work is to study other models with singular potential and with dimensional-coupling constant. Our feeling is that in this model too there are nonintegral powers in the short-distance Wilson expansion; but the anomalous dimensions should not be coupling-constant dependent. It would be very interesting to verify this conjecture.

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