COMITATO NAZIONALE PER L'ENERGIA NUCLEARE
Laboratori Nazionali di Frascati

LNF-73/13
10 Aprile 1973

E. Etim : GRIBOV-LIPATOV RECIPROCITY FROM CONSERVATION CONSTRAINTS AND O(4) SYMMETRY. -

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E. Etim[(x)]: GRIBOV-LIPATOV RECIPROCITY FROM CONSERVATION CONSTRAINTS AND O(4) SYMMETRY. -

ABSTRACT. -

The physical basis of the Gribov-Lipatov reciprocity is traced to energy-momentum conservation. N-plane analyticity and O(4) symmetry.

(x) - Work supported by the INFN.

The physical meaning of the remarkable relation[1]

(1)
$$F_1^S(q^2, \frac{1}{\omega}) = -\omega F_1^A(q^2, \omega)$$
$$F_2^S(q^2, \frac{1}{\omega}) = -\omega^3 F_2^A(q^2, \omega)$$

between scattering and annihilation structure functions is not very clear. One interpretation[2] is that, combined with analytic continuation, it expresses the equality between the probability distribution of finding a parton with longitudinal momentum fraction $1/\omega$ in a hadron and the analogous distribution for finding a hadron with longitudinal momentum fraction ω in a parton. A different interpretation by Ferrara et al.[3], also based on analytic continuation, attributes the reciprocity relation, in the form

(2)
$$F_2^S(\frac{1}{\omega}) = -\omega^3 F_2^S(\omega) ,$$

to a "twin" symmetry which, though operative at the O(3) level, leaves the quadratic Casimir

(3)
$$C_N^{(2)} = l_N(l_N - 4) + N(N+2) = 2N(N+1) - 4$$
$$l_N = N + 2 ,$$

of the SO(4,2) representations of the $(\frac{N}{2}, \frac{N}{2})$ Lorentz tensors appearing in the light cone expansion of the product of two e.m. currents, invariant under the substitution $N \rightarrow -N-1$. One argument against these interpretations is that analytic continuation of the structure functions is not relevant for the validity of eq. (1). Besides it is not eq. (2) but (1), with its important physical consequences, that is of real interest.

The purpose of this communication is to point out that the physical basis of the Gribov-Lipatov reciprocity is energy-momentum conservation and to show that it follows directly from invariance under SO(3,1) and the analyticity of moments of the structure functions in a complex N-plane for generally non-scaling and non-analytic structure functions. The possibility of realising the latter property would demand the extension of the analyticity domain of the said moments. The continuation into the complex N (spin) plane is analogous to the Sommerfeld-Watson continuation of partial wave amplitudes. An inductive proof of eq. (1) based directly on energy-momentum conservation and SO(3,1) invariance is also indicated.

Consider the process

$$\gamma(q) + h(p) \longrightarrow x \tag{4}$$

which from generalized unitarity[4] can describe deep inelastic scattering (S) $e^- + h(p) \rightarrow e^- + x$ $(q^2 < 0,\ p{\cdot}q > 0)$, deep inelastic annihilation (A) $e^+ + e^- \rightarrow \bar{h}(-p) + x$ $(q^2 > 0,\ p{\cdot}q < 0)$ and deep inelastic three-body annihilation (T) $e^+ + e^- + h(p) \rightarrow x$ $(q^2 > 0,\ p{\cdot}q > 0)$ as appropriate discontinuities of the spin averaged Compton amplitude

$$T_{\mu\nu}(q^2, p{\cdot}q) = i\int d^4x\, e^{iqx} \langle p | T(j_\mu(\tfrac{x}{2})\, j_\nu(-\tfrac{x}{2})) | p \rangle \tag{5}$$

We assume that the energy momentum tensor (spin N = 2 and scale dimension $l_N = 4$) contributes in the light cone expansion of the current product

$$j_\mu(x)\, j_\nu(0) = (\partial_\mu \partial_\nu - g_{\mu\nu}\Box^2) \sum_N B^{(N)}_{\alpha_1 \ldots \alpha_N}(x)\, O^{(N)}_{\alpha_1 \ldots \alpha_N}(0) \; +$$

(6)
$$+ (g_{\mu\alpha_1}\partial_\nu \partial_{\alpha_2} + g_{\nu\alpha_1}\partial_\mu \partial_{\alpha_2} - g_{\mu\alpha_1} g_{\nu\alpha_2}\Box^2 - g_{\mu\nu}\partial_{\alpha_1}\partial_{\alpha_2}) \; \times$$

$$\times \sum_N C^{(N)}_{\alpha_3 \ldots \alpha_N}(x)\, O^{(N)}_{\alpha_1 \ldots \alpha_N}(0)$$

where the coefficients $B^{(N)}(x)$ contribute to the structure function $F_1(q^2, \omega)$ and $C^{(N)}(x)$ to $F_2(q^2, \omega)$. Hence forth we consider only the latter. Inserting from eq. (6) into (5) and integrating over the phase space of h gives, upon taking the trace

(7)
$$\sum_{N'} \int \frac{d^3p}{2E}\, \hat{P}_{\alpha_1 \ldots \alpha_N}\, \hat{P}_{\beta_1 \ldots \beta_{N'}}\, C^{(N')}_{\beta_1 \ldots \quad N'}(q) =$$
$$= \frac{1}{|q^2|} \int \frac{d^3p}{2E} \left[\omega F_2(q^2, \omega)\right] \hat{P}_{\alpha_1 \ldots \alpha_N}$$

where

(8a)
$$C^{(N)}_{\alpha_1 \ldots \alpha_N}(q) = -i \frac{q^2}{|q^2|} \left(g_{\alpha_1\alpha_2} + 2 \frac{q_{\alpha_1} q_{\alpha_2}}{q^2}\right) \int d^4x e^{iqx} \cdot C^{(N)}_{\alpha \ldots N}(x)$$

(8b)
$$\hat{P}_{\alpha_1 \ldots \alpha_N} = P_{\alpha_1} P_{\alpha_2} \ldots P_{\alpha_N} - \text{(TRACES)}$$

The combination of 4-momentum in eq. (8b) transform irreducibly under the Lorentz group. Projection of eq. (7) onto a suitable basis of the spin N representation yields the "partial wave"

(9)
$$C_N(q^2)\, G^{(1)}_N(\hat{q}) = \frac{4}{|q^2|} \int \frac{d^3p}{2E}\, G^{(1)}_N(\hat{p}) \left[\; F_2(q^2, \omega)\right]$$

where $\hat{p}$, $\hat{q}$ stand for angular coordinates and $G_N^{(1)}(z)$ is a Gegenbauer polynomial. The expansion coefficients $C_N(q^2)$ are the familiar moments of the structure functions[5]. The advantage of the above derivation is that it makes explicit the symmetry properties of these coefficients. In fact from the symmetry

(10) $$C_N^{(1)}(z) = -G_{-(N+2)}^{(1)}(z)$$

of the Gegenbauer polynomials[6] it follows that the $C_N(q^2)$ verify

(11) $$C_N(q^2) = C_{-(N+2)}(q^2)$$

for structure functions defined over the interval $0 \leq |\omega| < \infty$. Thus for the three-body annihilation process one gets

(12a) $$C_N^T(q^2) = C_{-(N+2)}^T(q^2)$$

This is not true for the scattering and annihilation moments; however if their corresponding structure functions are related by analytic continuation one gets from eq. (11)

(12b) $$C_N^S(q^2) = C_{-(N+2)}^A(q^2)$$

This is readily established by rewriting eq. (9) as

(13) $$C_N(q^2) = \int_0^{\infty} d\omega \left[\omega^2 F_2(q^2, \omega) \right] G^{(1)}\left(\frac{\omega}{x_o}\right)$$

and making use of the asymptotic behaviour of $G_N^{(1)}\left(\frac{\omega}{x_o}\right)$[6]

$$(14)\quad G_N^{(1)}\left(\frac{\omega}{x_o}\right) \xrightarrow[x_o=\sqrt{\frac{4M^2}{|q^2|}}\to 0]{} -\frac{1}{\sqrt{\pi}}\left[2^{-(N+1)}\left(\frac{\omega}{x_o}\right)^{-(N+2)} - 2^{N+1}\left(\frac{\omega}{x_o}\right)^{N}\right]$$

In general analytic continuation is not valid and hence also eq. (12b). From eq. (13) we then have for the annihilation and scattering coefficients

$$(15a)\quad C_N^A(q^2) = \left(\frac{4M^2}{|q^2|}\right)^{-\frac{N}{2}} \int_0^1 d\omega \left[\omega^{3/2} F_2^A(q^2,\omega)\right] Q^{1/2}_{-(N+3/2)}(\omega)$$

$$(15b)\quad C_N^S(q^2) = -\left(\frac{4M^2}{|q^2|}\right)^{\frac{N+2}{2}} \int_1^{\infty} d\omega \left[\omega^{3/2} F_2^S(q^2,\omega)\right] Q^{1/2}_{N+1/2}(\omega)$$

where $G_N^{(1)}(z)$ has been expressed in terms of the Legendre functions[6]. The canonical q^2-dependence[5] of the moments together with the symmetry in eq. (12b), in the case of analytic continuation, is manifest from the above equations[7]. From energy-momentum conservation the integral in eq. (15a) converges for $N=1$ and that in eq. (15b) for $N=2$; hence they converge for $N>1$ and $N>2$ respectively. Consequently the functions

$$(16a)\quad C^A(k,q^2) = \int_0^{\infty} dz\, f_A(q^2,z)\, e^{-i(k-\frac{3}{2}i)z}$$

$$(16b)\quad C^S(k,q^2) = \int_0^{\infty} dz\, f_s(q^2,z)\, e^{-i(k-\frac{3}{2}i)z}$$

defined in the complex plane $N=ik$ are both analytic in the lower half plane $\mathrm{Im}(k) < 3/2$. Eqs. (16) are gotten from (15) by a formal transition from Mellin to Fourier transforms with

(17) $$C^A(N,q^2) = C_N^A(q^2), \qquad C^S(N,q^2) = C_{N+1}^S(q^2)$$

and

(18) $$f_A(q^2,z) = e^{-3/2\,z}\,F_2^A(q^2,e^{-z})$$
$$f_S(q^2,z) = e^{3/2\,z}\,F_2^S(q^2,e^{z})$$

Along Im(k)= -1 the functions $C^A(k,q^2)$ and $C^S(k,q^2)$ coincide from energy-momentum conservation[8], whence by analytic continuation they coincide everywhere in Im(k) $\leqslant$ 3/2 giving

(19) $$C_N^A(q^2) = C_{N+1}^S(q^2)$$

This is our main result and the whole content of the Gribov-Lipatov relation. In fact taking the inverse Fourier transform in eq. (16) and making use of (18) one recovers the reciprocity relation

(20) $$F_2^S(q^2,\frac{1}{\omega}) = -\,\omega^3\,F_2^A(q^2,\omega)$$

If the domain of analyticity of the functions $C^A(k,q^2)$ and $C^S(k,q^2)$ can be extended into the upper half plane Im(k) $\geqslant$ 3/2, one verifies immediately that

(21a) $$C_N^A(q^2) = C_{-(N+3)}^A(q^2)$$

(21b) $$C_N^S(q^2) = C_{-(N+1)}^S(q^2)$$

and making use of these in the inverse Fourier transform yields their functional analogue

(22) $$F_2^S(q^2, \frac{1}{\omega}) = -\omega^3 F_2^S(q^2, \omega)$$

It is important to notice the different index symmetries involved in eqs. (12a), (21a) and (21b); only the last of these is of the O(3) variety[3].

In conclusion we have established, by exploiting energy-momentum conservation and analyticity in the complex N-plane, that the operators $O^{(N)}_{\alpha_1 \ldots \alpha_N}(x)$ in the light cone expansion of eq. (6), whether all conserved (as would be the case in a scaling theory) or not, describe the same physics as their associated "generalised momenta"[9]

(23) $$\hat{P}_{\alpha_1 \ldots \alpha_{N-1}}(t) = \int d^3x \, O^{(N)}_{\alpha_1 \ldots \alpha_{N-1} (\alpha_N = 0)}(x, t)$$

As very vividly expressed in eq. (19) this is the content of the Gribov-Lipatov reciprocity[9], the other relations, like eqs. (12), (21) and (22) being simple consequences of an underlying SO(3, 1) symmetry. We would like to emphasise the role of this symmetry by remarking that eq. (20) can also be obtained by induction from eq. (19) which is certainly true for N=1. The essential ingredient for the proof is the reciprocity relation

(24) $$Q^{1/2}_{-\nu}(\frac{1}{z}) = (4)^{\nu} \left[z^2 \, Q^{1/2}_{\nu}(z) \right]$$

of the Legendre functions for large z[6].

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(7) - It is convenient to absorbs these canonical q^2-dependence into the $C_N(q^2)$ so that they are given just by the integrals in equations (15).

(8) - $C_1^A(q^2)$ and $C_2^S(q^2)$ are constants.

(9) - i. e. within our scheme of operator product expansions. Note that the spin step-down $N \to N-1$ from $O^{(N)}_{\alpha_1 \ldots \alpha_N}$ to $\hat{P}_{\alpha_1 \ldots \alpha_{N-1}}$ is just what is involved in eq. (19).