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ON RENORMALIZABILITY OF NOT RENORMALIZABLE THEORIES

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On Renormalizability of Not Renormalizable Theories.

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If the interaction is not renormalizable, the renormalization program fails in perturbation theory (1): self-energies and vertices are not the only primitively divergent subgraphs, but an infinite number of divergent subgraphs is present.

One needs a number of subraction costants which increases with the order in the coupling costant. The theory is specified by an infinite number of renormalized coupling costant. The theory is specified by an infinite number of renormalized coupling constants: no prediction can be made unless in a very-weak-coupling-constant regime.

However there is the possibility that divergences be present only in perturbation theory but absent in the full solution. This would be very natural if the Green functions were not C^∞ in the coupling constant g at $g = 0$ (2), *i.e.* if terms like g^a or $g^2 \log g^2$ were present. Contributions of this kind automatically produce spurious divergences when expanded in powers of g. This type of divergences has nothing to do with possible real ultraviolet divergences in a nonperturbative treatment.

In this letter we present an example which shares this property. We consider the unphysical case of a $g\varphi^3$ theory in D dimensions (3), where D is a noninteger number greater than 6. The theory is superrenormalizable if $D < 6$, renormalizable if $D = 6$ and nonrenormalizable if $D > 6$.

We study the sum of all the ladder diagrams for the scattering amplitude in a particular kinematical situation. Although each diagram in perturbation theory is divergent and not well defined for rational dimensions greater than 6, we are able to give a meaning to the sum of all the diagrams: contributions of the form g^{8-D} appear in the sum.

The typical function we meet is

$$H(g, D) = \int_0^\infty \frac{l^{-t}}{1 + g^2 t((6-D)/2)}\, \mathrm{d}t = \sum_{N=0}^\infty g^{2N}\left(1 - N\frac{6-D}{2}\right). \tag{1}$$

(1) F. J. Dyson: *Phys. Rev.*, **75**, 486 (1949).

(2) P. J. Redmond and J. L. Uretski: *Phys. Rev. Lett.*, **1**, 147 (1958).

(3) G. 't Hooft and M. J. G. Veltman: *Nucl. Phys.*, **44** B, 189 (1972); C. G. Bollini and J. J. Giambiagi: *Phys. Lett.*, **40** B, 566 (1972); K. G. Wilson: *Phys. Rev. Lett.*, **28**, 548 (1972).

The expansion of the function in powers of g is clearly divergent for $D > 6$. No meaning can be given to the series for rational D (there exists always an N such that

$$\Gamma\left[1 - N\frac{D-6}{2}\right]$$

is divergent). Each term of the series is well defined for irrational D: the series is not convergent but can be formally summed. The final anseur is finite also for rational dimensions.

We compute the sum of the following diagrams:

$$I + II + III + IIII + \ldots \tag{2}$$

when all the internal mass are set to zero and two of the external momentum are also zero.

We write therefore

$$A(0, 0, p, -p) = A(p^2) = \sum_{N=0}^{\infty} g^{2(N+1)} A_N(p^2)\,, \tag{3}$$

where

$$A_0(p^2) = \frac{1}{(p^2)}\,, \qquad A_{N+1}(p^2) = \int \frac{A_N(k^2)}{(k^2)^2(k+p)^2}\, \mathrm{d}^D k\,. \tag{4}$$

We recall that all the integrals in noninteger dimension space can be evaluated reducing them to Gaussian integrals and using the formula [3]

$$\int \exp[-ak^2]\, \mathrm{d}^D k = \left(\frac{\pi}{a}\right)^{D/2}. \tag{5}$$

A long but simple evaluation yields

$$A(p^2) = g^2 \sum_{N=0}^{\infty} \frac{\Gamma(-2/\varepsilon)\Gamma(-1/\varepsilon)\Gamma(-1+\varepsilon)\Gamma(1/\varepsilon+2)\Gamma(1-N\varepsilon)}{\Gamma(N-2/\varepsilon)\Gamma(N-1/\varepsilon)\Gamma(1/\varepsilon+2+N)\Gamma(2+N)\Gamma(-1+N\varepsilon+\varepsilon)} \cdot [g^2(p^2)^\varepsilon \pi^{D/2}\varepsilon^4]^N\,, \qquad \varepsilon = \frac{D-6}{2}\,. \tag{6}$$

Letting the integral representations

$$\Gamma(z) = \int_0^\infty \exp[-t]\, t^{z-1}\mathrm{d}t\,, \qquad \frac{1}{\Gamma(z)} = \frac{i}{2\pi}\int_C (-\alpha)^{-z}\exp[-\alpha]\,\mathrm{d}\alpha\,, \tag{7}$$

in (6) and exchanging the sum with the integral we find

$$A(p^2) = \frac{ig^2\Gamma(-2/\varepsilon)\Gamma(-1/\varepsilon)\Gamma(-1+\varepsilon)\Gamma(1/\varepsilon+2)\Gamma(1-N\varepsilon)}{(2\pi)^5} \cdot \int_0^\infty \mathrm{d}t \int_C \mathrm{d}\alpha_1 \int_C \mathrm{d}\alpha_2 \int_C \mathrm{d}\alpha_3 \int_C \mathrm{d}\alpha_4 \int_C \mathrm{d}\alpha_5 (-\alpha_1)^{2/\varepsilon}(-\alpha_2)^{1/\varepsilon}(-\alpha_3)^{-2-1/\varepsilon} \cdot (-\alpha_4)^{-2}(-\alpha_5)^{1-\varepsilon} \frac{1}{1 - \dfrac{g^2(p^2)^\varepsilon \pi(D/2)\varepsilon^4 t^{-\varepsilon}}{\alpha_1\alpha_2\alpha_3\alpha_4(-\alpha_5)^\varepsilon}} \exp[-(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+t)]\,. \tag{8}$$

Its is clear that convergence troubles can come only from the t-integration. This integration produces terms nonanalitic in the coupling constant.

Unfortunately the first gamma-functions are divergent when D goes to integer values, so no meaning can be given to our result if the dimension is integer.

This computation suggests that standard renormalizability criteria are true only in perturbation theory, and not relevant in the full theory. A nonrenormalizable theory is simply a theory which does not admit a Taylor expansion in the coupling costant. Of course we have looked only to a particular case, but we feel that similar mechanisms may be operating in realistic situations.

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