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SEMI-GROUP APPROACH TO MULTIPERIPHERAL DYNAMICS

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Abstract: We consider the multiperipheral integral equation at vanishing four-momentum transfer in the form given by Chew and de Tar and we remark that the angular integrations can be interpreted as a convolution of measures defined in a semi-group S contained in the Lorentz group. We study the geometrical properties and a class of Banach-space representations of this semi-group. By projection on these representations, we perform a partial wave analysis of the multiperipheral equation. Under some physically very natural conditions, we prove that the projection integrals converge, the partial wave amplitudes are analytic in a half-plane of the complex angular momentum and the kernel of the partial wave equation represents a bounded operator. We give a preliminary discussion of the inversion problem, i.e., of the construction of the amplitude from its partial wave projections.

1. Introduction

In this paper we introduce some group theoretical concepts which are useful in the mathematical treatment of the multiperipheral models.

Since many years, it has been recognized that the equations of multiperipheral dynamics can be partially diagonalized using their symmetry with respect to the orthochronous (three- or four-dimensional) Lorentz group and that, by this procedure, the complex (three- or four-dimensional) angular momentum could be introduced in a consistent way. The physical content of this procedure is already present in the original papers by ABFST [1–3] and many authors [4–14] have successively developed the formalism putting in evidence its group-theoretical foundations.

In spite of important similar features, one should distinguish between equations of the Bethe-Salpeter type [4, 12] for the whole amplitude and multiperipheral equations for the absorptive part of the amplitude. In the second case, the only one we shall discuss in the present paper, the symmetry with respect to time inversion is lost due to the on-shell conditions, but other interesting features appear. We disregard the symmetry with respect to the space inversion [10] and we deal only with the proper three- or four-dimensional Lorentz groups or, more specifically, with the corresponding spinor groups SU(1,1) or SL(2C). Only the SL(2C) symmetric case will be treated in detail, but most of the concepts we shall develop can be adapted to the SU(1,1) symmetric equations.

It is well known that, if we perform the partial wave analysis of the multiperipheral equations by means of the standard techniques of harmonic analysis, some difficulties arise due to the fact that only functions suitably decreasing at infinity (namely at high energy) can be projected on the matrix elements of the irreducible representations of the symmetry group. For instance, the projection integral on the representations D^l of SU(1,1) converges only if l belongs to a strip of the kind $|\mathrm{Re}\, l + \frac{1}{2}| < c$, where c is connected with the asymptotic behaviour of the function to be projected. This strip disappears (c becoming negative) if the function is not suitably decreasing.

If one is only interested in the complex angular-momentum expansion of a given amplitude, these problems can be overcome formally by means of the methods of distribution theory [15]. Explicit group-theoretical expansions for functions polynomially bounded in the energy have been constructed in refs. [16–23]. These functions are represented as continuous superpositions of matrix elements of (not necessarily unitary) representations of O(2,1) or O(3,1), the weight function being analytic in a half-plane of the complex angular momentum, in analogy with the Froissart-Gribov amplitudes. Some of these expansions can be used for the diagonalization of multiperipheral equations, but the group-theoretical origin of this fact is obscure and, as a consequence, a systematic use of these formalisms in the treatment of multiperipheral models of the most general kind has not yet been clearly formulated.

In dealing with the diagonalization problem, it is convenient to consider two kinds of multiperipheral equations separately.

(a) *Equations of the ABFST type* [1–3]. The production amplitude is dominated by the exchange of spin-zero objects. If, for simplicity, we drop the radial variables, which are not affected by the group-theoretical treatment, we get equations of the kind

$$R(x,x') = K(x,x') + \int K(x,x'')R(x'',x')\,\mathrm{d}x'' , \tag{1.1}$$

where x,x', x'' are points on a one-sheet hyperboloid (in three or four dimensions) and $\mathrm{d}x$ is a Lorentz invariant measure on this hyperboloid. The kernel K and the resolvent R have the support property (x_0 indicates the time component)

$$K(x,x') = R(x,x') = 0 , \quad \text{if} \quad x_0 - x'_0 \leqslant 0 , \tag{1.2}$$

which ensures that the integration in eq. (1.1) is actually performed in a compact region of the hyperboloid.

The partial wave decomposition of this equation is completely understood and all the convergence difficulties can be avoided in a natural way [6, 13, 14]. In fact, in this case, one has to project on the harmonic functions on the hyperboloid, instead of projecting on the matrix elements of the representations of the symmetry group. If the harmonic functions are properly chosen, due to the support conditions (1.2), the projection integrals converge in a half-plane of the complex angular momentum and define there an analytic function even if the amplitudes increase as a power of the energy.

(b) *Multiperipheral equations of a more general kind* [7, 24, 25]. If we drop again the radial variables, they take the convolution form

$$R(a) = K(a) + \int K(aa'^{-1}) R(a')\,\mathrm{d}a' , \tag{1.3}$$

where a, a' are elements of SU(1,1) or of SL(2C) and $\mathrm{d}a$ is the invariant measure on the group. We remark that a convolution integral is defined only if the functions which appear in it have suitable decrease or support properties. Therefore, we expect that also in this case the support properties of the functions K and R will play an important role.

The partial wave analysis of eq. (1.3) has been investigated in refs. [7–11], using the matrix elements of the irreducible representations of SL(2C) [respectively SU(1,1)] with respect to a particular pseudobasis related to a non-compact subgroup. It turns out that the projected equation splits into two independent equations and one of them contains only quantities analytic in a half-plane of the complex angular momentum and is meaningful also when the amplitudes are polynomially increasing.

In the present paper we clarify the group-theoretical origin of this situation, exploiting the properties of a semi-group, naturally suggested by the Bali-Chew-Pignotti (BCP) group theoretical variables [26], which contains the supports of the functions $K(a)$ and $R(a)$ appearing in eq. (1.3). In this way, we obtain a clear, natural, rigorous and basis-independent treatment of the partial wave projection of eq. (1.3). In particular, using suitable normed spaces, we can give a proof of all the desirable convergence properties, a proof which was very difficult in the previously used basis-dependent formalism.

The relevance of the concept of semi-group in this kind of problems is clarified by the analogy with the classical Laplace transformation which, as well known, transforms a convolution product into an ordinary product *. In fact, a Laplace transform is analytic in a half-plane when the original function, besides being bounded by an

* Similar considerations are contained in ref. [22].

exponential, has its support in the real positive half-line, which is indeed a semi-group contained in the additive group of the real numbers. We remark that the product of Laplace transforms of this kind is always defined and analytic in a half-plane, in accord with the fact that the convolution of functions (or distributions) having their supports in the positive half-line always exists and is associative. In general, the Laplace transform of a function on the real line, is defined (if it exists) in a strip in the complex plane. The product of two Laplace transforms does not necessarily exist, as the definition strips may be non-overlapping, in accord with the fact that the convolution of two functions on the real line does not necessarily exist.

In order to introduce our notation, we recall briefly the definition of the BCP variables [26, 7, 25]. We consider a process with two incoming and n+2 outgoing particles and we label the external four-momenta and the four-momentum transfers as in fig. 1. We assume for simplicity that all the masses are equal to m. Then we have

$$t_i = Q_i^2 < 0\,, \quad i = 0,1,\ldots,n\,. \tag{1.4}$$

We introduce the elements $a_0, a_1, \ldots, a_{n+1}$ of SL(2C) with the properties

$$\begin{aligned} P_B &= L(a_{n+1})(m,0,0,0)\,, \\ Q_n &= L(a_{n+1}a_z(\chi_{n+1}))(0,0,0,\sqrt{-t_n})\,, \end{aligned} \tag{1.5}$$

$$\begin{aligned} Q_i &= L(a_i)(0,0,0,\sqrt{-t_i})\,, \\ Q_{i-1} &= L(a_i a_z(\chi_i))(0,0,0,\sqrt{-t_{i-1}})\,, \end{aligned} \qquad i = 1,2,\ldots,n\,, \tag{1.6}$$

$$\begin{aligned} Q_0 &= L(a_0)(0,0,0,\sqrt{-t_0})\,, \\ P_A &= L(a_0 a_z(\chi_0))(m,0,0,0)\,, \end{aligned} \tag{1.7}$$

where $L(a)$ is the 4 X 4 Lorentz matrix corresponding to the element a of SL(2C) and $a_z(\chi)$ corresponds to a boost along the z-axis with rapidity χ. It follows from the on-shell conditions that one must take

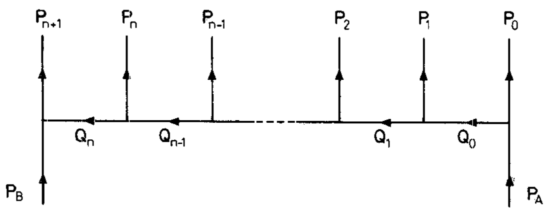

Fig. 1. Four-momenta appearing in a production process.

$$\sinh \chi_{n+1} = (2m)^{-1}\sqrt{-t_n}\,,$$
$$\sinh \chi_o = (2m)^{-1}\sqrt{-t_o}\,,$$
$$\chi_i = \chi(t_i, t_{i-1})\,, \qquad i = 1,2,\ldots,n\,, \tag{1.8}$$

where

$$\cosh \chi(t,t') = (m^2-t-t')(4tt')^{-\frac{1}{2}}\,,$$
$$\chi(t,t') > 0\,. \tag{1.9}$$

From eqs. (1.5)–(1.7), we see that the elements

$$g_i = a_z(-\chi_{i+1})a_{i+1}^{-1}a_i\,, \quad i = 0,1,\ldots,n\,, \tag{1.10}$$

belong to SU(1,1). The BCP variables $t_o, g_o, \ldots, t_n, g_n$ provide a complete description of the kinematical configuration. In particular, the square of the c.m. energy is given by

$$s = 2m^2(1 + L_{oo}(a))\,, \tag{1.11}$$

where

$$a = a_{n+1}^{-1}a_o a_z(\chi_o) = a_z(\chi_{n+1})g_n a_z(\chi_n)\ldots g_o a_z(\chi_o)\,. \tag{1.12}$$

The elements which appear in the right-hand side of eq. (1.12) either belong to SU(1,1) or are of the form $a_z(\chi)$ with $\chi > 0$. In sect. 2, we show that the elements of SL(2C) of these two kinds generate a semi-group. We indicate by S the closure of this semi-group, which is also a semi-group contained in SL(2C), but not coinciding with it. The geometric properties of S are discussed in sect. 2.

The multiperipheral models we are considering [25] are based on the assumption that the squares of the moduli of the production amplitudes are given by

$$|M_n|^2 = \sum_\pi F_n(P_A, P_B, P_{\pi(n+1)}, \ldots, P_{\pi(0)})\,, \tag{1.13}$$

where the sum is extended to all the $(n+2)!$ permutations π of the final particles and F_n has the factorized form

$$F_n = \hat{B}(t_n, g_n)\hat{K}(t_n, g_{n-1}, t_{n-1})\ldots$$
$$\times\ldots\hat{K}(t_1, g_o, t_o)\hat{A}(t_o)\,. \tag{1.14}$$

The quantities $\hat{B}$, $\hat{K}$ and $\hat{A}$ could be (possibly infinite dimensional) matrices and do not depend on n. The validity of the assumption (1.13), (1.14) is discussed in detail in ref. [25].

Under these assumptions, by means of a technique developed in ref. [7] (see also ref. [25]), one can express the total cross section and the r-particle inclusive distributions in terms of the function

$$R(t,a,t') = K(t,a,t') + \int K(t,aa'^{-1},t'')\, K(t'',a',t')\,\mathrm{d}^6a'\,\mathrm{d}t'' + \dots\ , \tag{1.15}$$

where the kernel K is given by

$$K(t,a,t') = |16\pi^2 t'|^{-1}[T(m^2,t,t')]^{\frac{1}{2}}$$
$$\times\ \hat{K}(t,a_z(-\chi(t,t'))a,t')\,\delta^3_-(a_z(-\chi(t,t'))a)\ , \tag{1.16}$$

$$T(a,b,c) = a^2+b^2+c^2-2ab-2bc-2ac\ . \tag{1.17}$$

We have indicated by $\delta^3_-(a)$ a measure on SL(2C) concentrated on the subgroup SU(1,1) and with the property

$$\int_{\mathrm{SL(2C)}} f(a)\,\delta^3_-(a)\,\mathrm{d}^6a = \int_{\mathrm{SU(1,1)}} f(g)\,\mathrm{d}^3g\ , \tag{1.18}$$

where d^3g is the invariant measure on SU(1,1).

If we fix t and t', we see that K is a measure on SL(2C) with its support in S. From eq. (1.15), it follows that R has the same property and satisfies an integral equation of the kind (1.3), where the variables t, t', t'' and the corresponding integrations are understood.

If $B^\Lambda(a)$ is a linear representation of S, namely an operator valued function defined on S with the property

$$B^\Lambda(ab) = B^\Lambda(a)B^\Lambda(b)\ , \quad a,b \in \mathrm{S}\ , \tag{1.19}$$

and if the projection integrals

$$B^\Lambda(K) = \int_{\mathrm{S}} B^\Lambda(a)K(a)\,\mathrm{d}^6a\ , \tag{1.20}$$

$$B^\Lambda(R) = \int_{\mathrm{S}} B^\Lambda(a)R(a)\,\mathrm{d}^6a\ , \tag{1.21}$$

exist, from eqs. (1.3) and (1.15) we get (the variables t, t', ... and the corresponding integrations are always understood)

$$B^\Lambda(R) = B^\Lambda(K) + B^\Lambda(K)B^\Lambda(K) + \dots\ , \tag{1.22}$$

$$B^\Lambda(R) = B^\Lambda(K) + B^\Lambda(K)B^\Lambda(R)\ , \tag{1.23}$$

which are the corresponding partial wave equations.

In sect. 3, we show that the convolutions which appear in eqs. (1.3) and (1.15) are always well defined, due to the support properties of $K(a)$ and $R(a)$. In the same section, we discuss also some conditions for the existence of the integrals (1.20) and (1.21) and for the possibility of writing the projected equations (1.22) and (1.23).

It is clear that a representation of SL(2C) is also a representation of S, but, as we shall see, not all the representations of S can be extended to representations of the whole group SL(2C). Therefore, even if the function $\mathcal{R}(a)$ is such that all the projection integrals on the representations of SL(2C) diverge, one can still hope to find representations $\mathcal{B}^{\Lambda}$ of S such that the projection integral (1.21) is well defined. The problem is to find a sufficiently large class of these representations, in such a way that the quantities $\mathcal{B}^{\Lambda}(\mathcal{R})$ permit one to reconstruct the original function $\mathcal{R}(a)$ univocally.

In sect. 4 we construct a class of representations $\mathcal{B}^{M\lambda}$, of S, labelled by the integral or half-integral number M and by the arbitrary complex number λ. These representations of S are contained in the well-known representations $\mathcal{D}^{M\lambda}$ of SL(2C) which, when restricted to S, become reducible. We remark that the representations $\mathcal{B}^{M\lambda}$ and $\mathcal{B}^{-M,-\lambda}$ are not equivalent, so that the projected amplitude has not the symmetry under the reflection $(M,\lambda) \rightarrow (-M,-\lambda)$, which is present in the usual SL(2C) projection [16, 27] and prevents the projected amplitude from being analytic in a half-plane of the variable λ.

The properties of the representations $\mathcal{B}^{M\lambda}$ are further analyzed in sect. 5, where we introduce a norm in the representation space and we compute the corresponding norms of the representation operators. In sect. 6 we discuss the convergence properties of the projection integrals (1.20) and (1.21) and we show that, under certain conditions which are satisfied in the physically relevant models, the projection integrals define operator valued functions which are analytic in a half-plane of the complex variable λ, in analogy with the classical one-sided Laplace transform. Also the convergence of the integrations over the radial variables, which are understood in the equations written above, is investigated in detail.

In sect. 7, we formulate the problem of finding an inverse formula which gives the resolvent $\mathcal{R}$ in terms of its projections $\mathcal{B}^{M\lambda}(\mathcal{R})$ We discuss the difficulties of this problem which will be reconsidered in a forthcoming paper.

2. A semi-group contained in SL(2C)

We shall use for the elements of SL(2C) the following notation

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} , \tag{2.1}$$

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{2.2}$$

$$u_z(\mu) = \exp\left(-\tfrac{1}{2} i\mu\sigma_z\right) , \tag{2.3}$$

$$a_z(\zeta) = \exp\left(\tfrac{1}{2}\zeta\sigma_z\right) , \tag{2.4}$$

and similar definitions for $u_x(\mu)$, $u_y(\mu)$, $a_x(\zeta)$ and $a_y(\zeta)$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

The 4 × 4 matrix $L(a)$, which belongs to SO(3,1) and corresponds to the element a of SL(2C), is defined in such a way that if

$$V = (V_o, V_1, V_2, V_3) \tag{2.6}$$

is a four-vector, the relation

$$V' = L(a)\, V\,, \tag{2.7}$$

is equivalent to

$$V'_o e + V'_1 \sigma_x + V'_2 \sigma_y + V'_3 \sigma_z = a(V_o e + V_1 \sigma_x + V_2 \sigma_y + V_3 \sigma_z) a^+ \,. \tag{2.8}$$

We remember that the matrix $L(a)$ has the properties

$$\sum_{\beta=0}^{3} L_{\alpha\beta}(a) L_{\alpha'\beta}(a) g_{\beta\beta} = \sum_{\beta=0}^{3} L_{\beta\alpha}(a) L_{\beta\alpha'}(a) g_{\beta\beta} = g_{\alpha\alpha'}\,, \tag{2.9}$$

$$L_{oo}(a) \geqslant 1\,. \tag{2.10}$$

where $g_{\alpha\alpha'}$ is the metric tensor.

Now we define the semi-group S which will form the object of our investigations.

Proposition 1: The subset S of SL(2C) defined by the inequalities

$$L_{33}(a) \geqslant 1\,, \quad L_{30}(a) \geqslant 0\,, \quad L_{03}(a) \geqslant 0\,, \tag{2.11}$$

is a semi-group, i.e., it is closed under multiplication. The interior S^o of S is also a semi-group. It is composed of the points of S such that

$$L_{33}(a) > 1\,. \tag{2.12}$$

The product of an element of S and an element of S^o belongs to S^o. The closure of S^o is S.

Proof: First we have to show that, if a and b belongs to S, the matrix $L(ab)$ satisfies the conditions (2.11). In fact, using the properties (2.9) and (2.10) and the Schwarz inequality we obtain

$$\begin{aligned} L_{33}(ab) &= L_{30}(a)L_{03}(b) + L_{31}(a)L_{13}(b) + L_{32}(a)L_{23}(b) \\ &\quad + L_{33}(a)L_{33}(b) \geqslant L_{30}(a)L_{03}(b) + L_{33}(a)L_{33}(b) \\ &\quad - [1 + (L_{30}(a))^2 - (L_{33}(a))^2]^{\frac{1}{2}}\, [1 + (L_{03}(b))^2 - (L_{33}(b))^2]^{\frac{1}{2}} \\ &\geqslant L_{33}(a)L_{33}(b) \geqslant 1\,, \end{aligned} \tag{2.13}$$

$$L_{30}(ab) = L_{30}(a)L_{00}(b) + L_{31}(a)L_{10}(b) + L_{32}(a)L_{20}(b)$$
$$+ L_{33}(a)L_{30}(b) \geqslant L_{30}(a)L_{00}(b) + L_{33}(a)L_{30}(b)$$
$$- [1 + (L_{30}(a))^2 - (L_{33}(a))^2]^{\frac{1}{2}} [(L_{00}(b))^2 - (L_{30}(b))^2 - 1]^{\frac{1}{2}}$$
$$\geqslant L_{33}(a)L_{30}(b) \geqslant 0 . \tag{2.14}$$

In a similar way we prove that

$$L_{03}(ab) \geqslant L_{03}(a)L_{33}(b) \geqslant 0 . \tag{2.15}$$

Therefore ab belongs to S.

From eq. (2.9), we have

$$(L_{30}(a))^2 \geqslant (L_{33}(a))^2 - 1 , \quad (L_{03}(a))^2 \geqslant (L_{33}(a))^2 - 1 , \tag{2.16}$$

and therefore we see that if eq. (2.12) holds, a is an interior point of S. Conversely, if $L_{33}(a) = 1$, it is easy to show that for arbitrarily small negative ζ the element $aa_z(\zeta)$ does not belong to S and therefore a cannot be an interior point of S. If a belongs to S and b belongs to S^o, it follows from eq. (2.13) that ab belongs to S^o. In particular, we see that S^o is a semi-group. The last statement of the Proposition follows from the remark that if a belongs to S, $aa_z(\zeta)$ belongs to S^o for positive ζ.

It is useful to define S directly in terms of the matrix elements of a. The conditions are

$$\det a = a_{11}a_{22} - a_{12}a_{21} = 1 ,$$
$$2L_{33}(a) = |a_{11}|^2 + |a_{22}|^2 - |a_{12}|^2 - |a_{21}|^2 \geqslant 2 ,$$
$$2L_{03}(a) = |a_{11}|^2 - |a_{22}|^2 - |a_{12}|^2 + |a_{21}|^2 \geqslant 0 ,$$
$$2L_{30}(a) = |a_{11}|^2 - |a_{22}|^2 + |a_{12}|^2 - |a_{21}|^2 \geqslant 0 . \tag{2.17}$$

It is convenient to collect for later use some inequalities which have been proved above or can be proved in a similar way.

Proposition 2: If $a, b \in SL(2C)$, we have

$$L_{00}(a) \geqslant |L_{\alpha\beta}(a)| , \quad \alpha,\beta = 0,1,2,3 , \tag{2.18}$$

$$L_{00}(ab) \geqslant L_{03}(a)L_{30}(b) . \tag{2.19}$$

If moreover $a, b \subset S$, we have also

$$L_{33}(ab) \geqslant L_{33}(a)L_{33}(b) , \tag{2.20}$$

$$L_{30}(ab) \geqslant L_{33}(a)L_{30}(b) , \tag{2.21}$$

$$L_{03}(ab) \geqslant L_{03}(a) L_{33}(b) . \tag{2.22}$$

Proposition 3: The semi-group S^{o} coincides with the set of all the elements of SL(2C) which can be written in the form

$$a = g a_z(\zeta) g' , \quad g, g' \in SU(1,1) , \quad \zeta > 0 . \tag{2.23}$$

Proof: We remember that SU(1,1) is just the subgroup of SL(2C) which contains all the elements g such that $L(g)$ does not act on the z-coordinate. It follows that

$$SU(1,1) \subset S . \tag{2.24}$$

It is easy to verify directly that

$$a_z(\zeta) \in S^{o} , \quad \text{if} \quad \zeta > 0 . \tag{2.25}$$

It follows that the elements of the form (2.23) belong to S^{o}.

Now we consider an arbitrary element a of S^{o}. From the condition (2.12) we see that the Minkowski three-vectors

$$\begin{aligned} &(L_{03}(a), L_{13}(a), L_{23}(a)) , \\ &(L_{30}(a), L_{31}(a), L_{32}(a)) , \end{aligned} \tag{2.26}$$

are timelike and therefore we can find two elements h and h' of SU(1,1) such that

$$\begin{aligned} &\sum_{\beta=0}^{2} L_{\alpha\beta}(h) L_{\beta 3}(a) = 0 , \quad \text{for} \quad \alpha = 1,2 , \\ &\sum_{\beta=0}^{2} L_{\beta\alpha}(h') L_{3\beta}(a) = 0 , \quad \text{for} \quad \alpha = 1,2 . \end{aligned} \tag{2.27}$$

It follows that

$$L(hah') = \begin{pmatrix} \Lambda_{00} & \Lambda_{01} & \Lambda_{02} & \sinh\zeta \\ \Lambda_{10} & \Lambda_{11} & \Lambda_{12} & 0 \\ \Lambda_{20} & \Lambda_{21} & \Lambda_{22} & 0 \\ \sinh\zeta & 0 & 0 & \cosh\zeta \end{pmatrix} , \quad \zeta > 0 . \tag{2.28}$$

From the conditions (2.9) and (2.10) we get

$$\begin{aligned} &\Lambda_{01} = \Lambda_{02} = \Lambda_{10} = \Lambda_{20} = 0 , \\ &\Lambda_{00} = \cosh\zeta . \end{aligned} \tag{2.29}$$

Therefore we have

$$hah' = a_z(\zeta)u_z(\varphi)\,, \quad \zeta > 0\,, \tag{2.30}$$

and it follows immediately that a has the form (2.23).

It is also easy to show that the semi-group S^{-1} composed of the inverses of the elements of S is defined by the inequalities

$$L_{33}(a) \geqslant 1\,, \quad L_{30}(a) \leqslant 0\,, \quad L_{03}(a) \leqslant 0\,. \tag{2.31}$$

The semi-group S^{o-1} is defined by the extra condition

$$L_{33}(a) > 1\,, \tag{2.32}$$

and is composed of the elements of the form

$$ga_z(\zeta)g'\,, \quad g,g' \in SU(1,1)\,, \quad \zeta < 0\,. \tag{2.33}$$

By reduction to the canonical form (2.28), we see that if $L_{33}(a) > 1$, a belongs to S^o or to S^{o-1}. It follows immediately that if $L_{33}(a) < -1$, a belongs to one of the two sets (they are not semi-groups)

$$\begin{aligned} &u_y(\pi)S^o = S^{o-1}u_y(\pi)\,,\\ &u_y(\pi)S^{o-1} = S^o u_y(\pi)\,. \end{aligned} \tag{2.34}$$

If

$$|L_{33}(a)| < 1\,, \tag{2.35}$$

following a procedure analogous to the one used to get eq. (2.28), we can write

$$L(hah') = \begin{pmatrix} \Lambda_{00} & 0 & \Lambda_{02} & 0 \\ 0 & \cos\varphi & 0 & \sin\varphi \\ \Lambda_{20} & 0 & \Lambda_{22} & 0 \\ 0 & -\sin\varphi & 0 & \cos\varphi \end{pmatrix}, \quad \begin{matrix} h,h' \in SU(1,1), \\ \\ 0 < \varphi < \pi\,, \end{matrix} \tag{2.36}$$

and therefore

$$a = gu_y(\varphi)g'\,, \quad g,g' \in SU(1,1)\,, \quad 0 < \varphi < \pi\,. \tag{2.37}$$

We do not consider in detail the case $|L_{33}(a)| = 1$.

The following result is interesting, though we shall not use it in the following.

Proposition 4: A semi-group S′ contained in SL(2C) and containing S coincides with SL(2C) or with S. In other words, S is a maximal semi-group contained in SL(2C).

Proof: Assume that there is an element $b \in S'$ but not belonging to S. Then the open set

$$bS^o \cap [SL(2C) - S] \subset S' \tag{2.38}$$

is not empty, as it contains $ba_z(\zeta)$ for small positive ζ. Then we can choose in the set (2.38) an element a such that $|L_{33}(a)| \neq 1$. We consider three cases:

(i) if $a \in S^{-1}$, then $aa^{-1} = e$ is an interior point of S′; as the group SL(2C) is connected, S′ must coincide with it [28];

(ii) if a belongs to one of the sets (2.34), we see that S′ must contain $u_y(\pi)$ and therefore also $S^{-1} = u_y(\pi) S u_y(\pi)$; therefore we are again in the case (i);

(iii) if a satisfies the condition (2.35), from eq. (2.37) we see that $u_y(\varphi)$ is an interior point of S′.

It follows that $u_y(\varphi) \in S'$ for any φ and in particular for $\varphi = \pi$. Then we can reason as in the case (ii). We have shown that in all the possible cases we get $S' = SL(2C)$ and the proposition is proved.

3. The convolution of measures on S

In order to agree with the notation usual in physics, we write a measure on SL(2C) in the form $F(a)\,d^6a$, where d^6a is the invariant measure suitably normalized and $F(a)$ is a generalized function. The convolution $F_1 * F_2 * \ldots * F_n$ of several measures is defined by the formula [29]

$$\int [F_1 * F_2 * \ldots * F_n](a) g(a)\, d^6a = \int F_1(a_1) F_2(a_2) \ldots F_n(a_n) g(a_1 a_2 \ldots a_n)\, d^6a_1\, d^6a_2 \ldots d^6a_n\,, \tag{3.1}$$

where $g(a)$ is a continuous function with compact support.

The convolution exists and is associative only if certain conditions are satisfied. If we indicate by $\operatorname{supp} F_i$ the support of the measure F_i, by φ the function

$$\varphi\colon [SL(2C)]^n \to SL(2C)\,,$$
$$\varphi(a_1, a_2, \ldots, a_n) = a_1 a_2 \ldots a_n\,, \tag{3.2}$$

and by $\varphi^{-1}(K)$ the inverse image of the set $K \subset SL(2C)$, a sufficient condition is that the set

$$\varphi^{-1}(K) \cap [\prod_{i=1}^{n} \operatorname{supp} F_i] \subset [SL(2,C)]^n \tag{3.3}$$

is compact whenever K is compact [29].

A special case of this condition, which is useful for our problem, is the following.

Proposition 5: If the measures F_i on SL(2C) have the property

$$\operatorname{supp} F_i \subset c_i S\,, \quad c_i \in S^{\circ}\,, \tag{3.4}$$

their convolution exists and is associative.

Proof: We use the condition stated above, remembering that a set in SL(2C) is compact if and only if it is closed and bounded, namely contained in a set defined by $L_{00}(a) \leqslant C$. Using proposition 3, we put

$$c_i = g_i a_z(\xi_i) h_i\,, \quad g_i, h_i \in \mathrm{SU}(1,1)\,, \quad \xi_i > 0\,. \tag{3.5}$$

If a_i belongs to supp F_i, we can write

$$a_i = g_i a_z(\xi_i) b_i g_{i+1}^{-1}\,,$$
$$b_i \in \mathrm{S}\,, \quad i = 1,2,\dots,n\,, \quad g_{n+1} = e\,. \tag{3.6}$$

It follows that

$$a_1 a_2 \dots a_n = g_1 a_z(\xi_1) b_1 a_z(\xi_2) b_2 \dots a_z(\xi_n) b_n\,. \tag{3.7}$$

If the product (3.7) belongs to the compact set K, we have

$$L_{00}(a_1 a_2 \dots a_n) \leqslant C \tag{3.8}$$

and using the inequalities (2.11), (2.18), (2.20) and (2.21), we obtain

$$L_{33}(a_z(\xi_i) b_i a_z(\xi_{i+1})) \leqslant C\,, \quad i = 1,2,\dots,n-1\,,$$
$$L_{30}(a_z(\xi_n) b_n) \leqslant C\,. \tag{3.9}$$

Using the representation property and the inequality (2.11), we get

$$L_{00}(b_i) \leqslant C\,(\sinh \xi_i \sinh \xi_{i+1})^{-1}\,, \quad i = 1,2,\dots,n-1\,,$$
$$L_{00}(b_n) \leqslant C\,(\sinh \xi_n)^{-1}\,. \tag{3.10}$$

We see that the families of group elements $\{a_i\}$ which satisfy eqs. (3.6) and (3.10) form a compact set in $(\mathrm{SL}(2C))^n$. Therefore the closed set (3.3) is also compact, as we wanted to show.

Remark that the proposition we have just proved is sufficient to ensure the existence of the convolutions which appear in eqs. (1.3) and (1.15), without any assumption on the rate of growth of the measures which appear in these equations.

There is another sufficient condition for the existence of a convolution on S, which is useful for our purpose. We consider a strictly positive, finite, lower semicontinuous function $\rho(a)$ defined on S with the property

$$\rho(ab) \leqslant \rho(a)\,\rho(b)\,. \tag{3.11}$$

Then the measures on S such that the norm

$$\|F\|_\rho = \int_{\mathrm{S}} \rho(a)\,|F(a)|\,\mathrm{d}^6 a\,, \tag{3.12}$$

is finite form a Banach space M_ρ. Remark that, as S is closed, these measures can also be considered as measures on SL(2C) with support in S. We have

Proposition 6: The convolution of measures belonging to M_ρ exists and is associative. Moreover, we have the inequality

$$\|F_1 * F_2\|_\rho \leqslant \|F_1\|_\rho \|F_2\|_\rho \,. \tag{3.13}$$

This means that M_ρ is a complete normed algebra (Banach algebra) with respect to convolution.

The proof of these results is simple and can be found in ref. [29], remarking that the fact that S is a semi-group (but not a group) does not introduce new difficulties.

If $\mathcal{B}(a)$ is a continuous representation of the semi-group S in a Banach space, and we put

$$\rho(a) = \|\mathcal{B}(a)\| \,, \tag{3.14}$$

this function has the properties required above. The following results can also be found in ref. [29].

Proposition 7: If the measures F_i belong to M_ρ, the integrals

$$\mathcal{B}(F_i) = \int_S \mathcal{B}(a) F_i(a) \mathrm{d}^6 a \,, \tag{3.15}$$

exist as bounded operators and we have

$$\|\mathcal{B}(F_i)\| \leqslant \|F_i\|_\rho \,, \tag{3.16}$$

$$\mathcal{B}(F_1 * F_2) = \mathcal{B}(F_1)\mathcal{B}(F_2) \,. \tag{3.17}$$

This proposition is just what we need in order to get the partial wave equations (1.22) and (1.23).

For the applications to multiperipheral equations of the most general kind [25], one should consider also the case in which the quantities $F_i(a)\,\mathrm{d}^6 a$ are operator valued measures [30] in a Banach space $\mathcal{N}$. In this case the quantity $|F_i(a)|\mathrm{d}^6 a$ stands for the norm of the operator valued measure [30] and the quantities $\mathcal{B}(F_i)$ are bounded operators in a completed tensor product [31, 32] of the representation space and of the Banach space $\mathcal{N}$.

This generalization of the propositions given above seems to be possible, but a detailed mathematical analysis should still be performed.

4. Some linear representations of S

In order to find linear representations of S, we consider the restrictions to S of the well-known irreducible representations of SL(2C). For the theory of these representations, see refs. [16, 33, 34]. We shall see that the representations of S found in this way are in general reducible and we shall study the closed invariant subspaces of the representation spaces and the restrictions of the representations to these subspaces.

Following a procedure coherently developed in ref. [33], we start by considering representations acting in spaces of infinitely differentiable functions. Hilbert or

Banach space representations can be built successively, completing these spaces with respect to suitable norms and extending the operators by continuity.

The representations of SL(2C) we shall consider are labelled by the complex parameter λ and by the integral or half-integral parameter M. Alternatively, one can use the parameters [16, 33]

$$n_1 = \lambda - M\,,\qquad n_2 = \lambda + M\,. \tag{4.1}$$

Following ref. [33] with slight changes of notation, we consider the spaces $\mathcal{F}^{M\lambda}$ of the infinitely differentiable functions $f(z)$ of the complex variable z such that also the function

$$\hat{f}(z^{-1}) = z^{1-n_1}\,\bar{z}^{\,1-n_2} f(z) \tag{4.2}$$

is infinitely differentiable with respect to z^{-1}. We say that a sequence of functions f_n converges to zero in $\mathcal{F}^{M\lambda}$ if $f_n(z), \hat{f}_n(z)$ and all their derivatives with respect to z and $\bar{z}$ converge to zero uniformly in any compact set of the complex plane. With this definition, $\mathcal{F}^{M\lambda}$ is a Frechet space.

From the Taylor expansion of $\hat{f}(z)$ near the origin and from eq. (4.2), we get the asymptotic expansion

$$f(z) \underset{z\to\infty}{\sim} z^{n_1-1}\bar{z}^{\,n_2-1} \sum_{j,k=0}^{\infty} d_{jk} z^{-j}\bar{z}^{\,-k}\,. \tag{4.3}$$

The representation operators $\mathcal{D}^{M\lambda}(a)$ are defined by

$$[\mathcal{D}^{M\lambda}(a)f](z) = (a_{12}z + a_{22})^{n_1-1}(\bar{a}_{12}\bar{z} + \bar{a}_{22})^{n_2-1} f(z_a)\,, \tag{4.4}$$

where

$$z_a = \frac{a_{11}z + a_{21}}{a_{12}z + a_{22}}\,. \tag{4.5}$$

Now we restrict the representations $\mathcal{D}^{M\lambda}$ to the semi-group S. When necessary to avoid confusion, this restriction will be indicated by $\mathcal{D}^{M\lambda}_{\mathrm{S}}$.

An invariant subspace can be found at once.

Proposition 8: The closed subspace $\mathcal{G}$ of $\mathcal{F}^{M\lambda}$ composed of the infinitely differentiable functions such that

$$f(z) = 0\,,\quad \text{if}\quad |z| \geqslant 1\,, \tag{4.6}$$

is invariant with respect to the representations $\mathcal{D}^{M\lambda}_{\mathrm{S}}$.

Proof: We have just to show that if $a \in \mathrm{S}$ and $|z| \geqslant 1$, from eq. (4.5) it follows $|z_a| \geqslant 1$. First we consider the special cases

$$a = u_z(\mu)\,, \quad z_a = z \exp(-i\mu)\,, \tag{4.7}$$

$$a = a_x(\xi)\,, \quad z_a = \frac{z \cosh \frac{1}{2}\xi + \sinh \frac{1}{2}\xi}{z \sinh \frac{1}{2}\xi + \cosh \frac{1}{2}\xi}\,, \tag{4.8}$$

$$a = a_z(\zeta)\,, \quad \zeta \geqslant 0\,, \quad z_a = z \exp \zeta\,. \tag{4.9}$$

We see at once that the required property holds in these three cases. If $a \in S^o$, it can be expressed as a product of elements of the kind considered above (proposition 3) and the required property still holds. For $a \in S$, the proof can be completed by a continuity argument.

We indicate by $\mathcal{B}^{M\lambda}$ the restriction of the representation $\mathcal{D}_S^{M\lambda}$ to the invariant subspace $\mathcal{G}$. It is a linear continuous representation of S which cannot in general be extended to a linear representation of SL(2C).

Now we limit ourselves to the case in which $\lambda - M$ is not integral and we consider the operator $A^{M\lambda}: \mathcal{F}^{M\lambda} \to \mathcal{F}^{-M,-\lambda}$ defined in ref. [33] by the formula

$$[A^{M\lambda}f](z) = \int (z-z_1)^{-n_1-1}(\bar{z}-\bar{z}_1)^{-n_2-1} f(z_1)\mathrm{d}^2 z_1\,, \tag{4.10}$$

where

$$\mathrm{d}^2 z_1 = \mathrm{d}\,\mathrm{Re}\, z_1\, \mathrm{d}\,\mathrm{Im}\, z_1\,. \tag{4.11}$$

If $\mathrm{Re}\,(n_1 + n_2) \geqslant 0$, the integral (4.10) diverges at $z = z_1$ and one has to take its finite part.

This operator has the intertwining property [33]

$$A^{M\lambda}\mathcal{D}^{M\lambda} = \mathcal{D}^{-M,-\lambda}A^{M\lambda}\,. \tag{4.12}$$

As shown in ref. [33], for not integral $\lambda - M$, the operator $A^{-M,-\lambda}$ is proportional to the inverse of $A^{M\lambda}$. Therefore $A^{M\lambda}$ defines an isomorphism of the spaces $\mathcal{F}^{M\lambda}$ and $\mathcal{F}^{-M,-\lambda}$ which establishes the equivalence of the representations $\mathcal{D}^{M\lambda}$ and $\mathcal{D}^{-M,-\lambda}$. Therefore we have

Proposition 9: The subspace $\mathcal{G}_1^{M\lambda} \subset \mathcal{F}^{M\lambda}$, defined as the image under $A^{-M,-\lambda}$ of $\mathcal{G} \subset \mathcal{F}^{-M,-\lambda}$ is closed and invariant under the representation $\mathcal{D}_S^{M\lambda}$ of S. The restriction of $\mathcal{D}_S^{M\lambda}$ to $\mathcal{G}_1^{M\lambda}$ is equivalent to the restriction of $\mathcal{D}_S^{-M,-\lambda}$ to $\mathcal{G}$, namely to $\mathcal{B}^{-M,-\lambda}$.

For the next developments, we need the following results.

Lemma: If $f \in \mathcal{G}$, the function $A^{M\lambda}f \in \mathcal{G}_1^{-M,-\lambda}$ is given for $|z| > 1$ by the convergent series

$$[A^{M\lambda}f]\,(z) = z^{-n_1-1}\bar{z}^{\,-n_2-1} \sum_{j,k=0}^{\infty} d_{jk} z^{-j}\bar{z}^{\,-k}\,, \tag{4.13}$$

where

$$d_{jk} = \frac{\Gamma(-n_1)\Gamma(-n_2)}{\Gamma(-n_1-j)j!\,\Gamma(-n_2-k)k!}\int (-z)^j(-\bar{z})^k f(z)\,\mathrm{d}^2 z\,. \tag{4.14}$$

Proof: Since

$$[A^{M\lambda}f](z) = z^{-n_1-1}\bar{z}^{-n_2-1}$$

$$\times \int_{|z_1|\leqslant 1} \left(1-\frac{z_1}{z}\right)^{-n_1-1}\left(1-\frac{\bar{z}_1}{\bar{z}}\right)^{-n_2-1} f(z_1)\,\mathrm{d}^2 z_1\,, \tag{4.15}$$

we have just to develop the binomials in the integral into power series and to integrate term by term.

The connection between the invariant subspaces G and $G_1^{M\lambda}$ is clarified by:

Proposition 10: If $\lambda - M$ is not integral, the subspaces G and $G_1^{M\lambda}$ have no common element except 0, namely

$$G \cap G_1^{M\lambda} = \{0\}\ . \tag{4.16}$$

The subspace

$$G + G_1^{M\lambda}\,, \tag{4.17}$$

is dense in $F^{M\lambda}$, but does not coincide with it.

Proof: If $f \in G \cap G_1^{M\lambda}$, then eqs. (4.13) and (4.14) hold. Moreover, $A^{M\lambda}f \in G$ and therefore the coefficients d_{jk} must be all equal to zero. As the factor in front of the integral (4.14) is not vanishing, we see that all the moments of f are zero. Therefore, we have $f = 0$.

The space $G + G_1^{M\lambda}$ contains all the functions of the form

$$v = g + A^{-M,-\lambda}f\,, \quad g,f \in G\,. \tag{4.18}$$

It follows that for $|z| > 1$, $v(z)$ is given by the convergent expansion (4.13) with the signs of n_1 and n_2 changed. As not all the elements of $F^{M\lambda}$ have this property, the subspace $G + G_1^{M\lambda}$ cannot coincide with the whole space $F^{M\lambda}$.

Finally, we have to prove that the closure of the subspace $G + G_1^{M\lambda}$ is $F^{M\lambda}$. First we show that this closure contains all the functions of $F^{M\lambda}$ which have the property

$$u(z) = z^{n_1-j-1}\,\bar{z}^{n_2-k-1}\,, \quad \text{for} \quad |z| \geqslant \tfrac{1}{2}\,, \tag{4.19}$$

where j and k are arbitrary non-negative integers. We consider an infinitely differentiable function $\chi(z)$ with the properties

$$\chi(z) = 0\,, \quad \text{for} \quad |z| \geqslant 1\,,$$

$$\chi(z) = 1\,, \quad \text{for} \quad |z| \leqslant \tfrac{1}{2}\,, \tag{4.20}$$

and the sequence of elements of $G + G_1^{M\lambda}$

$$v_\nu(z) = \chi(z)(u(z) - [A^{-M,-\lambda} f_\nu](z)) + [A^{-M,-\lambda} f_\nu](z)\,, \tag{4.21}$$

where

$$f_\nu(z) = \nu^2 \frac{\Gamma(n_1 - j)\Gamma(n_2 - k)}{\Gamma(n_1)\Gamma(n_2)} \frac{\partial^j}{\partial z^j} \frac{\partial^k}{\partial \bar{z}^k} \varphi(\nu z)\,. \tag{4.22}$$

The function $\varphi \in G$ has the property

$$\int \varphi(z)\,\mathrm{d}^2 z = 1\,. \tag{4.23}$$

From eq. (4.10) we get

$$[A^{-M,-\lambda} f_\nu](z) = z^{n_1-j-1}\,\bar{z}^{n_2-k-1}\, I_\nu(z^{-1})\,, \tag{4.24}$$

$$I_\nu(w) = \int (1 - wz)^{n_1-j-1} (1 - \overline{wz})^{n_2-k-1}\, \varphi(\nu z)\nu^2 \mathrm{d}^2 z\,. \tag{4.25}$$

When ν increases, this integral converges to one and all its partial derivatives converge to zero uniformly in every compact set of the w complex plane. From the preceding formulae we get

$$v_\nu(z) - u(z) = z^{n_1-j-1}\bar{z}^{n_2-k-1}\,(1 - \chi(z))(I_\nu(z^{-1}) - 1)\,, \tag{4.26}$$

and it is easy to see that this sequence of functions converges to zero in the topology of $F^{M\lambda}$ defined above.

In order to complete the proof we have to show that the finite sums of functions which have the property (4.19) are dense in $F^{M\lambda}$. If $h(z)$ is an arbitrary element of $F^{M\lambda}$, one can find a sequence of polynomials $P_\nu(w,\bar{w})$ such that

$$P_\nu(w,\bar{w}) \xrightarrow[\nu\to\infty]{} \hat{h}(w) = w^{n_1-1}\bar{w}^{n_2-1}\, h(w^{-1})\,, \tag{4.27}$$

uniformly together with all the partial derivatives for $|w| \leqslant 4$. Then the functions

$$u_\nu(z) = \chi(2z)\,[h(z) - z^{n_1-1}\bar{z}^{n_2-1} P_\nu(z^{-1},\bar{z}^{-1})] + z^{n_1-1}\bar{z}^{n_2-1} P_\nu(z^{-1},\bar{z}^{-1}) \tag{4.28}$$

are finite sums of functions with the property (4.19) and converge towards h in the topology $F^{M\lambda}$.

In order to express the operators $D^{M\lambda}(a)$ in terms of the operators $B^{M\lambda}(a)$ ($a \in S$), we introduce (for $\lambda - M$ not integral) the operator $Z^{M\lambda}$ defined in the dense subspace $G + G_1^{M\lambda} \subset F^{M\lambda}$ by the formula

$$Z^{M\lambda}(g + g_1) = g\,, \quad g \in G\,, \quad g_1 \in G_1^{M\lambda}\,. \tag{4.29}$$

The element g is determined uniquely by the sum $g + g_1$, due to the property (4.16). It can be shown that the linear operator $Z^{M\lambda}$ is not continuous.

From the definition (4.29) we have

$$(A^{M\lambda})^{-1}Z^{-M,-\lambda}A^{M\lambda}(g+g_1)=g_1\,,\quad g\in G\,,\quad g_1\in G_1^{M\lambda}\,, \tag{4.30}$$

since

$$A^{M\lambda}g\in G_1^{-M,-\lambda}\,,\quad A^{M\lambda}g_1\in G\subset F^{-M,-\lambda}\,. \tag{4.31}$$

From eqs. (4.29) and (4.30), we get the identity

$$[Z^{M\lambda}+(A^{M\lambda})^{-1}Z^{-M,-\lambda}A^{M\lambda}]f=f\,,\quad f\in G+G_1^{M\lambda}\,. \tag{4.32}$$

From this equation and eq. (4.12), we obtain

$$\mathcal{D}^{M\lambda}(a)f=[\mathcal{B}^{M\lambda}(a)Z^{M\lambda}+(A^{M\lambda})^{-1}\mathcal{B}^{-M,-\lambda}(a)Z^{-M,-\lambda}A^{M\lambda}]f\,,$$

$$a\in \mathrm{S}\,,\quad f\in G+G_1^{M\lambda}\,. \tag{4.33}$$

This formula will be used in sect. 7, where we discuss the inverse formula. The fact that the operator $Z^{M\lambda}$ is not continuous is the origin of the difficulties we shall encounter.

5. Banach-space extensions, operator norms and analyticity properties

Dealing with the representations $\mathcal{D}^{M\lambda}$ of SL(2C), it is useful to introduce in the space $F^{M\lambda}$ the norm [16, 33, 34]

$$\|f\|^2=\int |f(z)|^2(1+|z|^2)^{-2\,\mathrm{Re}\,\lambda}\,\mathrm{d}^2z\,. \tag{5.1}$$

With respect to this norm, the operators $\mathcal{D}^{M\lambda}(a)$ are bounded and their norms are given by

$$\|\mathcal{D}^{M\lambda}(a)\|=\exp|\zeta\,\mathrm{Re}\,\lambda|\,,\quad \cosh\zeta=L_{00}(a)\,. \tag{5.2}$$

They are also isometric if Re $\lambda=0$ or if $a\in\mathrm{SU}(2)$. The completion of $F^{M\lambda}$ with respect to this norm is a Hilbert space.

If we consider the representation $\mathcal{B}^{M\lambda}$ of S, one can define norms that are more convenient than the restriction to G of the norm (5.1) for the purpose of proving the existence of the projection integrals. We study in detail only norms of the kind

$$\|f\|=[\int |f(z)|^p\,\gamma(|z|^2)\,\mathrm{d}^2z]^{\frac{1}{p}}\,,\quad f\in G\,,\quad p\geqslant 1\,. \tag{5.3}$$

We remember that for $p<1$ this equation does not define a norm.

From eq. (4.5) we get

$$\mathrm{d}^2z=|a_{11}-a_{12}z_a|^{-4}\,\mathrm{d}^2z_a\,, \tag{5.4}$$

and therefore, from eq. (4.4)

$$\|\mathcal{B}^{M\lambda}(a)f\|^p = \int |f(z_a)|^p\, |a_{11} - a_{12}z_a|^{2p(1-\mathrm{Re}\,\lambda)-4} \times \gamma\left(\left|\frac{a_{22}z_a - a_{21}}{a_{11} - a_{12}z_a}\right|^2\right) \mathrm{d}^2 z_a\,, \quad a \in \mathrm{S}\,. \tag{5.5}$$

It follows immediately that

$$\|\mathcal{B}^{M\lambda}(a)\| = \sup_{f\in G} \left(\|\mathcal{B}^{M\lambda}(a)f\|\, \|f\|^{-1}\right)$$
$$= \sup_{|z|\leqslant 1} \left[(\gamma(|z|^2))^{-1}\gamma\left(\left|\frac{a_{22}z - a_{21}}{a_{11} - a_{12}z}\right|^2\right)|a_{11} - a_{12}z|^{2p(1-\mathrm{Re}\,\lambda)-4}\right]^{\frac{1}{p}}. \tag{5.6}$$

We restrict our considerations to the choice

$$\gamma(|z|^2) = [1 - |z|^2]^{-rp}\,, \quad r \geqslant 0\,. \tag{5.7}$$

Remark that if $r < 0$, the expression (5.6) is clearly infinite. The norm (5.6) takes the form

$$\|\mathcal{B}^{M\lambda}(a)\|_{pr} = \sup_{|z|\leqslant 1} \left[\left(\frac{1 - |z|^2}{|a_{11} - a_{12}z|^2 - |a_{22}z - a_{21}|^2}\right)^r \times |a_{11} - a_{12}z|^{2(r-v)}\right] = \rho_{rv}(a)\,. \tag{5.8}$$

We have put for simplicity

$$v = \mathrm{Re}\,\lambda + 2p^{-1} - 1\,. \tag{5.9}$$

It is easy to evaluate the expression (5.8) in some simple cases:

$$\rho_{rv}(u_z(\mu)) = 1\,, \tag{5.10}$$

$$\rho_{rv}(a_z(\chi)) = \exp(-v\chi)\,, \quad \chi \geqslant 0\,, \tag{5.11}$$

$$\rho_{rv}(a_x(\xi)) = \exp|(v - r)\xi|\,. \tag{5.12}$$

From eq. (5.10) we see that the rotations around the z-axis are always represented by isometric operators. From eq. (5.12) we see that for $r = v$ all the elements of SU(1,1) are represented by isometric operators. Therefore, taking into account eq. (5.11), we get

$$\rho_{vv}(a) = \exp(-v\chi)\,,$$
$$\cosh\chi = L_{33}(a)\,, \quad a \in \mathrm{S}^{\mathrm{o}}\,, \quad v \geqslant 0\,, \quad \chi > 0\,. \tag{5.13}$$

Another case in which the function (5.8) can be simply evaluated is the case $r = 0$:

$$\rho_{0v}(a) = \left(|a_{11}| - \frac{v}{|v|}|a_{12}|\right)^{-2v}\,, \quad a \in \mathrm{S}\,. \tag{5.14}$$

Eq. (5.13) for $v = 1$ gives

$$\sup_{|z| \leqslant 1} \left[\frac{1 - |z|^2}{|a_{11} - a_{12}z|^2 - |a_{22}z - a_{21}|^2} \right] = \exp(-\chi) , \tag{5.15}$$

and from eq. (5.8) we get the important inequality

$$\rho_{r+c,\, v+c}(a) \leqslant \exp(-\chi c) \rho_{rv}(a) , \quad c \geqslant 0 . \tag{5.16}$$

It is also possible to show that

$$\rho_{rv}(a_z(\chi) a_x(\xi)) \underset{\xi \to \infty}{\sim} \exp[(v - r)\xi] \Phi_{rv}(\chi) , \quad 0 \leqslant r \leqslant v , \tag{5.17}$$

$$\rho_{rv}(a_z(\chi) a_x(\xi)) \underset{\xi \to \infty}{\sim} \exp(-v\xi) \Phi_{rv}(\chi) , \quad 2v \leqslant r , \tag{5.18}$$

where

$$\Phi_{rv}(\chi) \underset{\chi \to 0}{\sim} 1 , \quad 0 \leqslant r \leqslant v , \tag{5.19}$$

$$\Phi_{rv}(\chi) \underset{\chi \to 0}{\sim} \chi^{2(v-r)} 2^{4(r-v)} r^{-r} (r - v)^{2(r-v)} (2v - r)^{2v-r} , \quad v \leqslant r \leqslant 2v , \tag{5.20}$$

$$\Phi_{rv}(\chi) \underset{\chi \to \infty}{\sim} e^{-v\chi} (4r)^r (2v)^{-2v} (2v - r)^{2v-r} , \quad 0 \leqslant r \leqslant 2v , \tag{5.21}$$

$$\Phi_{rv}(\chi) = e^{-v\chi} (1 - e^{-2\chi})^{-r} 2^{2v} r^r$$
$$\times (r - 2v)^{r-2v} (r - v)^{2(v-r)} , \quad 2v \leqslant r . \tag{5.22}$$

We are using the convention $0^0 = 1$.

If we consider the inequality

$$\rho_{rv}(a_z(\chi) a_x(\xi)) \leqslant \rho_{rv}(a_z(\chi) a_x(\xi + \hat{\xi}))\, \rho_{rv}(a_x(-\hat{\xi})) , \tag{5.23}$$

and we let $\hat{\xi}$ increase indefinitely, from eqs. (5.12) and (5.17) we get the majorization

$$\rho_{rv}(a_z(\chi) a_x(\xi)) \leqslant \exp[(v - r)\xi] \Phi_{rv}(\chi) , \quad v \leqslant r \leqslant 2v . \tag{5.24}$$

It is useful to introduce the Banach spaces G_{pr}, which are the completions of the space G with respect to the norms (5.3). They can be considered as spaces of functions defined on the circle $|z| < 1$ (functions which coincide almost everywhere being considered as equal). The representation operators $\mathcal{B}^{M\lambda}(a)$ can be extended by continuity to the space G_{pr} and their norm is still given by eq. (5.8). The action of the operator $\mathcal{B}^{M\lambda}(a)$ on the function $f \in G_{pr}$ is still given by [see eqs. (4.4) and (4.5)]

$$[\mathcal{B}^{M\lambda}(a)f](z) = |a_{12}z + a_{22}|^{2\lambda-2} \left(\frac{\bar{a}_{12}\bar{z} + \bar{a}_{22}}{a_{12}z + a_{22}}\right)^M f(z_a) \,. \tag{5.25}$$

We see from this equation that for fixed M, p, r and a, $\mathcal{B}^{M\lambda}(a)$ is an analytic operator valued function of the complex parameter λ. In conclusion, we have:

Proposition 11: For $p \geqslant 1$ and $r \geqslant 0$, we indicate by $\mathcal{G}_{pr}$ the space of the functions on the circle $|z| < 1$ which are L^p with respect to the measure $(1 - |z|^2)^{-rp} d^2z$. The operators $\mathcal{B}^{M\lambda}(a)$ defined by eq. (5.25) form a continuous representation of S in $\mathcal{G}_{pr}$. These operators form, for fixed values of the other variables, an entire operator valued function of λ and their norm is given by eq. (5.8).

Slightly modifying the notations of sect. 3, we indicate by M_{rv} the space of the measures on S such that the norm

$$\|F\|_{rv} = \int_S \rho_{rv}(a)\, |F(a)| d^6a \tag{5.26}$$

is finite.

We remark that, if F belongs both to M_{rv} and to $\mathrm{M}_{r'v'}$, where v is given by eq. (5.9) and

$$v' = \mathrm{Re}\,\lambda + 2(p')^{-1} - 1 \tag{5.27}$$

the projection integral

$$\mathcal{B}^{M\lambda}(F) = \int_S \mathcal{B}^{M\lambda}(a) F(a) d^6a \tag{5.28}$$

can be considered both as an operator in $\mathcal{G}_{pr}$ and as an operator in $\mathcal{G}_{p'r'}$. However, these two operators, which are in principle different, coincide in the intersection $\mathcal{G}_{pr} \cap \mathcal{G}_{p'r'}$, which is dense both in the space $\mathcal{G}_{pr}$ and $\mathcal{G}_{p'r'}$. Therefore, one of these operators determines the other uniquely or, in other words, they contain the same information about the measure F.

6. Projection of the multiperipheral equation

In order to apply the techniques developed above to the multiperipheral equations (1.3) or (1.15), we have to study the norms

$$\|K(t,t')\|_{rv} = \int_S \rho_{rv}(a) |K(t,a,t')| d^6a = |16\pi^2 t'|^{-1}$$
$$\times [T(m^2,t,t')]^{\frac{1}{2}} \int_{SU(1,1)} \rho(a_z(\chi(t,t'))g) |\hat{K}(t,g,t')| d^3g \,. \tag{6.1}$$

The last expression has been obtained using eqs. (1.16) and (1.18).

If this expression is finite, one can introduce the projected kernel

$$\mathcal{B}^{M\lambda}(K(t,t')) = \int_S \mathcal{B}^{M\lambda}(a)K(t,a,t')\,\mathrm{d}^6 a\,, \tag{6.2}$$

which, for fixed t and t', is a bounded operator in G_{pr}. Of course, the parameters λ, v and p have to satisfy the relation (5.9).

From eq. (5.16) we get the inequality

$$\|K(t,t')\|_{r+c,\ v+c} \leqslant \exp\,[-c\chi(t,t')]\,\|K(t,t')\|_{rv}\,, \qquad c \geqslant 0\,. \tag{6.3}$$

In order to obtain more detailed results, we use some assumptions which have been proposed and discussed in ref. [25]. Using eqs. (4.2), (6.3) and (7.1) of that paper, we see that it is natural to assume

$$|\hat{K}(t,g,t')| \leqslant d(t)d(t')\left(\frac{t'}{t}\right)^{\frac{1}{2}}\left[\frac{T(m^2,t,t')}{8m^2\sqrt{tt'}}\right]^{2\alpha}[1+\cosh\xi]^{2\alpha}\,, \qquad \alpha \geqslant 0\,, \tag{6.4}$$

where we have used for g the parametrization

$$g = u_z(\mu)a_x(\xi)u_z(\nu)\,, \tag{6.5}$$

and $d(t)$ represents a continuous function of t sufficiently fast decreasing for $t \to -\infty$. We remark that one can multiply the kernel $\hat{K}$ by a factor of the form $\gamma(t)(\gamma(t'))^{-1}$ without changing the product (1.14) (of course one has to modify also the quantities $\hat{A}$ and $\hat{B}$). In eq. (6.4), this arbitrary factor has been chosen in the most convenient way.

From eqs. (6.1) and (6.4), we get the majorization

$$\|K(t,t')\|_{rv} \leqslant (4\pi)^{-2}(8m^2)^{-2\alpha}d(t)d(t')(tt')^{-\alpha-\frac{1}{2}}$$

$$\times\ [T(m^2,t,t')]^{2\alpha+\frac{1}{2}}\,\tfrac{1}{2}\int_0^\infty (1+\cosh\xi)^{2\alpha}\rho_{rv}(a_z(\chi(t,t'))a_x(\xi))\sinh\xi\,\mathrm{d}\xi\,. \tag{6.6}$$

From eqs. (5.17) and (5.18), we see that this integral converges for

$$r - v > 2\alpha + 1 \geqslant 1\,,$$

$$0 \leqslant r \leqslant 2v\,, \tag{6.7}$$

and for

$$v > 2\alpha + 1 \geqslant 1\,,$$

$$2v \leqslant r\,. \tag{6.8}$$

Eq. (6.7) defines a region in fig. 2 which is the union of the regions A and B, while eq. (6.8) defines the region C.

In the case (6.7), which is the most interesting one, we can use eq. (5.24) and we get the more explicit majorization

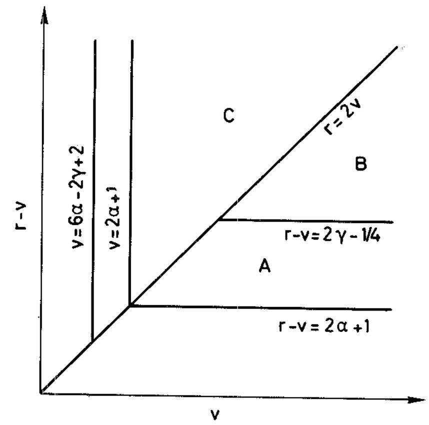

Fig. 2. The region A indicates the values of the parameters r and v which are suitable for the partial wave projection of the multiperipheral kernel.

$$\|K(t,t')\|_{rv} \leqslant 2^{-6}\pi^{-2}(2m)^{-4\alpha}d(t)d(t')(tt')^{-\alpha-\frac{1}{2}}$$
$$\times\ [T(m^2,t,t')]^{2\alpha+\frac{1}{2}}(r-v-2\alpha-1)^{-1}\Phi_{rv}(\chi(t,t'))\ ,$$
$$1 \leqslant 2\alpha+1 < r-v \leqslant v\ . \tag{6.9}$$

Summarizing, we have seen that if r and v satisfy the conditions (6.7) or (6.8), the kernel $K(t,t')$, for fixed t and t', belongs to the convolution algebra M_{rv}. It follows that, if we keep all the radial variables fixed, all the convolutions which appear in eq. (1.15) exist and belong to M_{rv}.

In order to discuss the existence of the integrals over the radial variables t, t', ..., we consider the quantities

$$N_{rv} = \left[\int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{0} \mathrm{d}t' \|K(t,t')\|_{rv}^2\right]^{\frac{1}{2}}, \tag{6,10}$$

$$N'_{rv}(t) = \left[\int_{-\infty}^{0} \|K(t,t')\|_{rv}^2\, \mathrm{d}t'\right]^{\frac{1}{2}}, \tag{6.11}$$

$$N''_{rv}(t') = \left[\int_{-\infty}^{0} \|K(t,t')\|_{rv}^2\, \mathrm{d}t\right]^{\frac{1}{2}}. \tag{6.12}$$

Using this definition, the Schwarz inequality and eq. (3.13), we get the inequality

$$\int \|K(t,t_1) * K(t_1,t_2) * \ldots * K(t_\nu,t')\|_{rv}\mathrm{d}t_1 \ldots \mathrm{d}t_\nu$$
$$\leqslant N'_{rv}(t) N_{rv}^{\nu-1} N''_{rv}(t')\ , \quad \nu \geqslant 1\ . \tag{6.13}$$

If this quantity is finite, the integrals

$$R^{(\nu)}(t,t') = \int K(t,t_1) * K(t_1,t_2) * \dots$$

$$\dots * K(t_\nu, t') \mathrm{d}t_1 \dots \mathrm{d}t_\nu \,, \tag{6.14}$$

exist ⋆, belong to M_{rv} for fixed values of t and t' and satisfy the inequality

$$\|R^{(\nu)}(t,t')\|_{rv} \leq N'_{rv}(t) N_{rv}^{\nu-1} N''_{rv}(t') \,. \tag{6.15}$$

Eq. (1.15) can be written in the form

$$R(t,t') = K(t,t') + \sum_{\nu=1}^{\infty} R^{(\nu)}(t,t') \,, \tag{6.16}$$

and we see that, if we assume

$$N_{rv} < 1 \,, \tag{6.17}$$

this series converges in the space M_{rv} and we have the inequality

$$\|R(t,t')\|_{rv} \leq \|K(t,t')\|_{rv} + N'_{rv}(t)(1 - N_{rv})^{-1} N''_{rv}(t') \,. \tag{6.18}$$

We remark that this formula permits one to get important information about the kernel R without computing it explicitly.

In this situation, it is easy to derive rigorously the multiperipheral integral equation (1.3), which can be written also in the form

$$R(t,t') = K(t,t') + \int K(t,t'') * R(t'',t') \mathrm{d}t'' \,, \tag{6.19}$$

and the partial wave equations (1.22) and (1.23).

In order to complete our discussion, we have to consider the integrals (6.10)–(6.12) in more detail. We start from the majorization (6.9), assuming that

$$d(t) \leq c(m^2 + |t|)^{-\gamma} \,, \quad \gamma > 0 \,. \tag{6.20}$$

Using also eqs. (5.20) and (5.21), we see that the integrals (6.11) and (6.12) converge in the region defined by the inequalities (6.7) and

$$v > 6\alpha - 2\gamma + 2 \,. \tag{6.21}$$

In order to ensure the convergence of the integral (6.10), we have to add to the conditions (6.7) and (6.21) the inequality

$$r - v < 2\gamma - \tfrac{1}{4} \,. \tag{6.22}$$

The inequalities (6.7), (6.21) and (6.22) define the region A in fig. 2. If, as we have done in fig. 2, we assume

⋆ Of course, we have also to assume that the integrand is measurable in a suitable sense [30]; we shall not discuss here this rather technical assumption.

$$\gamma \geqslant 2\alpha + \tfrac{1}{2} \,, \tag{6.23}$$

the condition (6.21) is already implied by the conditions (6.7). We see that the region A is not empty only if we assume *

$$\gamma > \alpha + \tfrac{5}{8} \,. \tag{6.24}$$

In order to discuss the condition (6.17), we remark that if N_{rv} is finite, from eqs. (6.3) and (6.10) follows

$$\lim_{c \to +\infty} N_{r+c,\, v+c} = 0 \,. \tag{6.25}$$

We see that, if the condition (6.24) holds, N_{rv} is smaller than one for v sufficiently large and r suitably chosen.

It is useful to summarize the results of this section.

Proposition 12: Under the assumptions (6.4), (6.20) and (6.24), if the parameters r and v satisfy the inequalities

$$2\alpha + 1 < r - v < 2\gamma - \tfrac{1}{4} \,, \tag{6.26}$$

$$\begin{aligned} &v \geqslant r - v > 2\alpha + 1 \,, \\ &v > 6\alpha - 2\gamma + 2 \,, \end{aligned} \tag{6.27}$$

the kernel $K(t,t')$ belongs to M_{rv} and the quantities N_{rv}, $N'_{rv}(t)$ and $N''_{rv}(t')$ are finite. Moreover, if

$$\mathrm{Re}\,\lambda = v + 1 - 2p^{-1} \,, \quad p \geqslant 1 \,, \tag{6.28}$$

the projected kernels $B^{M\lambda}(K(t,t'))$ exist as operators in G_{pr}. If r and v satisfy eq. (6.26) and are both sufficiently large, we have that N_{rv} is smaller than one, the kernel $R(t,t')$ belongs to M_{rv} and satisfies the integral equation (6.19). Moreover, if $\mathrm{Re}\,\lambda$ is given by eq. (6.28), the projected kernels $B^{M\lambda}(R(t,t'))$ exist as operators in G_{pr} and satisfy the projected integral equation (1.23).

We see that, in the case $\gamma \geqslant 2\alpha + \frac{1}{2}$, if we take $p = 1$ and we choose r in a suitable way, the projected kernels $B^{M\lambda}(K(t,t'))$ can be defined in the half-plane

$$\mathrm{Re}\,\lambda > 2\alpha \,, \tag{6.29}$$

while the resolvent projected kernel $B^{M\lambda}(R(t,t'))$ can be defined in another half-plane of the form

$$\mathrm{Re}\,\lambda > L \geqslant 2\alpha \,. \tag{6.30}$$

These results are in agreement with the results of refs. [7–11, 24, 36].

* As shown in ref. [35], one can weaken the condition (6.24) without losing some important properties of multiperipherism. Many successful models assume that $d(t)$ decreases exponentially with $|t|$, so that one can put $\gamma = +\infty$.

7. The problem of the inverse formula

In order to complete our analysis, we have to reconstruct the resolvent $\mathcal{R}(t,t')$ starting from its projections $\mathcal{B}^{M\lambda}(\mathcal{R}(t,t'))$. More generally, we have to solve the following problem. Assume that the measure F belongs to M_{rv} and that its projections $\mathcal{B}^{M\lambda}(F)$ are known, for Re λ given by eq. (6.28), as operators in $\mathcal{G}_{pr}$ (the parameters r and p are fixed). Is the measure F determined uniquely by $\mathcal{B}^{M\lambda}(F)$? Can F be represented by means of a simple formula (the inverse formula) containing $\mathcal{B}^{M\lambda}(F)$? In the present paper we shall solve these problems affirmatively only when F is a measure with compact support and we shall discuss the difficulties one meets in the general case.

We start from some well-known results [16, 33, 34] about the harmonic analysis on SL(2C). We consider the representation operators $\mathcal{D}^{M\lambda}(a)$ with Re $\lambda = 0$ as unitary operators acting in the Hilbert space defined by the norm (5.1). If φ_i are C^∞ functions on SL(2C) with compact support, the operators $\mathcal{D}^{M\lambda}(\varphi_i)$ are of the Hilbert-Schmidt type and we can write the Plancherel formula in the form

$$\int_{\mathrm{SL(2C)}} \varphi_1(a^{-1})\varphi_2(a)\,\mathrm{d}^6a = [\varphi_1 * \varphi_2](e)$$

$$= (4\pi^4 i)^{-1} \sum_{M=-\infty}^{\infty} \int_{-i\infty}^{+i\infty} \mathrm{Tr}\,[\mathcal{D}^{M\lambda}(\varphi_1)\mathcal{D}^{M\lambda}(\varphi_2)](M^2-\lambda^2)\,\mathrm{d}\lambda\,. \tag{7.1}$$

If we consider a measure F on SL(2C) with compact support and we put

$$\varphi_2 = \varphi_3 * F\,, \quad \varphi = \varphi_1 * \varphi_3\,, \quad \varphi_1 * \varphi_2 = \varphi * F\,, \tag{7.2}$$

from eq. (7.1) we get

$$\int_{\mathrm{SL(2C)}} \varphi(a^{-1})F(a)\,\mathrm{d}^6a = [\varphi * F](e) = [\varphi_1 * \varphi_2](e)$$

$$= (4\pi^4 i)^{-1} \sum_{M=-\infty}^{\infty} \int_{-i\infty}^{+i\infty} \mathrm{Tr}\,[\mathcal{D}^{M\lambda}(\varphi)\mathcal{D}^{M\lambda}(F)](M^2-\lambda^2)\,\mathrm{d}\lambda\,. \tag{7.3}$$

Remark that the trace exists, because the operator

$$\mathcal{D}^{M\lambda}(\varphi) = \mathcal{D}^{M\lambda}(\varphi_1)\mathcal{D}^{M\lambda}(\varphi_3) \tag{7.4}$$

belongs to the trace class and $\mathcal{D}^{M\lambda}(F)$ is bounded.

As the functions φ of the form $\varphi_1 * \varphi_3$ are dense in the space of the continuous functions on SL(2C) with compact support, eq. (7.3) determines the measure F uniquely.

Now we assume that the compact support of F is contained in S. Then we can define for any value of λ and M the quantities $\mathcal{B}^{M\lambda}(F)$ as continuous operators in $\mathcal{G}$ or as bounded operators in any of the spaces $\mathcal{G}_{pr}$. From eq. (4.33) we obtain the formula

$$\mathcal{D}^{M\lambda}(F)f = [\mathcal{B}^{M\lambda}(F)Z^{M\lambda} + (A^{M\lambda})^{-1}\mathcal{B}^{-M,-\lambda}(F)$$
$$\times Z^{-M,-\lambda}A^{M\lambda}]f\,, \quad f \in G + G_1^{M\lambda}\,. \tag{7.5}$$

As the subspace $G + G_1^{M\lambda}$ is dense in the representation Hilbert space, we see that the operators $\mathcal{B}^{M\lambda}(F)$ determine the operators $\mathcal{D}^{M\lambda}(F)$ uniquely and therefore they determine the measure F uniquely. More explicitly, we have using also eq. (4.12),

Proposition 13: If F is a measure with compact support in S and f_i is an orthonormal basis in the Hilbert space with the property

$$f_i \in G + G_1^{M\lambda}\,, \quad i = 1,2,\ldots\,, \tag{7.6}$$

the measure F is uniquely determined by its projections $\mathcal{B}^{M\lambda}(F)$ by means of the formula

$$\int_{\mathrm{SL(2C)}} \varphi(a^{-1})F(a)\,\mathrm{d}^6a = (4\pi^4 i)^{-1}\sum_{M=-\infty}^{\infty}\int_{-i\infty}^{+i\infty}(M^2-\lambda^2)$$
$$\times \sum_{i=1}^{\infty}(f_i, [\mathcal{D}^{M\lambda}(\varphi)\mathcal{B}^{M\lambda}(F)Z^{M\lambda} + (A^{M\lambda})^{-1}\mathcal{D}^{-M,-\lambda}(\varphi)$$
$$\times \mathcal{B}^{-M,-\lambda}(F)Z^{-M,-\lambda}A^{M\lambda}]f_i)\,\mathrm{d}\lambda\,. \tag{7.7}$$

where φ is the convolution of two C^∞ functions with compact support on SL(2C).

In the general case $F \in \mathrm{M}_{rv}$, the formula (7.7) is not even meaningful, as $\mathcal{B}^{M\lambda}(F)$ does not necessarily exist for Re $\lambda = 0$. In order to solve the general problem, one should first modify eq. (7.7), in such a way that its right-hand side is meaningful also for $F \in \mathrm{M}_{rv}$, and then to prove the validity of the modified formula in this more general case.

If one applies naively to eq. (7.7) the formal properties of the trace, one gets

$$\int \varphi(a^{-1})F(a)\,\mathrm{d}^6a = (2\pi^4 i)^{-1}$$
$$\times \sum_{M=-\infty}^{\infty}\int_{-i\infty}^{+i\infty}\mathrm{Tr}\,[\mathcal{D}^{M\lambda}(\varphi)\mathcal{B}^{M\lambda}(F)Z^{M\lambda}](M^2-\lambda^2)\,\mathrm{d}\lambda\,. \tag{7.8}$$

Then one can try to shift the integration path along the line defined by eq. (6.28). Unfortunately, as the operator $Z^{M\lambda}$ is unbounded, it is by no means clear that the trace which appears in eq. (7.8) exists and is an analytic function of λ. In particular, the operator $Z^{M\lambda}$ does not even exist for $\lambda - M$ integral, because the proposition 10, which allows its definition, is no longer valid in this case. It follows that singularities at these points are expected.

We remark that the same difficulties arise if one uses some explicit basis, as in ref. [9]. These problems will be treated in detail elsewhere.

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