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Tensor Representations of Conformal Algebra and Conformally Covariant Operator Product Expansion

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A conformally covariant formulation of operator product expansion is discussed as an expansion of the product of two representations into a direct sum of irreducible representations. The basic irreducible representations are analyzed and classified. The isomorphism between the conformal algebra and the $O(4, 2)$ algebra is used to obtain a manifestly covariant formalism. The implications of the isomorphism in the derivation of the representations is discussed. The covariant $O(4, 2)$ formalism directly relates dominant terms to nondominant terms in the light-cone limit. The essential coincidence of the problem of a conformal covariant operator product expansion to the problem of determining the form of the three-point function is stressed, together with the relevance of a selection rule for two-point functions following from exact (not spontaneously broken) conformal covariance. The role of Ward identities in a conformal covariant scheme is pointed out, and the mathematical implications on the n-point functions from causality are described.

1. INTRODUCTION

The notion of dilatation invariance has for many years been used in theoretical investigations of high energy physics [1–5]. In particular, Wilson [5] has made the proposal of applying dilatation invariance to the determination of the leading short-distance singularities in operator product expansions. Such an expansion is, for two currents, of the form

$$j_\mu(x)\, j_\nu(0) \simeq \sum c_{\mu\nu}^{(n)}(x)\, O^{(n)}(0) \tag{1}$$

where $c_{\mu\nu}^{(n)}(x)$ are singular c-numbers and the local operators $O^{(n)}(x)$ are a complete set of operators which provide a basis for the expansion.

Field-theoretic investigations in perturbation theory [5–9] and in model theories [5, 10–12] generally support the conjecture of the existence of the expansion.

Dilatations invariance, if applicable, allows for a determination of the degree of singularity of the c-number coefficient $c_{\mu\nu}^{(n)}(x)$ in Eq. (1).

The question of validity of dilatation invariance, if examined from a field-theory standpoint, is very intricate and complex. Very accurate and difficult investigations [10, 13–18] have been carried out on the most relevant problem of determining whether the operators O_n have definite "scale dimensions," and, if this were the case, of obtaining informations on the values of such dimensions.

The problem of scaling will not, however, be discussed in this note, in which we rather deal with the still hypothetical conjecture of invariance under the conformal algebra. Our approach will be entirely algebraic, and no theoretical justifications will be provided for the application of approximate conformal algebra, as an extension of the Poincare algebra plus dilatations, to physical processes, in an asymptotic limit.

From a algebraic point of view, the conformal algebra [19–39] becomes of interest as it provides for an extension of the Poincaré algebra into a higher dimension orthogonal algebra. On more physical ground, the extension from dilatations to the entire conformal algebra may be justified—though not at all in a compelling form—by some reasons which we briefly summarize here. First, one may recall that in Lagrangian field theories invariance under dilatations often implies invariance under conformal transformations. A sufficient, though not necessary condition, for such implication is, for instance, that there be no derivative couplings. Second, one has the important circumstance that conformal transformations leave the light-cone invariant.

The relation of the Bjorken limit [40] in momentum space to the light-cone limit in configuration space is well known (see, e.g., Ref. [41]). It is therefore suggestive to exploit the consequences of conformal invariance on an operator product expansion on the light-cone [42–46].

The requirement of covariance under the infinitesimal generators of $SU(2, 2)$ (the covering group of the conformal group) can be imposed directly on an operator product expansion on the light-cone [47–49]. The transformation properties of the infinite set of local operators which provide the expansion basis must then be preliminarly analyzed [48].

It is remarkable that conformal invariance alone directly solves the apparently unrelated problem of the causality support for the operator product expansion. In the expansion in Eq. (1) a problem of support arises when one commutes both sides of the equation with a third local observable $C(y)$. The right-hand side is then expected to vanish for $y^2 < 0$, but the vanishing of the left-hand side is guaranteed only for $y^2 < 0$ and $(x - y)^2 < 0$. However, for a manifestly conformally covariant expansion, the terms on the right-hand side are arranged so as to formally eliminate the problem [47–49]. This circumstance also adds some argument in favor of the conjecture of conformal invariance on the light cone.

In addition to offering a solution to the causality problem, conformal invariance also solves the independent problem of translation invariance on a hermitan basis [48].

The most elegant derivation of a manifestly conformal covariant operator product expansion can be given by exploiting the isomorphism between the conformal algebra and the $O(4, 2)$ orthogonal algebra. The derivation is uniquely extensible off the light-cone into the entire space-time [49–51]. Such expansions, manifestly conformal covariant over the entire space-time, should apply to the skeleton theory in Wilson's sense [5], provided such a theory enjoys the property of conformal invariance beyond the postulated scale invariance.

The manifestly conformal covariant formalism used in this paper is based on the isomorphism with $O(4, 2)$, and makes use of a six-dimensional pseudo-euclidean coordinate space. The most important problem to be solved is that of classifying those representations which contain infinite towers of irreducible $SL(2, C)$ tensor representations. For such a purpose, it is convenient to deal with homogeneous spinor functions defined on the six-dimensional hypercone and properly behaving under orbital and internal conformal transformations.

It is remarkable, and again pointing out to the possible physical relevance of the conformal algebra, that the case of "canonical dimensions," i.e., tensors satisfying the relation $\ell_n = 2 + n$ between their dimensions (in energy) and their tensor order, plays a peculiar role and is associated with an algebraic pathology. In such a case a degeneracy comes about in the eigenspace of tensor operators which commute with the generator K_λ, of special conformal transformations. The pathology is reflected in a definite way in the operator product expansion, when operators of canonical dimensions appear in the basic expansion set. At this point, we recall that scaling—as observed from deep inelastic electroproduction—directly implies the existence of a subset of operators O_n of dimension $\ell_n = n + 2$. Among them is the energy-momentum tensor, whose basic role in providing a theoretical justification of scaling has been strongly emphasized [4, 52, 53].

From the standpoint of conformal covariance, the product of two local operators can be thought of as an infinite sum of irreducible representations of the conformal algebra. One has therefore to deal with towers of operators which transform irreducibly under the algebra, and whose classification can be easily obtained—as we have said before—within the manifestly covariant six-dimensional formalism.

It is also interesting to prove the direct connection of the expansion to the conformally covariant vacuum expectation value for three local operators.

The content of this paper is as follows. In Section 2 we discuss the field-theoretical representations of conformal algebra, by exploiting the isomorphism with $O(4, 2)$. In Section 3 we concentrate on the tensor representations of $O(4, 2)$ on space-time.

In Section 4 we treat in detail the case of canonical dimensions and the related situation of conserved tensors. In Section 5 we summarize some results pertaining to operator product expansions, which provide the main motivation for our work. In Section 6 we give a discussion of the n-point function. The appendix provides some mathematical details.

2. Field-Theoretical Representations of Conformal Algebra

The conformal algebra is isomorphic to the orthogonal algebra $O(4, 2)$. Its action on space-time corresponds to the action of the algebra $O(4, 2)$ on the homogeneous space $O(4, 2)/IO(3, 1) \otimes D$.

The 15 generators of $O(4, 2)$, can be thought of as the components of a skew-symmetric tensor J_{AB}, defined according to

$$J_{\mu\nu} = M_{\mu\nu}, \quad J_{5\mu} = \tfrac{1}{2}(P_\mu - K_\mu), \quad J_{6\mu} = \tfrac{1}{2}(P_\mu + K_\mu), \quad J_{65} = D. \tag{2.1}$$

In Eq. (2.1), P_μ and $M_{\mu\nu}$ are the Poincaré generators, D the generator for dilatations, and K_μ that for special conformal transformations. The $O(4, 2)$ algebra is of rank three. Its Casimir operators are

$$C_{\text{I}} = J_{AB}J^{AB}, \tag{2.2}$$

$$C_{\text{II}} = \epsilon_{ABCDEF}J^{AB}J^{CD}J^{EF}, \tag{2.3}$$

$$C_{\text{III}} = J_A{}^B J_B{}^C J_C{}^D J_D{}^A. \tag{2.4}$$

The irreducible representations of the conformal algebra are therefore specified by the eigenvalues of these operators. Let us consider a pseudoeuclidean space in six dimensions with metric tensor $g_{AA} = (+---, -+)$ $A = 0, 1, 2, 3, 5, 6$. We introduce spinor operators $\Psi_{\{\alpha\}}(\eta)$ defined on the hypercone $\eta^A\eta_A = 0$ $(\eta_A = \eta_\mu, \eta_5, \eta_6)$ which transform according to

$$\delta\Psi_{\{\alpha\}}(\eta) = -i\epsilon^{AB}J_{AB\{\alpha\}}^{\{\beta\}} \quad \Psi_{\{\beta\}}(\eta) = -i\epsilon^{AB}(L_{AB}\delta_{\{\alpha\}}^{\{\beta\}} + S_{\{\alpha\}}^{\{\beta\}})\Psi_{\{\beta\}}(\eta), \tag{2.5}$$

where

$$L_{AB} = i(\eta_A\partial_B - \eta_B\partial_A)$$

and $S_{\{\alpha\}}^{\{\beta\}}$ is an irreducible finite-dimensional representation of the spinor group $SU(2, 2)$ which is locally isomorphic to $O(4, 2)$.

We assume that the functions $\Psi_{\{\alpha\}}(\eta)$ are homogeneous, i.e., they satisfy

$$\eta^A\partial_A\Psi_{\{\alpha\}}(\eta) = \lambda\Psi_{\{\alpha\}}(\eta). \tag{2.6}$$

Then one can show that the (local) operator

$$\begin{gathered} O_{\{\alpha\}}(x) = k^{-\lambda}(e^{-ix\cdot\pi})_{\{\alpha\}}^{\{\beta\}} \Psi_{\{\beta\}}(\eta) \\ \left(k = \eta_5 + \eta_6 , \qquad \pi_\mu = S_{6\mu} + S_{5\mu} , \qquad x_\mu = \frac{1}{k}\, \eta_\mu\right) \end{gathered} \tag{2.7}$$

transforms according to a representation of the conformal algebra on space-time induced from a representation $\Sigma_{\mu\nu}$, Δ, K_λ of the stability algebra at $x = 0$ given by

$$\Sigma_{\mu\nu} = S_{\mu\nu} ; \qquad \Delta = S_{65} - i\lambda; \qquad K_\lambda = S_{6\lambda} - S_{5\lambda} , \tag{2.8}$$

i.e.,

$$\begin{aligned} [O_{\{\alpha\}}(x), P_\mu] &= i\partial_\mu O_{\{\alpha\}}(x), \\ O_{\{\alpha\}}(x), M_{\mu\nu}] &= \{i(x_\mu\partial_\nu - x_\nu\partial_\mu)\, \delta_{\{\alpha\}}^{\{\beta\}} + \Sigma_{\{\alpha\}}^{\{\beta\}}\}\, O_{\{\beta\}}(x), \\ [O_{\{\alpha\}}(x), D] &= \{ix^\nu \partial_\nu \delta_{\{\alpha\}}^{\{\beta\}} + \Delta_{\{\alpha\}}^{\{\beta\}}\}\, O_{\{\beta\}}(x), \\ [O_{\{\alpha\}}(x), K_\lambda] &= \{i(2x_\lambda x_\nu \partial^\nu - x^2\partial_\lambda)\, \delta_{\{\alpha\}}^{\{\beta\}} \\ &\qquad + (K_\lambda - 2ix^\nu[g_{\lambda\nu}\Delta + \Sigma_{\lambda\nu}])_{\{\alpha\}}^{\{\beta\}}\}\, O_{\{\beta\}}(x). \end{aligned} \tag{2.9}$$

We are interested in classifying those representations of the conformal algebra which contain infinite towers of irreducible representations of $SL(2, C)$ of the type $(n/2, n/2)$, i.e., tensor representations.

In the following section the classification of these representations will be discussed.

3. Tensor Representations of 0(4, 2) on Space-Time

We start with the following lemmas.

Lemma 1. *Every irreducible (infinite dimensional) representation of the conformal algebra which contains a ladder of irreducible $SL(2, C)$ representations of the type $[(n+k)/2, (n+k)/2]$ $k = 0, 1, 2,...$ can be specified by an irreducible Lorentz tensor $(n/2, n/2)$ of definite dimension l_n and annihilated by K_λ, i.e., by an irreducible representation of $SL(2, c) \otimes D$.*

The proof of Lemma 1 follows from the fact that the Casimir operators (2.2–4) are given in such representations by

$$\begin{aligned} C_{\mathrm{I}} &= 2l_n(l_n - 4) + 2n(n+2), \\ C_{\mathrm{II}} &= 0, \\ C_{\mathrm{III}} &= n(n+2)[3 + l_n(l_n - 4)], \end{aligned} \tag{3.1}$$

as can be seen by evaluating their eigenvalues on a tensor operator annihilated by K_λ. Therefore, these representations are specified by the couple: n (nonnegative integer), and l_n (which can assume any value).

LEMMA 2. *Every irreducible representation of the conformal algebra, which, according to Lemma* 1, *is uniquely specified by an irreducible tensor representation of* $SL(2, C) \otimes D$ *(i.e., by the order of the tensor n and its dimension* l_n*), can be uniquely enlarged (for* $l_n \neq 2 + n$*) to a tensor representation of* $O(4, 2)$ *acting on* $O(4, 2)/IO(3, 1) \otimes D$. *The tensors* $\Psi_{A_1 \cdots A_n}(\eta)$ *are specified by the following properties*:

(3a) *They are homogeneous of degree* l_n;

(3b) *They are irreducible, i.e., symmetric and traceless*;

(3c) *They satisfy two sets of supplementary conditions*: $\eta^{A_1}\Psi_{A_1 \cdots A_n}(\eta) = 0$ *and* $\partial^{A_1}\Psi_{A_1 \cdots A_n}(\eta) = 0$ *(generalized Lorentz conditions).*

One can immediately see that these properties of $\Psi_{A_1 \cdots A_n}$ *are equivalent to the following.*

(3d) *They are irreducible with respect to the orbital part of the algebra* $O(4, 2)$;

(3e) *They are irreducible with respect to the spin part of the algebra* $O(4, 2)$;

(3f) *They are irreducible with respect to the whole algebra, i.e.,* $L \cdot S$ *is a constant on these representations.*

Proof. Let us consider a tensor operator $\Psi_{A_1 \cdots A_n}(\eta)$ under $O(4, 2)$ transformations on the cone $\eta^2 = 0$, of homogeneity degree $\lambda_n = -l_n$, i.e., satisfying

$$\eta^A \partial_A \Psi_{A_1 \cdots A_n}(\eta) = \lambda_n \Psi_{A_1 \cdots A_n}(\eta). \tag{3.2}$$

It is easy to show that the orbital quadratic Casimir operator on the cone reduces to

$$L_{AB}L^{AB} = 2\eta^A \partial_A (4 + \eta^A \partial_A) \tag{3.3}$$

(the other two Casimir operators vanish on orbital representations), and therefore,

$$L_{AB}L^{AB}\Psi_{A_1 \cdots A_n}(\eta) = 2\lambda_n(4 + \lambda_n)\Psi_{A_1 \cdots A_n}(\eta) = 2l_n(l_n - 4)\Psi_{A_1 \cdots A_n}(\eta). \tag{3.4}$$

Thus the equivalence of 3a and 3d is proved; 3b and 3e are obviously equivalent.

To prove the equivalence between 3c and 3f we observe that

$$J^{AB}J_{AB} = L^{AB}L_{AB} + S_{AB}S^{AB} + 2L^{AB}S_{AB}, \tag{3.4'}$$

and by explicitly performing the calculations one obtains

$$(L \cdot S\Psi)_{A_1\cdots A_n}(\eta) = -2 \sum_{i=1}^{n} (\eta_{A_i}\, \partial^B \Psi_{A_1\cdots \hat{A}_i \cdots A_n B}(\eta) - \eta^A\, \partial^{A_i} \psi_{A_1 \cdots \hat{A}_i \cdots A_n A}(\eta))$$

$$= 2 \sum_{i=1}^{n} \eta^A\, \partial^{A_i} \Psi_{A_1\cdots \hat{A}_i \cdots A_n}(\eta) = -2n\Psi_{A_1\cdots A_n}(\eta) \tag{3.5}$$

($\hat{A}_i$ means that the index A_i is omitted), where we have used the two supplementary conditions in 3c. Moreover, $S_{AB}S^{AB} = 2n(n+4)$, as can be proved by applying this operator to the highest order $SL(2, C)$ tensor for which $S_{65} = 0$ and $\pi \cdot K = n$ (C_{III}(internal) $= 3n(n+2)$ and C_{II}(internal) $= 0$).

Let us now prove Lemma 2.

Remembering that

$$O_{\alpha_1\cdots\alpha_n}(x) = k^{l_n}(e^{-ix\cdot\pi})^{A_1\cdots A_n}_{\alpha_1\cdots\alpha_n} \Psi_{A_1\cdots A_n}(\eta) \tag{3.6}$$

and that

$$(1/2)L^{AB}L_{AB} = l_n(l_n - 4), \tag{3.7}$$

$$(1/2)S_{AB}S^{AB} = (1/2)S_{\mu\nu}S^{\mu\nu} + \pi \cdot K - S_{65}^2 + 8iS_{65} \tag{3.8}$$

$$S_{AB}L^{AB} = S_{\mu\nu}L^{\mu\nu} - 2S_{65}L_{65} + \pi \cdot \mathscr{K} + K \cdot \mathscr{P}, \tag{3.9}$$

where

$$\mathscr{K}_\mu = L_{6\mu} - L_{5\mu}, \qquad \mathscr{P}_\mu = L_{6\mu} + L_{5\mu},$$

we have

$$e^{-ix\cdot\pi}\, \tfrac{1}{2} J_{AB}J^{AB} e^{ix\cdot\pi} = l_n(l_n - 4) + \tfrac{1}{2}\tilde{S}_{\mu\nu}\tilde{S}^{\mu\nu} + \pi \cdot \tilde{K} - \tilde{S}_{65}^2 + 4i\tilde{S}_{65} + \tilde{S}_{\mu\nu}\tilde{L}^{\mu\nu} - 2\tilde{S}_{65}\tilde{L}_{65} + \pi \cdot \tilde{\mathscr{K}} + \tilde{K} \cdot \tilde{\mathscr{P}}, \tag{3.10}$$

where

$$\begin{aligned}
\tilde{S}_{\mu\nu} &= e^{-ix\pi}S_{\mu\nu}e^{ix\pi} = S_{\mu\nu} + (x_\mu\pi_\nu - x_\nu \cdot \pi_\mu),\\
\tilde{K}_\mu &= K_\mu + 2x^\nu(g_{\mu\nu}S_{65} + S_{\mu\nu}) + 2x_\mu x \cdot \pi - x^2\pi_\mu,\\
\tilde{S}_{65} &= S_{65} + x \cdot \pi,\\
\tilde{L}_{\mu\nu} &= L_{\mu\nu} + x_\nu\pi_\mu - x_\mu\pi_\nu,\\
\tilde{\mathscr{P}}_\lambda &= \mathscr{P}_\lambda - \pi_\lambda,\\
\tilde{\mathscr{K}}_\lambda &= \mathscr{K}_\lambda + 2il_n x_\lambda - 2x_\lambda x \cdot \pi + x^2\pi_\lambda,\\
\tilde{L}_{65} &= il_n - x \cdot \pi,\\
\tilde{S}_{65} &= S_{65} + 2x \cdot \pi + (x \cdot \pi)^2.
\end{aligned} \tag{3.11}$$

Using translation invariance, it follows that

$$\tfrac{1}{2}J_{AB}J^{AB}O_{\alpha_1\cdots\alpha_n}(0) = l_n(l_n - 4) + \tfrac{1}{2}S_{\mu\nu}S^{\mu\nu} + \pi \cdot K + K \cdot (\mathscr{P} - \pi)$$
$$= l_n(l_n - 4) + n(n + 2) \tag{3.12}$$

and

$$[O_{\alpha_1\cdots\alpha_n}(0), K_\lambda] = 0 \tag{3.13}$$

due to the first supplementary condition in 3c. To complete the proof of Lemma 2, the mapping between the covariant $O(4, 2)$ tensors $\Psi_{A_1\cdots A_n}(\eta)$ and the irreducible representations of conformal algebra on space-time of the type discussed in Section 2 and Lemma 1 must be shown.

It will be proved that if $l_n \neq 2 + n$, the components of the tensor

$$\Psi_{A_1\cdots A_n}(\eta) = k^{-l_n}(e^{ix\cdot\pi}O)_{A_1\cdots A_n}(x) \tag{3.13}$$

are completely specified in terms of the divergences of the highest-spin tensor $O_{\alpha_1\cdots\alpha_n}(x)$.

To evaluate these components we observe that at $x = 0$ Eq. (3.13) becomes

$$\Psi_{A_1\cdots A_n}(\eta)_{x=0} = k^{-l_n}O_{A_1\cdots A_n}(0). \tag{3.14}$$

Therefore, the problem is that of evaluating the components $O_{\alpha_1\cdots\alpha_J xx\cdots}(0)$, where $(J = 1, 2,..., n)$ and $x \cdots$ stands for 5 or 6.

This problem can be completely solved by using the supplementary conditions 3c. In order to study the structure of the components of $O_{A_1\cdots A_n}(0)$ we observe that

$$(S_{65}O)_{A_1\cdots A_n} = i\sum_{i=1}^{n}(g_{5A_i}O_{A_1\cdots\hat{A}_i\cdots A_n6} - g_{6A_i}O_{A_1\cdots\hat{A}_i\cdots A_n5}), \tag{3.15}$$

and, using the first supplementary condition, at $x = 0$ we find

$$O_{5A_2\cdots A_n} = O_{6A_2\cdots A_n} \qquad \text{(we omit the index 0).} \tag{3.16}$$

Equation (3.15) becomes

$$(S_{65}O)_{A_1\cdots A_n} = i\sum_{i=1}^{n}(g_{5A_i} - g_{6A_i})\, O_{A_1\cdots\hat{A}_i\cdots A_n6},$$

so that

$$(S_{65}O)_{\alpha_1\cdots\alpha_k A_{k+1}\cdots A_n} = i \sum_{i=k+1}^{n} (g_{5A_i} - g_{6A_i})\, O_{\alpha_1\cdots\alpha_k A_{k+1}\cdots \hat{A}_i \cdots A_n 6}$$
$$= -i \sum_{i=k+1}^{n} (\delta_{5A_i} + \delta_{6A_i})\, O_{\alpha_1\cdots\alpha_k A_{k+1}\cdots \hat{A}_i \cdots A_n 6}\,, \tag{3.17}$$

where $A_{k+1} \cdots A_n$ are components 5 or 6, indifferently because of condition (3.16). We get

$$(S_{65}O)_{\alpha_1\cdots\alpha_k 6\cdots 6} = -i(n-k)\, O_{\alpha_1\cdots\alpha_k 6\cdots 6} \tag{3.18}$$

and, defining $L = iS_{65}$,

$$(LO)_{\alpha_1\cdots\alpha_k 6\cdots 6} = (n-k)\, O_{\alpha_1\cdots\alpha_k 6\cdots 6}\,, \tag{3.19}$$

so that all Lorentz tensors are eigenstates of the dimension. We also recall that

$$(S_{AB}O)_{A_1\cdots A_n} = i \sum_{i=1}^{n} (g_{AA_i} O_{A_1\cdots\hat{A}_i\cdots A_n B} - g_{BA_i} O_{A_1\cdots\hat{A}_i\cdots A_n A}),$$

$$(\pi_\mu O)_{A_1\cdots A_n} = i \sum_{i=1}^{n} [(g_{6A_i} + g_{5A_i})\, O_{A_1\cdots\hat{A}_i\cdots A_n \mu}$$
$$- g_{\mu A_i}(O_{A_1\cdots\hat{A}_i\cdots A_n 6} + O_{A_1\cdots\hat{A}_i\cdots A_n 5})], \tag{3.20}$$

$$(i\pi_\mu O)^{\mu}_{A_2\cdots A_n} = 2\left(4O_{6A_2\cdots A_n} + \sum_{i=2}^{n} g_{\mu A_i} O^{\mu}_{A_2\cdots\hat{A}_i\cdots A_n 6}\right)$$
$$= 2\left[4O_{A_2\cdots A_n 6} + \sum_{i=2}^{n} O_{A_2\cdots\hat{A}_i\cdots A_n 6}\right.$$
$$\left. + \sum_{i=2}^{n} (g_{5A_i} O_{5A_2\cdots\hat{A}_i\cdots A_n} - g_{6A_i} O_{6A_2\cdots\hat{A}_i\cdots A_n})\right] \tag{3.21}$$

for $A_j = 5$ or 6 $(J > K)$ we have:

$$(i\pi_\mu O)^{\mu}_{\alpha_2\cdots\alpha_k A_{k+1}\cdots A_n} = 2\left[(4 + n - 1)\, O_{\alpha_2\cdots\alpha_k A_{k+1}\cdots A_n 6}\right.$$
$$\left. - \sum_{i=k+1}^{n} (\delta_{5A_i} + \delta_{6A_i})\, O_{6\alpha_2\cdots\alpha_k A_{k+1}\cdots\hat{A}_i\cdots A_n 6}\right]$$
$$= 2(3 + k)\, O_{\alpha_2\cdots\alpha_k 6\cdots 6}\,. \tag{3.22}$$

Using (3.22) and the second supplementary condition 2c we get

$$\tfrac{1}{2}\partial_\mu O^\mu_{\alpha_2\cdots\alpha_k 6\cdots 6} + (2 + k - l_n)\, O_{\alpha_2\cdots\alpha_k 6\cdots 6} = 0, \qquad 2 \leqslant k \leqslant n. \tag{3.23}$$

Note that the second supplementary condition is well defined on the cone $\eta^2 = 0$, since

$$\eta_{A_1}\, \partial^A \Psi_{AA_2\cdots A_n}(\eta) = 0 \quad \text{is equivalent to} \quad (L^A_{A_1} - ig^A_{A_1})\, \Psi_{AA_2\cdots A_n}(\eta) = 0 \tag{3.24}$$

(the second equation is manifestly defined on $\eta^2 = 0$); Eq. (3.23) can easily be solved by iteration to give

$$O_{\alpha_1\cdots\alpha_{n-k}6\cdots 6} = 2^{-k}\, \frac{\Gamma(l_n - 2 - n)}{\Gamma(l_n - 2 - n + k)}\, \partial_{\mu_1}\cdots\partial_{\mu_k} O^{\mu_1\cdots\mu_k}_{\alpha_1\cdots\alpha_{n-k}}\, . \tag{3.25}$$

Equation (3.5) solves the problem.

For the correlated dimensions $l_n = l + n$, formula (3.25) simplifies to

$$O_{\alpha_1\cdots\alpha_{n-k}6\cdots 6} = 2^{-k}\, \frac{\Gamma(l - 2)}{\Gamma(l - 2 + k)}\, \partial_{\mu_1}\cdots\partial_{\mu_k} O^{\mu_1\cdots\mu_k}_{\alpha_1\cdots\alpha_{n-k}}\, . \tag{3.26}$$

4. Canonical Dimensions and Conserved Tensors

We investigate the particular case $l_n = 2 + n$, which we call "canonical dimensions." This case contains, as a subcase, as shown in Ref. (47), the situation of local conservation of the four-tensors $O_{\alpha_1\cdots\alpha_n}(x)$

$$\partial^{\alpha_1} O_{\alpha_1\cdots\alpha_n}(x) = 0. \tag{4.1}$$

We recall that only for $l_n = 2 + n$ is Eq. (4.1) conformal covariant. Moreover, if $l_n = 2 + n$ and $\partial^{\alpha_1} O_{\alpha_1\cdots\alpha_n}(x) \neq 0$ then the new tensors

$$\hat{O}_{\alpha_1\cdots\alpha_{n-1}}(x) = \partial^\alpha O_{\alpha\alpha_1\cdots\alpha_{n-1}}(x) \tag{4.2}$$

are annihilated by K_λ at $x = 0$.

This particular behavior is responsible for the fact that (3.25) does not make sense for canonical dimensions. However, subcase (4.1) is consistent with supplementary condition 3c (generalized Lorentz condition). In fact, in this case 3c tells us that (4.1) is indeed a conformal covariant equation. Supplementary condition 3c cannot be imposed, however, in case (4.2) where we have to use a different method. In the particular situation we are considering, we note that on space-time the irreducible representations of conformal algebra (see Section 3) spanned by $O_{\alpha_1 \cdots \alpha_n}(x)$ exhibit a peculiarity. There appears in fact a degeneracy of the eigenspace $K_\lambda = 0$. There are in such a case two irreducible representations of the stability algebra at $x = 0$, given by the couple of tensors $O_{\alpha_1 \cdots \alpha_n}$ and $\partial^\alpha O_{\alpha\alpha_1 \cdots \alpha_{n-1}}$, which induce the same irreducible representation on space-time. In fact, these two tensors, irreducible when restricted to the stability algebra $x = 0$, behave reducibly under the full conformed algebra, since

$$i\hat{O}_{\alpha_1 \cdots \alpha_{n-1}}(0) = [O_{\alpha_1 \cdots \alpha_n}(0), P^{\alpha_n}]. \tag{4.3}$$

This pathology has a well-defined counter part on the manifestly covariant tensors. In fact, one sees that, starting with a tensor $\Psi_{A_1 \cdots A_n}(\eta)$ which is not conserved, one can always define a new tensor

$$\tilde{\Psi}_{A_1 \cdots A_n}(\eta) = \Psi_{A_1 \cdots A_n}(\eta) - \tfrac{1}{2} \sum_{i=1}^{n} \eta_{A_i}\, \partial^B \Psi_{BA_1 \cdots \hat{A}_i \cdots A_n}(\eta) \tag{4.4}$$

which is conserved, provided $\partial^B \Psi_{BA_1 \cdots \hat{A}_i \cdots A_n}$ satisfies supplementary condition 3c (as it does not carry "canonical dimensions").

This ensures us that the new tensor is irreducible under $O(4, 2)$ transformations, according to 3a–f of Section 3. Note that the two tensors which define $\tilde{\Psi}_{A_1 \cdots A_n}(\eta)$ are not separately irreducible, and the combination in (4.4) is uniquely fixed by the requirement of irreducibility. We also observe that the tensor $\Psi_{A_1 \cdots A_{n-1}}(\eta) = \partial^B \Psi_{BA_1 \cdots A_{n-1}}(\eta)$ satisfies both supplementary conditions in 3c; its components can thus be evaluated with the same method of the previous section. One finds

$$\hat{O}_{\alpha_1 \cdots \alpha_{n-1-k} 6 \cdots 6} = 2^{-k} \frac{1}{\Gamma(2 + k)}\, \partial_{\mu_1} \cdots \partial_{\mu_k} \hat{O}^{\mu_1 \cdots \mu_k}_{\alpha_1 \cdots \alpha_{n-1-k}}, \tag{4.5}$$

and

$$\hat{O}_{\alpha_1 \cdots \alpha_{n-1}} = \partial^\alpha O_{\alpha\alpha_1 \cdots \alpha_{n-1}}, \qquad \hat{O}_{A_1 \cdots A_{n-1}}(0) = \partial^A O_{AA_1 \cdots A_{n-1}}(0). \tag{4.6}$$

The previous discussion enables us to consider only subcase (4.1) when dealing with a covariant tensor $\Psi_{A_1 \cdots A_n}(\eta)$ carrying canonical dimensions $l_n = 2 + n$. In fact, the covariant formulation of case (4.2) is the same as (4.1) with the substitution (4.5).

The trouble with conserved tensors lies in the fact that components 5 or 6 are not physical, in the sense that they cannot be determined in terms of the fundamental components $\Psi_{\alpha_1 \cdots \alpha_n}$. This is due to the fact that the second supplementary condition is identically verified in this case; the lower order tensors which correspond to some component 5 or 6 cannot be built up in terms of divergences of the highest order n-tensor (they vanish!). Moreover, these components cannot be chosen to be zero as this would destroy the covariance. So we must conclude that, for conserved covariant tensors of canonical dimensions there is not in general a one-to-one correspondence between irreducible representations on space-time and covariant $O(4, 2)$ tensors. This situation is deeply connected to the problem discussed in the paper by Mack and Salam [33], where they stress that a conformal covariant coupling, to be acceptable, must not mix unphysical components with physical ones. More generally, we call unphysical components those components which cannot be expressed in terms of the physical ones (and which cannot be put equal to zero). The above discussion implies that each conformally covariant formulation on space-time, which involves conserved tensors with $l_n = 2 + n$, must be such that it does not couple physical components ($\Psi_{\alpha_1 \cdots \alpha_n}$) to unphysical ones ($\Psi_{\alpha_1 \cdots \alpha_j x x \cdots}$, $x = 5$ or 6).

We finally conclude this section by observing that, for $l_n = 2 + n$ the covariant tensor

$$\tilde{\Psi}_{A_1 \cdots A_{n-1}}(\eta) = \partial^A \Psi_{A A_1 \cdots A_{n-1}}(\eta), \tag{4.7}$$

whose components on space-time are given in (4.5), is a genuine tensor with all components physical, according to the previous discussion. It is a covariant irreducible tensor, which has to be considered independently. This reflects the fact that, in space-time $O_{\alpha_1 \cdots \alpha_{n-1}}(x) = \partial^\alpha O_{\alpha\alpha_1 \cdots \alpha_{n-1}}(x)$ is a tensor annihilated by K_λ at $x = 0$.

5. Decomposition of an Operator Product into Irreducible Components

In this section we summarize some physical applications of the present investigation.

Wilson [5] suggested that a product of two local operators $A(x)\, B(x')$ (which is not a well-defined object when $x = x'$ and is furthermore expected to be singular for light-like distances) can be expanded as an infinite sum of local operators. Moreover, the (c-number) coefficients of such expansion are responsible for the singular behavior for $(x - x')^2 \to 0$. Such singular behavior is determined, in the limit, from dilatation symmetry. In Ref. (47) it was suggested that additional information on the expansion could be obtained by assuming the more restrictive covariance under the full conformal algebra. The assumption is motivated by the fact that, for a wide class of interacting lagrangian field theories, conformal symmetry is a consequence of dilatation symmetry. We can think of the product of two local operators as a (infinite) sum of irreducible representations of the conformal algebra, i.e., as sum of towers of operators which transform irreducibility under this algebra. These towers, according to our previous discussions, are each specified by a Lorentz irreducible tensor $O_{\alpha_1 \cdots \alpha_n}(x)$ with dimension l_n and annihilated by K_λ.

If we consider the operator product expansion of two (conformal) scalars, $A(\eta)$, $B(\eta')$, in a supposedly existing skeleton theory, using exact conformal covariance we have [49]

$$A(\eta)\, B(\eta') = \sum_{\tau_i} \sum_{n=0}^{\infty} E_i(\eta \cdot \eta')\, D^{(n,i)A_1\cdots A_n}(\eta, \eta')\, \Psi^{(i)}_{A_1\cdots A_n}(\eta'), \tag{5.1}$$

where

$$E_n(\eta \cdot \eta') = (\eta \cdot \eta')^{-1/2(l_A + l_B - \tau_i)}.$$

($\eta \cdot \eta' = -\frac{1}{2}kk'(x - x')^2$) are singular c-numbers; -1_A, -1_B, -1_n are the homogeneity degrees of the operators A, B, Ψ_n, respectively; $\tau_i = l_n^{(i)} - n$ is the "twist" of the particular representation (i.e., the difference between the dimension and the order of the highest Lorentz representation contained in the conformal tensor $\Psi^{(i)}_{A_1\cdots A_n}(\eta)$) and it determines the strength of the light cone singularity for $\eta \cdot \eta' \to 0$.

For greater generality we have included sums over the twists and over the tensor orders, since, for $\eta \cdot \eta' \neq 0$, many representations could be relevant. An additional sum over the representations which have the same twist and order may also appear and is not explicitly written in (5.1).

The "orbital" operator $D^{(n,i)}(\eta, \eta')^{A_1\cdots A_n}$ behaves as a differential operator with respect to the η' coordinate, is defined on $\eta^2 = \eta'^2 = 0$, and is completely determined up to a factor by conformal covariance to be the formal power (see Ref. (49))

$$[\eta \cdot \eta' \Box_6' - 2\eta \cdot \partial'(1 + \eta' \cdot \partial')]^{\frac{1}{2}l_B - l_A - l_n - n}. \tag{5.2}$$

This operator can be shown to exist for integer powers. It can be easily continued analytically to any power by means of the following identity:

$$D^h(\eta, \eta') = \frac{\Gamma(l_n - 1 + h)}{\Gamma(l_n - 1)} (-2)^h \sum_{J=0}^{\infty} \left(\frac{\eta \cdot \eta'}{2}\right)^J \frac{1}{J!} \frac{\Gamma(-h + J)}{\Gamma(-h)} \frac{\Gamma(l_n - 1)}{\Gamma(l_n - 1 + J)}$$
$$\times (-\eta \cdot \partial')^{h-J} \Box_6'^J \qquad \left(h = -\frac{l_A - {}_B + l_n + n}{2}\right). \tag{5.3}$$

Using the coordinates $x_\mu = k^{-1}\eta_\mu$, $k = \eta_5 + \eta_6$ on the cone $\eta^2 = 0$ and the identities

$$D(\eta, \eta') = 2\frac{k}{k'}\left[(L - (x - x')^\mu\, \partial_\mu')(1 - L) - \left(\frac{x - x'}{2}\right)^2 \Box_4'\right]$$
$$\left(L = -k' \frac{\partial}{\partial k'}\right), \tag{5.4}$$

$$(-\eta \cdot \partial')^\beta = \left(\frac{k}{k'}\right)^\beta \frac{\Gamma(1 + \beta)}{\Gamma(l)}\, {}_1F_1(-\beta; l; (x - x')^\mu\, \partial_\mu') \tag{5.5}$$

(where $-l$ is the homogeneity degree of the function on which $-\eta \cdot \partial'$ is applied), we are able to rewrite (5.1) as

$$A(x)\, B(x') = \sum_{n=0}^{\infty} \left(\frac{1}{(x-x')^2}\right)^{(1/2)(l_A+l_B+n-l_n)} C_n^{AB} \int_0^1 du\, u^{(1/2)(l_A-l_B+l_n+n)-1}$$
$$\times (1-u)^{(1/2)(l_B-l_A+l_n-n)-1}\, e^{u(x-x')\cdot\partial'}$$
$$\times {}_0F_1\left(l_n - 1; -\left(\frac{x-x'}{2}\right)^2 u(1-u)\,\Box'\right) x^{A_1} \cdots x^{A_n} \Psi_{A_1\cdots A_n}(x'), \tag{5.6}$$

where $x^A = k^{-1}\eta^A$ and

$$\Psi_{A_1\cdots A_n}(x') = (e^{i\pi\cdot x'} O)_{A_1\cdots A_n}(x')$$

(see 3.13). We have omitted the sum over the twists for simplicity. Details on the derivation of (5.6) from (5.3–5) are given in appendix A. Expansion (5.6) is manifestly covariant under the conformal algebra on space-time. The representations appearing in (5.6) are exactly the tensor representations constructed by means of the isomorphism exploited in Section 3. Moreover, according to Section 4 we have to separately discuss the contribution of tensors with $l_n = 2 + n$ to the expansion (5.6).

In that case we have that, in general, the n-tensor representations cannot appear. This is due to the fact that conformal covariance necessarily couples together physical components $\tilde{O}_{\alpha_1\cdots\alpha_n}(k)$ to nonphysical ones $O_{\alpha_1\cdots\alpha_{n-k}xx\cdots}(x)$ ($x = 5$ or 6).

As we have seen before, we call the latter nonphysical as they cannot be expressed in terms of the physical components. Moreover, for nonconserved tensors, $\hat{O}_{\alpha_1\cdots\alpha_{n-1}}(x) = \partial^\alpha O_{\alpha\alpha_1\cdots\alpha_{n-1}}(x) \neq 0$, we recall that the tensors $\partial^B O_{BA_1\cdots A_{n-1}}(x)$ are acceptable, as they do not have unphysical components, and they are expected to appear in the expansion. They have twist $l_n - n = 4$ and therefore contribute to (5.6) with a term like

$$\left[\frac{1}{(x-x')^2}\right]^{1/2(l_A+l_B)-2} C_n^{AB} \int_0^1 du\, u^{(1/2)(l_A-l_B)+n+1}(1-u)^{(1/2)(l_B-l_A)+1}\, e^{u(x-x')\cdot\partial'}$$
$$\times {}_0F_1\left(n+3; -\left(\frac{x-x'}{2}\right)^2 u(1-u)\,\Box'\right) x^{A_1} \cdots x^{A_{n-1}} \hat{\Psi}_{A_1\cdots\ n-1}(k'), \tag{5.7}$$

where

$$\hat{\Psi}_{A_1\cdots A_n}(x) = (e^{ix\cdot\pi} \hat{O})_{A_1\cdots A_{n-1}}(k)$$

and

$$\hat{O}_{\alpha_1\cdots\alpha_{n-1-k}6\cdots 6}(x) = 2^{-k} \frac{1}{\Gamma(2+k)}\, \partial_{\mu_1\cdots\mu_k} \partial_\alpha O^{\alpha\mu_1\cdots\mu_k}(x)_{\alpha_1\cdots\alpha_{n-1-k}} \tag{5.8}$$

according to (4.5). In free-field theory this term is always absent, as the n-tensors are conserved in space-time.

It can, however, be shown that, for canonical dimensions $l_n = 2 + n$, an exceptional case occurs. This is the case $l_A = l_B$. In this case the nonphysical components do not appear in expansion (5.6), i.e., the covariant expansion contains those conformal tensors whose only nonvanishing components are $O_{\alpha_1 \cdots \alpha_n}(0)$. The existence of this exceptional case is due to the fact that the coefficients of the nonphysical components $O_{\alpha_1 \cdots \alpha_{n-k} xx \cdots}(0)$ ($x = 5$ or 6) appear in the expansion with a multiplicative factor which is an integer power of $(l_A - l_B)$. Such a factor can easily be understood from consistency with translation invariance. It is the same factor which cancels out the divergences for $l_A = l_B$ and $l_n \neq 2 + n$. So, for $l_A = l_B$, the contributions of the n-tensor representation of conformal algebra split into two independent contributions to the operator product expansion:

$$\begin{aligned} A(x)\,A(x')_n \sim {} & \left[\frac{1}{(x-x')^2}\right]^{l_A-1} C_n \int_0^1 du\, u^n(1-u)^n e^{u(x-x')\cdot\partial'} \\ & \times {}_0F_1\left(n+1; -\left(\frac{x-x'}{2}\right)^2 u(1-u)\,\Box'\right) x^{A_1} \cdots x^{A_n} \Psi_{A_1 \cdots A_n}(x') \\ & + \left[\frac{1}{(x-x')^2}\right]^{l_A-2} C_n' \int_0^1 du\, u^{n+1}(1-u)^{n+1} e^{u(x-x')\cdot\partial'} \\ & \times {}_0F_1\left(n+3; -\left(\frac{x-x'}{2}\right)^2 u(1-u)\,\Box'\right) \\ & \times x^{A_1} \cdots x^{A_{n-1}}\, \partial^B \Psi_{BA_1 \cdots A_{n-1}}(x'), \end{aligned} \tag{5.9}$$

where

$$x^{A_1} \cdots x^{A_n} \Psi_{A_1 \cdots A_n}(x') = (x-x')^{\alpha_1} \cdots (x-x')^{\alpha_n}\, O_{\alpha_1 \cdots \alpha_n}(x'),$$

$$\begin{aligned} & x^{A_1} \cdots x^{A_{n-1}}\, \partial^B \Psi_{BA_1 \cdots A_{n-1}}(x') \\ & = \sum_{J=0}^{n-1} \binom{n-1}{J} \frac{1}{\Gamma(2+J)} \left[-\tfrac{1}{2}(x-x')^2\right]^J (x-x')^{\alpha_1} \cdots (x-x')^{\alpha_J} \\ & \times \partial'_{\mu_1} \cdots \partial'_{\mu_J} \partial_\mu O^{\mu\mu_1 \cdots \mu_J}_{\alpha_1 \cdots \alpha_{n-J}}(x'). \end{aligned} \tag{5.10}$$

However, in the light-cone limit we always regain the expansion derived in Ref. (48).

At $x' = 0$ we have

$$\begin{aligned} \underset{x^2 \to 0}{A(x)\, B(0)_n \sim} &\left(\frac{1}{x^2}\right)^{(1/2)(l_A+l_B+n-l_n)} C_n^{AB} \int_0^1 du\, u^{(1/2)(l_A-l_B+l_n+n)-1} \\ &\times (1-u)^{(1/2)(l_B-l_A+l_n+n)-1}\, x^{\alpha_1} \cdots x^{\alpha_n} O_{\alpha_1 \cdots \alpha_n}(ux) \\ &= \left(\frac{1}{x^2}\right)^{(1/2)(l_A+l_B+n-l_n)} C_n^{AB} x^{\alpha_1} \cdots x^{\alpha_n} \\ &\times {}_1F_1(\tfrac{1}{2}(l_A - l_B + l_n + n);\, l_n + n;\, x \cdot \partial)\, O_{\alpha_1 \cdots \alpha_n}(0), \end{aligned} \tag{5.11}$$

which is causal and conformal covariant in this limit. We also note that in this expansion no troubles arise for $l_n = 2 + n$. This is clearly due to the fact that the nonphysical components are present in nonleading light-cone singularities or, similarly, possible divergences in the expansion, for $l_n \to 2 + n$, are carried by nonleading terms for $x^2 \to 0$. Algebraically this is due to the fact that conformal invariance on the light-cone is less stringent than strict conformal invariance. Conformal covariance on the light-cone only requires the Clebsh–Gordon coefficients to satisfy an iterative formula; but, when strict conformal covariance is imposed, the same coefficients must satisfy a larger set of equations, which couple them to the coefficients of nonleading singularities, and, in particular, to possible nonphysical components. This is reflected in the fact that for $l_n = 2 + n$ some equations become incompatible and the spin-n representation drops out.

Finally, it is interesting to observe that expansion (5.6) can also be understood from the strict conformal invariant three-point function [49, 54]

$$\begin{aligned} &\langle 0 \mid C(y)\, A(x)\, B(0) \mid 0 \rangle \\ &= C_{ABC} \left(\frac{1}{x^2}\right)^{(1/2)(l_A+l_B-l_C)} \left(\frac{1}{y^2}\right)^{(1/2)(l_C+l_B-l_A)} \left(\frac{1}{(y-x)^2}\right)^{(1/2)(l_C-l_A-l_B)} \end{aligned} \tag{5.12}$$

The following integral representation holds:

$$\begin{aligned} &\left(\frac{1}{y^2}\right)^{(1/2)(l_C+l_B-l_A)} \left(\frac{1}{(y-x)^2}\right)^{(1/2)(l_C+l_A-l_B)} \\ &\quad\times \frac{\Gamma(l_C)}{\Gamma\left(\frac{l_C+l_A-l_B}{2}\right) \Gamma\left(\frac{l_C+l_B-l_A}{2}\right)} \int_0^1 du\, u^{(1/2)(l_C+l_A-l_B)-1} \\ &\quad\times (1-u)^{(1/2)(l_C+l_B-l_A)-1} \left[\frac{1}{(y-ux)^2}\right]^{l_C} \left[1 + \frac{x^2 u(1-u)}{(y-ux)^2}\right]^{-l_C} \end{aligned} \tag{5.13}$$

(Riemann–Liouville fractional integral).

It may be interesting to note that (5.13) contains l_A, l_B only through the difference $l_A - l_B$, as it must for the q-number part of $A(x)\,B(0)$.

We expand in power series

$$\left(1 + \frac{x^2u(1-u)}{(y-ux)^2}\right)^{-l_C} = \sum_{h=0}^{\infty} \frac{1}{h!} \frac{\Gamma(l_C+h)}{\Gamma(l_C)} (-x^2)^h\, [u(1-u)]^h \left[\frac{1}{(y-ux)^2}\right]^h \tag{5.14}$$

and use the property

$$\begin{aligned}\langle 0 \mid \Box^h C(x)\, C(y) \mid 0\rangle &\sim \Box^h \left[\frac{1}{(x-y)^2}\right]^{l_C} \\ &= 4^h \frac{\Gamma(l_C+h)\,\Gamma(l_C-1+h)}{\Gamma(l_C)\,\Gamma(l_C-1)} \left[\frac{1}{(y-x)^2}\right]^{l_C+h}.\end{aligned} \tag{5.15}$$

We see that the contribution from (5.13) to (5.12) comes from the operator product expansion

$$\begin{aligned}A(x)\,B(0) = \left(\frac{1}{x^2}\right)^{(1/2)(l_A+l_B-l_C)} &\int_0^1 du\, u^{(1/2)(l_C+l_A-l_B)-1}(1-u)^{(1/2)(l_C+l_B-l_A)-1}\, e^{ux\cdot\partial} \\ &\times {}_0F_1\left(l_C-1; -\frac{x^2}{4}u(1-u)\,\Box\right) C(0)\end{aligned} \tag{5.16}$$

(we recall the selection rule of conformal invariance on the two point function), and (5.16) is nothing but (5.6) at $x' = 0$ for the scalar representation.

6. Constraints of Conformal Algebra on n-Point Functions

Let us consider the n-point correlation function

$$\langle 0 \mid A_1(x_1) \cdots A_n(x_n) \mid 0\rangle, \tag{6.1}$$

where the fields $A_i(x_i)$ belong to representations with $K_\lambda = 0$, i.e., $[K_\lambda, A_j(0)] = 0$ and we suppose for simplicity that they are spinless.

To deduce the constraints from conformal algebra on (6.1) it is convenient to make use of the manifestly covariant notation in terms of the coordinates η_A. We therefore consider the expression

$$\langle 0 \mid A_1(\eta_1) \cdots A_n(\eta_n) \mid 0\rangle. \tag{6.2}$$

From $O(4, 2)$ covariance it turns out that (6.2) can only depend on $\frac{1}{2}n(n-1)$ scalar products

$$\langle 0 \mid A_1(\eta_1) \cdots A_n(\eta_n) \mid 0\rangle = F(\eta_1 \cdot \eta_2, \ldots, \eta_{n-1} \cdot \eta_n), \tag{6.3}$$

where

$$\eta_i \cdot \eta_j = -\tfrac{1}{2}k_i k_j (x_i - x_j)^2.$$

In addition, using the homogeneity conditions

$$\eta_i \frac{\partial}{\partial \eta_i} A_i(\eta_i) = -l_i A_i(\eta_i), \tag{6.4}$$

where l_i are the scale dimensions of the local operators $A_i(x_i)$, we obtain that the n-point function depends on $\frac{1}{2}n(n-1) - n = \frac{1}{2}n(n-3)$ independent variables for $n \leqslant 6$.

This shows, in particular, that for $n = 2$ the solution is overdetermined and in fact one has

$$\begin{aligned} \langle 0 \mid A(x)\, B(y) \mid 0 \rangle &= C_A \left[\frac{1}{(x-y)^2}\right]^{l_A} && \text{if} \quad l_A = l_B \\ &= 0 && \text{if} \quad l_A \neq l_B . \end{aligned} \tag{6.5}$$

Writing

$$\langle 0 \mid A(\eta)\, B(\eta') \mid 0 \rangle = F(\eta \cdot \eta'), \tag{6.6}$$

one has the homogeneity conditions

$$\eta^A \partial_A F(\eta \cdot \eta') = -l_A F(\eta \cdot \eta'), \tag{6.7}$$

$$\eta'^A \partial_A' F(\eta \cdot \eta') = -l_B F(\eta \cdot \eta'), \tag{6.8}$$

which are consistent only for $l_A = l_B$.

Selection rule (6.5) can be generalized for two irreducible tensor fields. The result is the following general selection rule for vacuum expectation values.

THEOREM. *The vacuum expectation value*

$$\langle 0 \mid \Psi_{A_1 \cdots A_n}(\eta)\, \Phi_{B_1 \cdots B_m}(\eta') \mid 0 \rangle = F_{A_1 \cdots A_n B_1 \cdots B_m}(\eta, \eta' \tag{6.9}$$

is nonvanishing only if $l_A = l_B$ *and* $n = m$.

To prove the above theorem we note first that the scalar function

$$F(\eta \cdot \eta') = \eta'^{A_1} \cdots \eta'^{A_n} \eta^{B_1} \cdots \eta^{B_n} F_{A_1 \cdots A_n B_1 \cdots B_m}(\eta, \eta') \tag{6.10}$$

is a homogeneous function of degree $-l_A + m$, $-l_B + n$ in k and k' respectively. Therefore, Eq. (6.10) is a consistent expression only if

$$l_A - l_B = m - n. \tag{6.11}$$

The scalar $F(\eta \cdot \eta')$ is the contribution to the vacuum expectation value of Eq. (6.9) from the covariant

$$\eta_{A_1} \cdots \eta_{A_n} \eta'_{B_1} \cdots \eta'_{B_m} (\eta \cdot \eta')^{-l_A - n}, \tag{6.12}$$

The covariant (6.12) satisfies the trace and transversality conditions, with fixed degree of homogeneity related by Eq. (6.11). From (6.12) one can now obtain the whole set of allowed covariants through repeated application of the following operations:

(i) permuting η and η';
(ii) substituting one or more couples (η, η') with a g_{AB}-symbol;
(iii) substituting an equal number of couples (η, η) and (η', η') with a corresponding number of g_{AB}-symbols.

One can convert to space-time by the procedures described before and obtain

$$\langle 0 \mid O_{\alpha_1 \cdots \alpha_n}(x)\, O_{\beta_1 \cdots \beta_m}(x') \mid 0 \rangle$$

$$= k^{l_A} k'^{l_B} (e^{-ix'\pi})^{A_1 \cdots A_n}_{\alpha_1 \cdots \alpha_n} (e^{-ix\pi})^{B_1 \cdots B_m}_{\beta_1 \cdots \beta_m} \langle 0 \mid \Psi_{A_1 \cdots A_n}(\eta)\, \Phi_{B_1 \cdots B_n}(\eta') \mid 0 \rangle, \tag{6.13}$$

where the operator $e^{-ix\pi}$ acts as

$$(e^{-ix\pi})^{A_1 \cdots A_n}_{\alpha_1 \cdots \alpha_n} \bar{\eta}_{A_1} \cdots \bar{\eta}_{A_n} = k^n (\bar{x} - x)_{\alpha_1} \cdots (\bar{x} - x)_{\alpha_n}, \tag{6.14}$$

so that the covariants defined above, symmetric and satisfying the supplementary conditions, do not contribute to (6.13) unless $n = m$. Let us indeed consider, for example, the leading light-cone contribution to the vacuum expectation value (6.9) which arises from the covariant

$$\eta'_{A_1} \cdots \eta'_{A_n} \eta_{B_1} \cdots \eta_{B_m} F'(\eta \cdot \eta').$$

The $O(4, 2)$ covariance implies $l_A - l_B = n - m$ and for consistency with Eq. (6.11) one must have $m = n$ and $l_A = l_B$.

The theorem can also be derived directly in space-time by simply remarking that the vacuum-expectation value in Eq. (6.13) is nothing else than the contribution of the c-number term to the operator product expansion

$$O_{\alpha_1 \cdots \alpha_n}(x)\, O_{\beta_1 \cdots \beta_m}(0) = F_{\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_m}(x)\, I + \cdots, \tag{6.15}$$

where the dots stand for local operators which do not contribute to the vacuum expectation value, unless there is spontaneous breaking. The tensor $F_{\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_m}(x)$ has to satisfy the equation

$$\left(2x_\lambda x^\mu \frac{\partial}{\partial x_\mu} - x^2 \frac{\partial}{\partial x_\lambda} + 2l_A x_\lambda - 2ix^\nu \Sigma_{\lambda\nu}^{(n)}\right) F_{\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_m}(x) = 0, \qquad (6.16)$$

which is easily obtained by commuting both sides of Eq. (6.15) with K_λ. For $F_{\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_m}(x)$ we have

$$F_{\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_m}(x) = c x_{\alpha_1} \cdots x_{\alpha_n} x_{\beta_1} \cdots x_{\beta_m} \left(\frac{1}{x^2}\right)^{(l_A + l_B + m + n)/2} \qquad (6.17)$$
$$+ \text{ less singular terms in } x^2.$$

Equation (6.16) can be considered at each order in x^2 and it leads to a set of homogeneous relations between differently singular terms in the expansion (6.17). The relations are consistent for $l_A - l_B = 0$ and $m - n = 0$. If, however, one only retains the light-cone dominating terms in expansion (6.17) one regains the less stringent condition $l_A - l_B = n - m$.

Conversely, one can work out the light-cone relation, $l_A - l_B = n - m$, directly from the six-dimensional formalism.

Let us consider for example the scalar-vector case

$$\langle 0 \mid O_A(\eta)\, C(\eta') \mid 0 \rangle = C_1 \eta_A (\eta \cdot \eta')^{-l_B} + C_2 \eta_A' (\eta \cdot \eta')^{-l_A}. \qquad (6.18)$$

Both terms on the right-hand side of Eq. (6.18) satisfy in the limit $\eta \cdot \eta' \to 0$ the transversality condition. On space-time we have

$$\langle 0 \mid O_\mu(x)\, C(x') \mid O \rangle \xrightarrow[(x-x')^2 \to 0]{} C_2 (x - x')_\mu \left(\frac{1}{(x - x')^2}\right)^{l_A}. \qquad (6.19)$$

Equation (6.11) is satisfied in the limit. Full conformal invariance would imply $C_2 = 0$, and therefore the vanishing of the vacuum expectation value.

The derivation based on the operator product expansion, Eq. (6.15), brings into evidence the failure of the selection rule for the vacuum expectation values given in the theorem in cases when spontaneous breaking is present. Possible local scalar operators on the right-hand side of Eq. (6.15) may contribute to the vacuum expectation value and thus invalidate the selection rule. However, the physical situation must be discussed carefully. It is obvious that if, in a scale invariant theory, an operator $u(x)$ has vacuum expectation value different from zero, it can only have dimension zero. In fact its vacuum expectation value can only be a numerical constant. The presence in a conformally invariant theory of a dimensioned operator $u(x)$ with nonzero vacuum expectation value thus necessarily implies spontaneous breaking. However in the physical situation conformal invariance applies at most to the so-called skeleton theory, in Wilson's sense. Therefore, in order to prove spontaneous breaking of the conformal invariance of the skeleton theory one would have to show that a scalar operator exists whose vacuum expectation value remains nonvanishing in the skeleton limit, that is, when all symmetry breaking parameters vanish. More specifically, suppose for example, following Wilson, that the symmetry breaking in the hamiltonian is $\epsilon w + \epsilon_3 u_3 + \epsilon_8 u_8 + \epsilon_0 u_0$, where w is an $SU_3 \times SU_3$ scalar and u_i belongs to $(3, \bar{3}) + (\bar{3}, 3)$ of $SU_3 \times SU_3$. The vacuum expectation values $\langle 0 \mid u_i \mid 0 \rangle$ in the limit when $\epsilon_3, \epsilon_8, \epsilon_0$ vanish might depend on ϵ and vanish with ϵ. If however, when all parameters $\epsilon, \epsilon_3, \epsilon_8, \epsilon_0$ vanish, $\langle 0 | u_i \mid 0 \rangle$ does not vanish, then one has spontaneous breaking of the scale invariance of the skeleton theory.

For $n = 3$ the solution for the conformally covariant vacuum expectation value is completely determined. One has [54, 55]

$$\langle 0 \mid A(x)\, B(y)\, C(z) \mid 0 \rangle = C_{ABC} \left[\frac{1}{(x-z)^2}\right]^{(1/2)(l_A - l_B + l_C)} \times \left[\frac{1}{(z-y)^2}\right]^{(1/2)(l_B - l_A + l_C)} \left[\frac{1}{(x-y)^2}\right]^{(1/2)(l_A + l_B - l_C)}. \tag{6.20}$$

For $n = 4$ the solution can be expressed in terms of an arbitrary function of two suitable variables of dimension zero.

The construction can be done as follows. Consider the $(4 \times 3)/2 = 6$ independent scalar products

$$\eta_1 \cdot \eta_2, \quad \eta_1 \cdot \eta_3, \quad \eta_2 \cdot \eta_3, \quad \eta_1 \cdot \eta_4, \quad \eta_2 \cdot \eta_4, \quad \eta_3 \cdot \eta_4. \tag{6.21}$$

It can be easily verified that there exist only two independent quantities which are homogeneous of zero degree in the variables k_i. A possible choice is

$$\frac{(\eta_1 \cdot \eta_2)(\eta_3 \cdot \eta_4)}{(\eta_1 \cdot \eta_3)(\eta_1 \cdot \eta_4)} = \eta, \qquad \frac{(\eta_1 \cdot \eta_4)(\eta_2 \cdot \eta_3)}{(\eta_1 \cdot \eta_2)(\eta_3 \cdot \eta_4)} = \xi. \tag{6.22}$$

By choosing four of the six scalar products in (6.21) we can write

$$\begin{aligned}\langle 0 \mid A_1(\eta_1)\, A_2(\eta_2)\, A_3(\eta_3)\, A_4(\eta_4) \mid 0 \rangle &= A(\eta_1 \cdot \eta_2, \ldots, \eta_3 \cdot \eta_4) \\ &= (\eta_1 \cdot \eta_2)^{\alpha} (\eta_1 \cdot \eta_3)^{\beta} (\eta_1 \cdot \eta_4)^{\gamma} (\eta_2 \cdot \eta_3)^{\delta} f(\eta, \xi).\end{aligned} \tag{6.23}$$

The constraints of Eq. (6.4) on the other hand give

$$\begin{aligned} \alpha &= \tfrac{1}{2}(\lambda_2 + \lambda_1 - \lambda_3 - \lambda_4); & \beta &= \tfrac{1}{2}(\lambda_1 + \lambda_3 - \lambda_2 - \lambda_4); \\ \gamma &= \lambda_4; & \delta &= \tfrac{1}{2}(\lambda_2 + \lambda_3 + \lambda_4 - \lambda_1), \end{aligned} \tag{6.24}$$

where $\lambda_i = -l_i$.

In space-time we obtain directly

$$\langle 0 \mid A(x)\, B(y)\, C(z)\, D(t) \mid 0\rangle$$
$$= \left[\frac{1}{(x-y)^2}\right]^{(1/2)(l_A+l_B-l_C-l_D)} \left[\frac{1}{(x-z)^2}\right]^{(1/2)(l_A+l_C-l_B-l_D)}$$
$$\times \left[\frac{1}{(y-z)^2}\right]^{(1/2)(l_B+l_C+l_D-l_A)} \left[\frac{1}{(x-t)^2}\right]^{l_D}$$
$$\times f\left(\frac{(x-y)^2\,(z-t)^2}{(x-z)^2\,(y-t)^2}, \frac{(x-t)^2\,(y-z)^2}{(x-y)^2\,(z-t)^2}\right). \tag{6.25}$$

Let us next discuss the n-point functions in conjunction with Wilson's expansion. Dealing with an exact conformally covariant theory, we insert into Eq. (6.5) a conformally covariant Wilson expansion (of course, on the whole space-time, not limited to the neighborhood of the light-cone). Only the contribution from the c-number is retained if all local operators contributing to the expansion have vanishing vacuum expectation values. In other words, if we write in brief notation

$$A(x)\, B(y) = \sum_i C^{\alpha_1\cdots\alpha_n}\left(x-y, \frac{\partial}{\partial y}\right) O^{(n)}_{\alpha\cdots\alpha}(y), \tag{6.26}$$

only the term with $O = I$ (identity operator) contributes to (6.5). Note also that the identity operator is contained in (6.26) only if $l_A = l_B$.

Passing now to the three-point function, if we insert a Wilson's expansion into (6.20) we get for the left-hand side the expression

$$\sum_i C^{\alpha_1\cdots\alpha_n}\left(x-y, \frac{\partial}{\partial y}\right) \langle 0 \mid O^{(n)}_{\alpha_1\cdots\alpha_n}(y)\, C(z) \mid 0\rangle. \tag{6.27}$$

From the selection rule proved in this section for vacuum expectation values we have

$$\langle 0 \mid O^{(n)}_{\alpha_1\cdots\alpha_n}(y)\, C(z) \mid 0\rangle = 0,$$

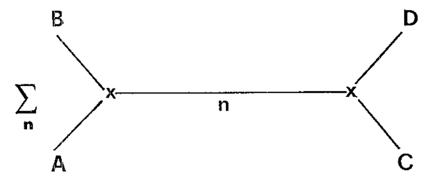
unless $n = 0$ and $l_n = l_c$, that is essentially unless $O^{(n)} = C$. Therefore the expression in (6.27) becomes

$$C\left(x-y, \frac{\partial}{\partial y}\right) \langle 0 \mid C(y)\, C(z) \mid 0\rangle, \tag{6.28}$$

where $C(x-y, \partial/\partial y)$ is the differential operator associated to $O^{(n)} = C$. The final expression (6.28) is nothing else than a different algebraic way of writing the right-hand side of Eq. (6.20), as proved in Ref. (56).

Finally, let us consider the interesting case of $n = 4$, that is the vacuum expectation value of four local operators (again we assume them to be scalars for simplicity).

In the four-point function $\langle 0 \mid A(x)\, B(y)\, C(z)\, D(t) \mid 0\rangle$ we have different ways of inserting Wilson's type expansions. The analogy with the different ways of performing partial wave expansions (in s, u, or t channel) is quite suggestive. To stress this analogy we speak of an s-channel decomposition when we insert expansions for $A(x)\, B(y)$ and $C(z)\, D(t)$; of a t-channel decomposition when we expand $A(x)\, D(t)$ and $B(y)\, C(z)$; and of a u-channel decomposition when we expand $A(x)\, C(z)$ and $B(y)\, D(t)$. The s-channel decomposition can be symbolized diagrammatically as



(6.29)

and it is obtained by first inserting an expansion for $A(x)\, B(y)$ giving

$$\langle 0 \mid A(x)\, B(y)\, C(z)\, D(t) \mid 0\rangle$$
$$= \sum_n C^{\alpha_1\cdots\alpha_n}_{ABO_n}\left(x - y, \frac{\partial}{\partial y}\right) \langle 0 \mid O^{(n)}_{\alpha_1\cdots\alpha_n}(y)\, C(z)\, D(t) \mid 0\rangle. \qquad (6.30)$$

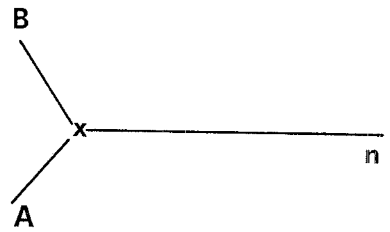
The vertex $\langle 0 \mid O^{(n)}_{\alpha_1\cdots\alpha_n}(y)\, C(z)\, D(t) \mid 0\rangle$ can be represented, again inserting a Wilson expansion and using the selection rule for vacuum expectation values, as

$$\langle 0 \mid O^{(n)}_{\alpha_1\cdots\alpha_n}(y)\, C(z)\, D(t) \mid 0\rangle$$
$$= C^{\beta_1\cdots\beta_n}_{CDO_n}\left(z - t, \frac{\partial}{\partial t}\right) \langle 0 \mid O_{\alpha_1\cdots\alpha_n}(y)\, O_{\beta_1\cdots\beta_n}(t) \mid 0\rangle. \qquad (6.31)$$

Finally, one has an expansion of the form

$$\langle 0 \mid A(x)\, B(y)\, C(z)\, D(t) \mid 0\rangle = \sum_n C^{\alpha_1\cdots\alpha_n}_{ABO_n}\left(x - y, \frac{\partial}{\partial y}\right) C^{\beta_1\cdots\beta_n}_{CDO_n}\left(z - t, \frac{\partial}{\partial t}\right)$$
$$\times \langle 0 \mid O_{\alpha_1\cdots\alpha_n}(y)\, O_{\beta_1\cdots\beta_n}(t) \mid 0\rangle. \qquad (6.32)$$

Note that the two differential operators commute, as they act on different variables. It is now apparent that in diagram (6.2) the vertex



stands precisely for the conformally covariant vertex $\langle 0 \mid ABO^{(n)}_{\alpha_1 \cdots \alpha_n} \mid 0\rangle$, and similarly for nDC, while the line n stands for the two-point function $\langle 0 \mid O_{\alpha_1 \cdots \alpha_n} O_{\beta_1 \cdots \beta_n} \mid 0\rangle$. The causality requirement in the theory leads to the (crossing) relations

$$\begin{aligned}
&\sum_n C^{\alpha_1 \cdots \alpha_n}_{ABO_n}\left(x - y, \frac{\partial}{\partial y}\right) C^{\beta_1 \cdots \beta_n}_{CDO_n}\left(z - t, \frac{\partial}{\partial t}\right) \langle 0 \mid O_{\alpha_1 \cdots \alpha_n}(y)\, O_{\beta_1 \cdots \beta_n}(t) \mid 0\rangle \\
&\quad = \sum_n C^{\alpha_1 \cdots \alpha_n}_{ACO_n}\left(x - y, \frac{\partial}{\partial z}\right) C^{\beta_1 \cdots \beta_n}_{BDO_n}\left(y - t, \frac{\partial}{\partial t}\right) \langle 0 \mid O_{\alpha_1 \cdots \alpha_n}(z)\, O_{\beta_1 \cdots \beta_n}(t) \mid 0\rangle \\
&\quad = \sum_n C^{\alpha_1 \cdots \alpha_n}_{ADO_n}\left(x - t, \frac{\partial}{\partial t}\right) C^{\beta_1 \cdots \beta_n}_{BCO_n}\left(y - z, \frac{\partial}{\partial z}\right) \langle 0 \mid O_{\alpha_1 \cdots \alpha_n}(t)\, O_{\beta_1 \cdots \beta_n}(z) \mid 0\rangle
\end{aligned} \tag{6.33}$$

which equate the result of the s-channel, u-channel, and t-channel expansions. We observe finally that the conformally covariant four-point function is not fully determined from conformal invariance. In fact the coefficients which appear in the sum at each n are undetermined.

Finally, we would like to comment on the role of possible Ward identities associated to conservation or partial conservation of local tensors (for instance $\theta_{\mu\nu}$, or the internal symmetry currents $j_\mu{}^\alpha$) within a conformal invariant theory. In this connection it is relevant to observe that a three-point function, such as $\langle 0 \mid j_\mu(x)\, A(y)\, B(z) \mid 0\rangle$, with j_μ satisfying a conservation equation $\partial^\mu j_\mu = 0$, has to satisfy a number of constraints, from conformal covariance and local conservation, which exceed the number of possible parameters. Conformal invariance in general gives the vertex $\langle 0 \mid O_{\alpha_1 \cdots \alpha_n}(x)\, A(y)\, B(z) \mid 0\rangle$ in terms of a single covariant. The conservation equation $\partial^{\alpha_1} O_{\alpha_1 \cdots \alpha_n} = 0$ imposes additional constraints which tell us that the vertex vanishes unless $l_A = l_B$. The strength of such a constraint is quite evident. Its use in conjunction with the set of relations that conformal invariance implies among vacuum expectation values appears as very promising.

7. Conclusions

The present work mainly deals with the conformal covariant formulation of operator product expansion. In such a context an operator product expansion can formally be regarded as the decomposition of the operator product into a direct sum of irreducible tensor representations of the conformal algebra. One is therefore led to a study of the representations which are relevant for the expansion and to their classification. The discussion can be most conveniently carried out in conformally covariant notation by exploiting the isomorphism of the conformal algebra with the orthogonal algebra $O(4, 2)$. We have thus studied in detail such an isomorphism and its implications for the infinite dimensional operator representations of the conformal algebra which provide a basis for the operator product expansion. We have noted, in particular, the peculiar structure of those representations which are associated to tensors of canonical dimensions, i.e., those tensors which on the light cone are responsible for the observed scaling in deep inelastic electroproduction. The isomorphism described above with the $O(4, 2)$ algebra, also applies, of course, for such representations, but it exhibits well-defined pathologies, which have to do with a degeneracy appearing in the representations of the stability subalgebra of the conformal algebra.

We have stressed the natural relationship between the conformally covariant form of the operator product expansion and the conformally covariant three-point function. Essentially, the relationship arises from an orthogonality property of the conformally covariant two-point functions, which appears as a very powerful selection rule, strongly restricting any conformally covariant solution. When spontaneous breaking occurs the selection rule is violated in a definite way. Excluding spontaneous breaking, it is important to consider the simultaneous validity of conformal covariance and of Ward identities following from local conservation (or partial conservation) relations. For a n-point function the causality restrictions of the theory imply sets of equalities, resembling crossing relations, to be satisfied by the coefficients of the irreducible representations which contribute to the expansions.

Appendix

In this appendix we show the existence and uniqueness of (5.6). We consider the operator

$$D(\eta, \eta') = \frac{g^{BC}\eta^A\eta^D}{\eta \cdot \eta'} L_{AB}L_{CD} = [(\eta \cdot \eta') \Box_6' - 2\eta \cdot \partial'(1 + \eta' \cdot \partial')]. \quad \text{(A.1)}$$

This operator is obviously the only differential operator defined over $\eta'^2 = 0$, regular for $\eta \cdot \eta' = 0$ and homogeneous of degree one in k/k'. Equation (5.3) can be derived by induction. In fact, we have for $h = 1, 2$,

$$\begin{aligned} D(\eta, \eta') &= (1 - L)\left[\frac{(\eta \cdot \eta') \Box_6'}{1 - L} - 2(\eta \cdot \partial')\right] \\ &= (1 - l_n)\left[\frac{(\eta \cdot \eta') \Box_6'}{1 - l_n} - 2\eta \cdot \partial'\right], \end{aligned} \tag{A.2}$$

that coincides with (5.3) for $h = 1$, and

$$\begin{aligned} D^2(\eta, \eta') &= (1 - L)^2 \left[\frac{(\eta \cdot \eta') \Box_6'}{1 - L} - 2(\eta \cdot \partial')\right]\left[\frac{(\eta \cdot \eta') \Box_6'}{1 - L} - 2(\eta \cdot \partial')\right] \\ &= (1 - l_n)(1 - l_n - 1)\left[\frac{2(\eta \cdot \eta')(\eta \cdot \partial') \Box_6'}{1 - l_n} + 4(\eta \cdot \partial')^2 \right. \\ &\quad \left. + \frac{(\eta \cdot \eta')^2 \Box_6'^2}{(1 - l_n)(1 - l_n - 1)}\right], \end{aligned} \tag{A.3}$$

which again coincides with (5.3) for $h = 2$.

To derive (A.3) we have made use of the following relations

$$(1 - L)^n = (1 - l_n) \cdots (1 - l_n - n + 1), \tag{A.4}$$

$$(\eta \cdot \partial')(\eta \cdot \eta') = (\eta \cdot \eta')(\eta \cdot \partial'), \tag{A.5}$$

$$\Box_6'(\eta \cdot \eta') \Box_6' = 2(\eta \cdot \partial') \Box_6' + (\eta \cdot \eta') \Box_6'^2. \tag{A.6}$$

We will now make use of the Newton binomial expansion for the formal n-th power and write

$$D^n(\eta, \eta') = (1 - L)^n \sum_{J=0}^{n} \binom{n}{J} \frac{(\eta \cdot \eta')^J [2(\eta \cdot \partial')]^{h-J} \Box_6'^J}{(1 - L)^{\cdot}}. \tag{A.7}$$

Using relations (A.4)–(A.6), it is possible by a rather lengthy but trivial calculation to show that Eq. (A.7) also holds for $n + 1$. This completes the proof of the assertion, since Eq. (A.7) is nothing but Eq. (5.3) of the text.

Similarly it is possible to show the validity of Eq. (5.5). We are now faced with the problem of the uniqueness of the expansion. The conformal covariance on $D^{(n)A_1 \cdots A_n}(\eta, \eta')$ implies

$$D^{(n)A_1 \cdots A_n} = \sum_{k=0}^{n} C_{nk} \eta^{A_1} \cdots \eta^{A_{n-k}} \eta'^{A_{n-k+1}} \cdots \eta'^{A_n} [D(\eta, \eta')]^{h+k}. \tag{A.8}$$

We want to show that it is possible to essentially restrict the sum, in (A.8) to $k = 0$, since all terms for different k are proportional to each other.

In order to do this we first show the identity

$$\eta'^{A} D^{m}(\eta, \eta') \Psi_{A_1 \cdots A_n}(\eta') = 2m[1 - (l_n + m - 1)] \eta^{A_1} D(\eta, \eta')^{m-1} \Psi_{A_1 \cdots A_n}(\eta'). \tag{A.9}$$

Equation (A.9) is a consequence of the two supplementary conditions. In fact we have, for $m = 1$,

$$\begin{aligned} &\eta'^{A_1} D(\eta, \eta') \Psi_{A_1 \cdots A_n}(\eta') \\ &\quad = \{[D(\eta, \eta') \eta'^{A_1} + 2(\eta \cdot \partial') \eta'^{A_1} - 2(\eta \cdot \eta') \partial'^{A_1} + 2\eta^{A_1}(1 - L)\} \Psi_{A_1 \cdots A_n}(\eta') \\ &\quad = 2\eta^{A_1}(k - l_n) \Psi_{A_1 \cdots A_n}(\eta'). \end{aligned} \tag{A.10}$$

By simple but tedious algebra, Eq. (A.9) can be proved for $m = 2$ and assuming its validity for arbitrary m, it can be proved for $m + 1$, too. This completes the proof of our last statement.

Added in proof. Recent work related to the results of the present paper is: G. Mack and K. Symanzik, *Comm. Math. Phys.* **27** (1972), 247; S. Ferrara, R. Gatto, A. F. Grillo, and G. Parisi, *Phys. Lett.* **38B** (1972), 333; and G. Mack and I. Todorov, *Phys. Rev.* (to be published). A convenient algorithm, the shadow operator formalism, is also available to deal with such problems (S. Ferrara, R. Gatto, A. F. Grillo, and G. Parisi, *Lett. Nuovo Cimento* **4** (1972), 115).

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