A.F. Grillo : CONFORMAL INVARIANCE IN QUANTUM FIELD THEORY. -

Laboratori Nazionali di Frascati del CNEN
Servizio Documentazione

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THEORY. -

(Invited talk at the Annual Meeting of the Italian Physical Society (SIF)
held in Cagliari, October-November 1972).

CONTENTS. -

# 1. - INTRODUCTION. -

Since the discovery of scaling behaviour in deep-inelastic elec troproduction[1], quite a lot of theoretical investigations have been devo ted to the study of the origin of this phenomenon.

Various models, such as parton model, have been invented, that give a partially satisfactory (or unsatisfactory) explanations of ex perimental results: but the more important achievement which emerges is the emphasis that has been put on fundamental properties of field theo ry such as dilatation[2-6] and conformal[6-9] invariance.

In the deep inelastic experiments, physics is tested at very high energies, i.e. at short distances, where masses become negligible: so the limiting theory we are interested in is a zero mass theory.

As a consequence, when masses become less and less important, one is free to rescale the energy units in the matrix elements: this is the content of the dilatation invariance[2,3]. In a physical (massive) field theory things are not so simple, and one has to invent quite sophisticated methods to study the consequences of broken dilatations[4,5].

A crucial concept related to dilatations is that of scale dimensions of a field, a number that . only for a free field is the same as the naive mass dimension: in general, whenever it can be defined, it specifies the transformation of the field under dilatations (see the appendix for mathe matical details). Here it is worth noting that commutation relations being non linear relations between operators can fix the scale dimension; in particular canonical commutation relations put the (canonical) dimension equal to the free one.

It is an important idea, due to Wilson[3], that the renormalization procedure of any sensible field theory could eventually give an anomalous part to the dimension of fields: this comes from the infinite strength re normalization, and is a parameter which is determined by the interaction and in some sense characterizes the dynamics.

In a massless theory, apart from pathologies and with the cau- tions we will discuss in the next Section, there are other transformations that can be symmetries for the theory, namely the conformal transfor- mations[6-9]. Again, for an exact mathematical definition we refer to the Appendix; it is however interesting to note that dilatations and conformal transformations, together with Poincarè transformations, form the largest group that leaves the light-cone invariant.

The full conformal group puts more restrictions on physical quan tities, such as matrix elements and so on, than only dilatations, so one can hope to derive much more informations on the structure of the theory we are considering. And in fact this is the case: vacuum expectation va-

lues are severely constrained[10], and in some cases uniquely determined, in terms of parameters such as anomalous dimensions and normalization constants.

These quantitites are in principle fixed by the dynamics, so their computation is in general impossible; but conformal invariance allows to make statements also on these.

In fact it has been noted that the Bethe-Sapeter equations that contain the dynamics become purely algebraic equations (at least in some cases) in these parameters. Being non linear, one can hope that they have at most a discrete set of solutions: this is the aim of the so called bootstrap approach[11-16], where the name means that self-consistency of the two sides of B.S. equations fixes the value of physical constants.

We have however to pay a very heavy price for this, for, as soon as we allow dimensions to be anomalous, we find that the LSZ limit no longer exists: in fact we loose the particle description[17], and, since scattering states are not present, also the S matrix cannot be defined.

In particular there is no hope of obtaining informations on the matrix element

$$\langle p | J_\mu (x) J_\nu (0) | p \rangle$$

which appears in deep inelastic scattering.

One way for circumventing this difficulty is that of applying conformal invariance to matrix elements of off-shell operators, such as the electromagnetic currents in $\pi^0 \rightarrow \gamma \gamma$ [18] or $e^+ e^- \rightarrow e^+ e^- + \text{anything}$[19].

The other way out is to not consider conformal invariance for the whole matrix elements, but only for products of operators and expanding the products in sums of local operators[3, 21-25]. so obtaining conformal covariant operator products expansions which exibit many interesting properties.

All we have said for conformal invariance is implicitely assumed to hold in a zero mass theory: however the relevance for the physical massive theory can be examined at the same level as for dilatations[8, 9].

In the following Sections we will consider more in details these aspects.

## 2. - CONFORMAL INVARIANCE AS AN ASYMPTOTIC SYMMETRY IN QUANTUM FIELD THEORY. -

In the following Sections 3 and 4 a number of results will be derived by assuming the validity of Conformal Invariance right from the beginning.

It is then necessary to justify the assumption, since it is easy to check that dilatation (and also conformal) invariance cannot be valid in the presence of masses: in fact, since (see Appendix)

$$(2.1) \qquad \left[ P^2, D \right] = 2i \, P^2$$

no mass eigenstates can exist in a dilatation invariant theory.

The problem then arises of understanding in what sense results concerning an unrealistic invariant theory can reflect themselves in the (massive) real-world physics. We want to present here only the relevant results, without giving any derivation.

Consider a definite theory, e.g. the self-interacting $g \, \emptyset^4$ scalar theory. Formal manipulations, namely the use of canonical commutation relations for the interacting field, led to the proof that it is possible, at least in not too complicate cases, to redefine the energy momentum ten‐sor $\theta_{\mu\nu}$ in such a way that[27]

$$(2.2) \qquad D = \int D_o(x) \, d^3x; \qquad K_\mu = \int K_{o\mu}(x) \, d^3x$$

and

$$(2.3) \qquad D_\mu = x^\lambda \, \theta_{\lambda\mu} \qquad K_{\mu\nu} = (2x_\mu x^\lambda - g^{\mu\lambda} x^2) \, \theta_{\nu\lambda}$$

In order to generate a symmetry these currents have to be conserved

$$(2.4) \qquad \partial^\nu K_{\mu\nu} = 2 x_\mu \theta_\lambda{}^\lambda = 2 x_\mu (\partial^\lambda D_\lambda)$$

so that in a conformally invariant theory the energy momentum tensor has to be traceless.

At the same level one can show that

$$(2.5) \qquad \theta_\lambda{}^\lambda (x) \sim m^2 \emptyset^2$$

so that in the naive Ward identities following from eqs. (2.2) and (2.3) the Weinberg power counting argument enables us to neglect the "soft" breaking term 2.5 at very short distances in the Euclidean region. The behaviour of Green's functions is then fixed, in the same limit, to be

4.

$$(2.6) \quad G_n(\lambda^{-1} x_1, \ldots \lambda^{-1} x_n) = \langle 0|T\left[\phi(\lambda^{-1} x_1)\ldots\phi(\lambda^{-1} x_n)\right]|0\rangle \underset{\lambda \to \infty}{\sim} \lambda^{-n} G_n(x_1 \ldots x_n)$$

the dimensions of fields having their canonical value and all things behaving canonically.

Equation (2.6) should be valid order by order in perturbation theory: it is however a well known fact that, on the contrary, logarithmic terms appear that invalidate eq. (2.6) in <u>every order</u> of perturbation theory[3-6, 28].

The reason of this clash is quite evident: the formal manipulations used in deriving eq. (2.6) are valid only in view of the fact that the theory has been cut off to eliminate divergences. Now, a cut-off is a large mass, that largely breaks dilatation invariance.

So we do not expect that the renormalization procedure will maintain the canonical dimensions of fields or that the naive Ward identities continue to be valid.

For dilatational invariance the new Ward identities have been found[(x)] by Callan[14] and Symanzik[4, 5]:

$$(2.7) \quad \sum_{i=1}^{n}\left[x_\mu^i \frac{\partial}{\partial x_\mu^i} - (1+\gamma(g))\right]\langle 0|T\left[\phi(x_1)\ldots\phi(x_n)\right]|0\rangle =$$

$$= \int d^4 x \langle 0|T\left[\phi(x_1)\ldots\phi(x_n)\theta(x)\right]|0\rangle$$

where $\theta(x)$ which is a sort of <u>effective</u> breaking term

$$(2.8) \qquad \theta(x) = -\eta(g) m^2 \phi^2(x) - \beta(g)\phi^4(x)$$

depends, through the functions $\gamma(g)$, $\beta(g)$, $\eta(g)$, on the detailed dynamics of the theory.

If the r.h.s. of equation (2.7) would have been negligible at short distances, then the anomalous dimension of $\phi$ could have been defined as

$$(2.9) \qquad d = 1 + \gamma(g)$$

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(x) - Obviously this is the form of the naive W.I. when $\beta(g) = \gamma(g) = 0$; $\eta(g) = -1$ which would have been obtained by using formal manipulations.

In general this is not the case, since $\theta(x)$ is quadrilinear in $\phi$ and then the Weinberg argument no longer holds.

If however the physical value of the coupling constant g were at a zero of $\beta(g)$ (perhaps for some self-consistency reason, as proposed by Wilson[29]), then the theory would asymptotically be dilatation invariant with anomalous dimensions given by eq. (2.9). This hypothesis turns out to be too restrictive: we do not need to know if the real world is actually sitting on a zero of $\beta$ but simply that such a zero does exist[4,5], even if it is different from the actual value.

From the explicit solution of the Callan-Symanzik equation (2.7) it is then possible to see that in this case the asymptotic form of $G_n$ approaches the solution of the homogeneous equation, which scales with anomalous dimensions fixed by the position of the zero.

As a remark, we note that it is possible to accomodate in this scheme also "dilatation multiplets"[23,30-32] as defined in the Appendix: their Callan Symanzik equation becomes a matrix equation and, interestingly enough, allows for logarithmic terms in the solution, without destroying the underlying dilatation invariance. The dynamical requirements in order to have operators transforming in this way are, however, rather obscure.

Coming back to the Callan Symanzik equation, it has been noted[4-5] that in the presence of a zero of $\beta$ its asymptotic form is identical to the equation derived by Gell-Mann and Low for the renormalization group[32].

This in turn implies that the asymptotic deep Euclidean limit is equivalent to the zero mass Gell-Mann Low limit (i.e. a zero mass limit performed in the renormalization procedure in such a way that infrared divergences are not introduced).

The way is now open to give meaning to asymptotic conformal invariance: the reason why the formal argument leading from dilatation to conformal symmetry can in principle not to be valid is that it is not possible to exclude "a priori" that the relation (2.4) is modified by non-soft terms, i.e. terms containing the fourth power of the field.

This happeans to be exactly the case: but it has been proved by Schroer[8] and later by Parisi[9], that one can identify $\theta(x)$ with $\theta_\lambda^\lambda$, without any extra term, and that the "true" Ward identities read

$$\sum_{i=1}^{n} \left[ x_{i\mu}(x_i^\lambda \frac{\partial}{\partial x_i^\lambda}) - x_i^2 \frac{\partial}{\partial x^{i\mu}} - 2(1+\gamma(g)) \right] \langle 0| T\left[ \phi(x_1)\ldots\phi(x_n) \right] |0\rangle =$$

(2.10)

$$= 2 \int d^4 x\, x_\mu\, \langle 0| T\left[ \phi(x_1)\ldots\phi(x_n)\theta(x) \right] |0\rangle$$

So, asymptotic conformal invariance is regained in much the same way as dilatation invariance is: namely only if a zero of $\beta$ (g) exists, the zero mass limit of the theory can be conformal invariant. In the follo wing sections we will always make the more restrictive hypothesis that we are studying a theory exactly at the zero of $\beta$ .

## 3. - SOME RESULTS FROM EXACT CONFORMAL INVARIANCE: WIGHTMAN FUNCTIONS AND OPERATOR PRODUCTS. -

Consider now a zero mass (Gell-Mann Low) field theory in which conformal invariance is a "true" symmetry (i.e. $\beta = 0$).

The transformation laws of fields are then determined, and this puts many restrictions on the associated Green's functions[10].

An instructive example comes from the study of two point func- tions (e.g. Wightman functions)

$$(3.1) \qquad G_2(x, y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle = G_2(x-y)$$

Simple algebra, using dilatation invariance, shows that

$$(3.2) \qquad G_2(x-y) = c \left[ (x-y)^2 + i \, \varepsilon \, (x_0 - y_0) \right]^{-1}$$

with c an unspecified constant, and l the dimension of $\phi$.

When $l = 1$, the two point function is the free one, so the field itself is free[33, 34]: we are thus interested in examining the case in which $l \neq 1$. The spectrum condition requires $l \geqslant 1$: so, in all interesting cases the matrix elements of fields decrease asymptotically (in the LSZ sense) faster than the particle factors in the reduction formulas, and con sequently particle interpretation is lost (infraparticle theory[17] in the language of Schroer).

Taking the discontinuity of $G_2$ we have the V.E.V. of the com mutator, which, if dimensions are not integer, has the form

$$(3.3) \qquad G_c(x-y) = c \, \theta(x-y)^2 \, \varepsilon(x_0 - y_0) \left[ (x-y)^2 \right]^{-1}$$

while it is proportional to derivatives of delta-functions when the dimen sions are integer.

This fact exhibits the most serious difficulty coming from con

formal invariance: in fact it is obvious that, if it a exists unitary opera tor $U(g)$ that represents conformal transformations $g$ (which can mix spa ce-like with time-like intervals) we can have, in the commutator

$$(3.4) \qquad U(g) \left[ \phi(x), \phi(y) \right] U^{-1}(g) = S(g, x) S(g, y) \left[ \phi(g^{-1}x), \ \phi(g^{-1}y) \right]$$

a violation of causality, if $l \neq$ integer (i. e. the commutator has support also inside th light cone).

The conclusion is that no unitary $U(g)$ exists at least in a strict sense; a more careful examination reveals that what is not left invariant is the boundary (i $\varepsilon$ ) prescription[35], while the functional form of vacuum expectation values is left invariant.

The concept of "weak conformal invariance" as proposed by Hor tacsu, Schroer and Seiler has its basis from this circumstance. They pro pose not to consider conformal transformations as a symmetry for opera tors, but rather for Green's functions continued into the complex domain of coordinates (e. g. by Wick rotation) where no causality problem arises.

The quantities defined in the pseudo-euclidean space-time are then obtained as different boundary values of the above defined functions.

In the case of operator quantities one has to prove that the appli cation of conformal invariance does not lead to wrong conclusions, by ta king the appropriate matrix elements: in this way one can prove, as we shall see later on, that in fact the kind of invariance relevant for operator products is the weak one[36].

With this in mind, we can proceed in the study of the two point function. Since its functional form is completely determined by dilatation invariance, the requirement of being also conformal covariant is either identically verified, or it implies that $c = 0$.

A selection rule in fact can be shown to hold, namely that the (conformal invariant) two-point function of every pair of operators is zero, if they do not belong to the same representation of the conformal group[37, 38].

Translated for Lorentz tensors (i. e. symmetric and traceless) this means that:

$$(3.5) \qquad \langle 0 | O_{\alpha_1 \ldots \alpha_n}(x) O_{\beta_1 \ldots \beta_m}(0) | 0 \rangle = 0$$

if $n \neq m$ or $l_n \neq l_m$.

8.

The properties of two point functions are then completely speci_
fied by conformal invariance, so also the currents conservation must be
implicit, whenever it is required by the theory. It is a simple matter
to prove that, in fact, the generalized conservation laws[36],

(3.6)
$$\partial_x^{\alpha_1} \langle 0 | O_{\alpha_1 \ldots \alpha_n}(x) O_{\beta_1 \ldots \beta_n}(y) | 0 \rangle = 0$$

hold, if and only if, the dimensions of the tensor operators are "canoni_
cal" i.e.

(3.7)
$$l_n = 2 + n$$

in other words the same dimensions as they have in a free field theory.

Conservation in the operator form then follows[20]

(3.8)
$$\partial^{\alpha_1} O_{\alpha_1 \ldots \alpha_n}(x) = 0$$

if the metric of the Hilbert space of the theory is positive definite; in
the following we will see the implications of this result.

The above relations are a particular case of a more general
statement that can be phrased in the following way: any n-point Green's
function that transforms covariantly under the conformal group, depends
on an arbitrary function[10] of n(n-3)/2 (if n 6) or 4n-15 variables of
the form

(3.9)
$$\omega_i = \frac{(x_i - x_{i+1})^2 (x_{i+2} - x_{i+3})^2}{(x_i - x_{i+2})^2 (x_{i+1} - x_{i+3})^2}$$

The aforementioned selection rule is a particular case of this
result: in fact, for n = 2, we have n(n-3)/2 = -1, i.e. the function is over-
-determined.

Analogous, and more stringent, conclusions can be stated for
fields belonging to "dilatation multiplets" from the study of their two point
functions. Even the definition of such fields in fact loses sense, if one
insists that the following conditions be verified: $D | 0 \rangle = 0$ and $K_\mu | 0 \rangle = 0$
and positivity in the Hilbert space. So one is led to the conclusion that
in this case conformal invariance is (possibly) realized in a spontaneously
broken way.

When n = 3 (n(n-3)/2 = 0) the Green's function is completely deter-

mined, apart from an arbitrary multiplicative constant: for scalar fields we have, for example

$$(3.10) \quad \langle 0|\phi(x_1)\phi(x_2)\phi(x_3)|0\rangle = c'\left[(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2\right]^{-1/2}$$

The four point function ($n = 4$, $n(n-3)/2 = 2$) is fixed up to an arbitrary function of two "harmonic ratios"

$$\langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = \left[(x_1-x_3)^2(x_2-x_4)^2\right]^{-1} F(\eta, \varrho)$$

where the variables $\eta$, $\varrho$ are of the kind described above.

It is not possible to put further restrictions on F, unless one makes some dynamical assumptions, i.e. one specifies better the model one is considering. As an example we can consider the model proposed by Migdal[38], in which the field $\phi(x)$ is composed by n free fields

$$(3.11) \quad \phi(x) = \psi_1(x). \ldots \psi_n(x)$$

and analytical continuation is performed in n; this model, which is interesting for other reasons, is trivial from this point of view, in that the model is soluble and all the n-point functions are determined.

Another possible approach will be pointed out in the next Section: we emphasise that a conformally invariant four point function has no clear cut relation with quantities such as for example the structure functions in deep inelastic scattering, due to the lack of an LSZ particle interpretation in all the physically interesting cases.

A bridge to this kind of applications is provided by the well-known expansion[9,39,40] for the product of two or more local operators

$$(3.12) \quad A(x)B(0) \underset{x^2 \to 0}{\sim} \sum_n c_n(x^2) x^{\alpha_1} \ldots x^{\alpha_n} O_{\alpha_1 \ldots \alpha_n}(0)$$

where the c-number functions $c_n(x^2)$ have singularities at short and light-like distances governed by the dimensions of the fields

$$(3.13) \quad c_n(x^2) = a_n(x^2+i\varepsilon x_0)^{-\frac{1}{2}(1_A+1_B-1_n+n)}$$

Conformal invariance can then be required to hold on both sides of eq. (3.12), so giving relations inside "towers" of operators forming an

irreducible representation of the conformal group[41,42], i. e. an operator $O_{\alpha_1 \ldots \alpha_n}(x)$ transforming as in equation and its derivatives: at the leading order in $x^2$ these relations take the form [+](x)

$$A(x)B(0) \sim \sum_n c_n(x^2) x^{\alpha_1} \ldots x^{\alpha_n} \int_0^1 d\lambda \; \lambda^{a-1}(1-\lambda)^{b-1} O_{\alpha_1 \ldots \alpha_n}( \; x)$$

(3.14)

$$a = \frac{1}{2}(1_A - 1_B + 1_n + n), \qquad\qquad b = \frac{1}{2}(1_B - 1_A + 1_n + n)$$

The expansion (3.14) automatically solves some problems left open by eq. (3.12): for instance that of causality, in the sense that

(3.15)
$$\left[A(x)B(0), C(z)\right] = 0 \qquad \text{if} \qquad \begin{array}{l} (x-z)^2 < 0 \\ z^2 < 0 \end{array}$$

on both sides of the Wilson expansion: this is a dynamical requirement for equation (3.12) while it is automatically verified for every n in the conformal invariant expression.

Operator product expansions can also be derived that account for conformal invariant relations between leading and non-leading terms in $x^2$: their derivation is very difficult, since one cannot use only the conformal algebra, but rather the action of finite transformations[44,45] or the 6-dimensional formalism[43]. The two approaches have been proved to lead to equivalent results, at least in the Euclidean region[46]; a lot of care must be taken in order to translate these results into the ordinary Minkowski space[47].

The use one can make of these "whole space" expansions[o] is

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(+) - The validity of the Wilson expansion has been proved also in the Callan-Symanzik framework, if a zero of the $\beta(g)$ exists. Also the equation (3.13) can be verified in this case, as a relation between leading and next-to-leading terms (in x) when the short distance limit is approached in the Euclidean region[24,25].

(x) - The $c_n(x^2)$ have the form (3.13), note that the constants an are not specified by conformal invariance.

(o) - For the validity of these expansions we have to require that either the physical world is at a zero of $\beta(g)$ or that $\beta(g) \equiv 0$, which happens to be the case in the two-dimensional Thirring model

mainly related to the study of the structure of the four[37] (and higher order[48]) functions: the insertion of operator product expansions in V.E.V.'s in fact allows one to reduce a higher order function to a sum of smaller order ones. However the causality requirements are not easy to verify for a "whole space" expansion, so the best one can hope to prove is that e.g. the four point function is causal[49].

An important feature one can easily check is that the invariance of the operator product expansions is equivalent to the invariance of the two resp. three point functions plus the selection rule[43,50]: in this sense Wilson expansions are "weakly" conformal invariant.

As a consequence normalization factors are connected each other: for example, from eqs. (3.2), (3.10) and (3.13) we obtain

$$c' = c \, a_0$$

This fact is important for phenomenological applications, as these normalization factors can be possibly measured in experiments.

Let us illustrate this possibility by briefly describing the result derived, in a slightly different framework, by Crewther[18]. The use of Wilson expansions for e.m. and axial currents

$$T(J_\mu(x)J_\nu(0)) = R(g_{\mu\nu}x^2 - 2x_\mu x_\nu)I(\pi x^2)^{-4} + K \, \varepsilon_{\mu\nu\varrho\sigma}x^\varrho A^\sigma(0)(3\pi^2 x^4)^{-1}$$

(3.16)

$$T(A_\mu(0)A_\nu(y)) = R'I(g_{\mu\nu}y^2 - 2y_\mu y_\nu)(\pi y^2)^{-4} \quad \text{(I is the identity operator)}$$

and conformal invariance for the 3-point function allows one to write the anomalous Adler's constant (which is fixed by PCAC and $\pi^0 \to \gamma \, \gamma$ decay: $S \sim 0.5$)

$$(3.17) \qquad S = -\frac{1}{12} \, \pi^2 \, \varepsilon^{\mu\nu\alpha\beta} \int d^4x \, d^4y \, x_\alpha \, y_\beta \langle 0| J_\mu(x)J_\nu(0) \, \partial^\lambda A_\lambda(y) | 0 \rangle$$

as

$$(3.18) \qquad 3S = K \, R'$$

where K can be measured in polarized deep inelastic scattering and R' is connected through asymptotic chiral invariance to the electron-positron total cross section.

Similar relations can also be found for other off mass-shell processes[19], like $e^+e^- \to e^+e^- + \text{anything}$.

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Wilson expansion can also be used to transfer properties coming from conformal invariance to on mass-shell matrix elements such as the pion form factor[12, 51]; unfortunately, application to the most interesting case, namely the deep inelastic electroproduction, is hopeless, since in the matrix, element between equal momentum states all the derivative of operators disappear and an infinite sum over inequivalent representations of the conformal group remains

$$(3.19) \qquad \langle p|J_\mu(x)J_\nu(0)|p\rangle = \sum_{\substack{x^2 \to 0 \\ n}} c_n(x^2)\langle p|O_{\alpha_1 \cdots \alpha_n}(0)|p\rangle \, x^{\alpha_1} \ldots x^{\alpha_n}$$

The observed scaling in deep inelastic experiments implies that an infinite number of local tensors contribute with "canonical dimension" $l_n = 2 + n$, so that

$$(3.20) \qquad c_n = a_n(x^2 + i\varepsilon x_0)^{-1/2(l_A + l_B - 2)}$$

Conformal invariance in turn implies that the operators are conserved. We have then this situation: dilatation invariance at short distances implies conformal invariance - scaling plus conformal invariance constrain the limiting (Gell-Mann Low) theory to posses an infinite number of conserved tensor quantities. It has been proved[52] that the infinite conservations laws in simple models (as the $g\phi^4$) mean that the theory, though formally interacting, is free.

However this is not always the case in more complicated theories, as the above mentioned model proposed by Migdal[38]; there the conservation laws are the consequences of the conservation of the "type" of fundamental constituents $\psi_n(x)$.

It is however necessary to note that we should perhaps have put a question mark on the statement of the "observed scaling": in fact recent semi-empirical[53, 54] analysis have shown that the observed data are also consistent with anomalous dimensions of the type

$$(3.21) \qquad l_n = 2 + n + a\left[1 - \frac{12}{(n+1)(n+2)}\right]$$

which have been suggested also from theoretical arguments[55-57].

## 4. - THE BOOTSTRAP APPROACH. -

The considerations developed in the preceding Sections are in some sense the "kinematics" of conformal invariance: quantities such as anomalous dimensions or normalizations are not specified and not con strained by the theory itself.

An interesting tool for studying dynamical properties of the theo ry is provided for by the bootstrap approach, first proposed by Migdal[13] and Polyakov[11] and by Parisi and Peliti[14]. We can consider this ap proach as complementary to the study of "kinematical" properties of a conformal invariant field theory, in the sense that it starts from the kno wledge of the structure of the Wightman (specifically 2 - and 3 - point) func tions.

Let us take as an example the case of the 3-point bootstrap equa tions, closely following the original treatment of Migdal[13].

Consider a theory with a conformal invariant threelinear interac tion, such as the $g\phi^3$ in 6-dimensional space-time (or the ps-n interaction in four dimensions): two and three point functions are functionally determi ned and we can make the hypothesis that all n-point functions have a ske leton expansion[15, 58]

$$(4.1) \qquad \text{}$$

that is an expansion in Feynman diagrams without self energy parts and vertex corrections, in which propagators and vertices are the "true" con formal invariant ones.

This kind of expansion can then be used also for the Bethe-Salpe ter kernel that appears in the renormalized Schwinger-Dyson equation for the three point function (inhomogeneous terms are absent since they would violate dilatation invariance)

$$(4.2) \qquad \text{}$$

where  means that the B.S. kernel is two particle irreducible.

We know that conformal invariance fixes the form of propagators and vertex functions appearing in the expansion (4.1) up to normalization factors. It can be shown that all the normalization factors can be reabsor bed in the normalization of the vertex function (the "coupling constant"). So we are left with an integral equation that, whenever the integrals in

14.

volved in eq. (4.2) do not introduce divergences, so spoiling its meaning, becomes an algebraic equation in the unknown coupling constant and anomalous dimensions[16,58] which are the only parameters left: since the equation is non-linear one can hope for at most a discrete set of solutions.

As for the propagator, since unitarity is not verified term by term in the skeleton expansion, this requirement can be adopted as a basis for bootstrap equations (note that it is sufficient to know the imaginary part of the propagator)[14]:

(4.3)    Im $\,$ ⊸———⊸ $\;=\;\sum_n\;$ 

Also a Bethe-Sapeter approach can be used[16].

The main point for the validity of the previous statements deeply relies on the absence of divergences: this has been proved by Mack and Todorov[15] for some range of the value of the dimensions. Up to now there is no solution of the bootstrap equations in 4 dimensions, whereas solutions have been found in the framework of calculations of critical indices[16], and, through analytical continuation in the dimension of the space-time, in $6 + \varepsilon$ dimensions[56].

The other hard problem in this approach is the verification of the Ward identities when the vertex functions contain currents or the energy momentum tensor: a detailed treatment of this point can be found in the paper by Mack and Symanzik[16].

Finally, as has been pointed out by Mack[53], the bootstrap approach is also important because it can be thought as a constructive way to the Gell-Mann Low limiting zero mass theory: it is clear in fact that conformal invariance is maintained in every step of the procedure so getting at the end a dynamical and self consistent invariant theory.


5. - TWO DIMENSIONAL MODELS. -

Two dimensional field theory has often been used as a laboratory for the study of quantum field theory: as an example the Thirring model[59,60], being soluble in its zero mass limit, allows to study many formal properties of q.f.t.

From the point of view of conformal invariance, such models stem from a twofold interest: the structure of conformal transformations is rather peculiar in two dimensions[61] and, secondly, in the Thirring model it can be proved that the Callan Symanzik function is identically zero[62], so the theory is exactly conformal invariant in the Gell-Mann

Low limit.

As for the peculiarity of conformal group in two dimensions we can see, from the Appendix, that in the defining equations $D = 2$ is an exceptional case (see Eq. A.3b) : in fact $\lambda(x)$ is no longer constrained to be linear in $x$, but has rather to verify the equation

(5.1) $$\square \; \lambda(x) = 0 \; .$$

The full conformal group becomes then infinite dimensional : i. e. every harmonic function can generate conformal transformations. Of course this algebra contains a sub-algebra which is the restriction of that in 4 dimensions ( $0(4,2)$ ), that is $0(2,2)$ : this algebra plays a special role, as we shall see latter.

The structure of the generators, the transformation laws of the fields and so on, can be easily derived in general, but are better illustrated in the framework of the Thirring model.

Here, as well known, we have a self interacting "spinor" field with equations of motion of the form

(5.2) $$\not{\partial} \; \psi(x) = g \; :\not{\partial}\psi: (x) \qquad (J_\mu = \; : \overline{\psi}(x) \, \gamma_\mu \, \psi(x) : )$$

Here the canonical approach, starting from the commutation relations of the fields can be substituded with some advantages by an alternative one that uses the currents

(5.3) $$J_\mu(x) \quad \text{and} \quad J_\mu^5(x) = \varepsilon_\mu{}^\nu \, J_\nu(x) = \; : \psi \, \gamma_5 \gamma_\mu \, \psi : (x)$$

$$\varepsilon_{o1} = - \, \varepsilon_{1o} = - 1$$

as dynamical quantities.

The relation between the axial and vector current, which is a consequence of the fact that the structure of the Dirac matrices is poorer than in 4 dimensions, is an important feature of the model. In fact, together with the conservation of both currents (which can be proven in the zero mass limit), it implies that the currents fulfill a free Klein Gordon equation, which makes fromal manipulations very easy to handle, and, more significantely, allows to prove that the Callan Symanzik function is zero for every value of the coupling constant[62]. This is a very crucial point : it means in fact that the short distance limiting theory is exactly dilatation and conformal invariant. Of course the conformal group here implied is the "restricted" ( $0(2,2)$ ) one, as we shall better see later.

Coming back to the "extended" algebra, simple computations allow to write its generators in terms of the currents[61], via the energy momentum tensor

$$(5.4) \qquad L(f) = \int dx^1 \, \theta_{o\nu}(x) \, f^{\nu}(x)$$

Here $f^{\nu}(x)$ is the generating function of infinitesimal conformal transformations, connected to $\lambda(x)$ through eq. (A.3a) and:

$$(5.5) \qquad \theta_{\mu\nu}(x) = :J_{\mu}(x) J_{\nu}(x): - \frac{1}{2} g_{\mu\nu} : J_{\alpha}(x) J^{\alpha}(x):$$

It is easy to see that, using the standard light cone variables

$$(5.6) \qquad u = x^0 + x^1; \qquad\qquad v = x^0 - x^1$$

the whole algebra splits into the direct product of the u and v part (in the same way as the 0(2,2) subalgebra splits into 0(2,1)$\otimes$0(2,1)) which in a particular basis can be expressed as

$$(5.7) \qquad \begin{aligned} L^u_n &= \int du \, \theta_+(u) \, u^{1-n} & (\theta_+ = \theta_{oo} + \theta_{o1}) \\ L^v_n &= \int dv \, \theta_-(v) \, v^{1-n} & (\theta_- = \theta_{oo} - \theta_{o1}) \end{aligned}$$

with commutation relations

$$(5.8) \qquad \left[ L^u_n, L^u_m \right] = (m-n) L^u_{n+m} + c\text{-number}; \qquad \left[ L^u_n, L^v_m \right] = 0$$

Amusingly enough, this is the same "gauge" algebra that appears in the dual models[63].

We have anticipated that only 0(2,2) is the invariance group: in fact, while the equations of motion are formally invariant under the whole algebra, Wightman functions are invariant only under 0(2,2), so resembling a situation in which the symmetry is spontaneously broken.

Finally, the fact that $\beta(g) \cong 0$, i.e. the exact invariance of the zero mass theory, allows to write a "whole space" operator product expan

sion; it turns out that the expansion originally proposed by Dell'Antonio, Frishman and Zwanzinger[64] is invariant not only under $O(2,2)$ transformations, but also under the whole algebra.

## 6. - CONCLUSIONS. -

In the preceeding sections we have tried to illustrate some of the most relevant results that follow from the assumed conformal invariance of field theory: they are encouraging, mainly in the direction of obtaining a self contained theory that can be thougt as a limit of realistic theories. However the relevance to the real world is not clear and normally requires addictional hypotesi, as for instance operator product expansion.

There are of course many other ways in which this argument can be studied: as an example we can quote the work presented by R. Nobili to this Conference[65], in which broken dilatation invariance (i.e. partial conservation of the dilatation current) is used to derive constraints an Green's functions.

Beyond these applications, there is a number of purely theoretical problems left open, and the most important one is perhaps that of the formulation of quantum electrodynamics in a conformal invariant way.

In fact, although conformal invariance found its first physical application as the symmetry group of the Maxwell equations, its extension to the quantized theory is very difficult due to the fact that conformal and gauge transformations are strictly connected.

An important achievement in this direction has been found by Adler[66], with the formulation of a conformal quantum electrodynamics for massless particles, not taking into account photon-self energy parts.

18.

APPENDIX. -

Conformal transformations are defined as the space-time tran sformations that leave the metric tensor (in the flat space) invariant up to multiplication by a scalar function of the coordinates:

$$(A.1) \quad x_\mu \rightarrow x'_\mu : \quad g'_{\mu\nu}(x) = \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\varrho}{\partial x'^\nu} g_{\lambda\varrho} = \Lambda(x) g_{\mu\nu}$$

Their name comes from the fact that they leave the cosine of infinitisimal angles invariant.

In the following, unless otherwise specified, we will consider a D-dimensional space-time: in it infinitesimal transformations

$$(A.2) \quad x'_\mu \simeq x_\mu + \delta x_\mu(x); \quad \Lambda(x) \simeq 1 - \lambda(x)$$

are constrained to obey the equations

$$(A.3a) \quad \partial_\mu \delta x_\nu + \partial_\nu \delta x_\mu = g_{\mu\nu} \lambda(x)$$

$$(A.3b) \quad \left[ (D-2) \partial_\lambda \partial_\varrho + g_{\lambda\varrho} \Box \right] \lambda(x) = 0$$

which, for $D \neq 2$, fix their form to be

$$(A.4) \quad \delta x_\mu = \delta a_\mu + \delta \omega_{\mu\nu} x^\nu + x_\mu \delta \varrho + (x^2 \delta c_\mu - 2(x \cdot \delta c) x_\mu)$$

Being $\delta \omega_{\mu\nu}$ antisymmetric, $\delta x$ depends on $(D+1)(D+2)/2$ parameters, which is appropriated for a $0(D,2)$ group: in the usually considered case $D=4$ and the group becomes $0(4,2)$. An interesting feature is that $0(4,2)$ does not act linearly in the Minkowski space, but rather in a 6-dimensional space[1a, 1b]: so conformal invariance can be induced through a set of linear relations and the results mapped in a well-defined way into the physical space-time.

Going back to eq. (A.4) we can easily identify the first two terms as infinitesimal Poincare transformations while the other two correspond to dilatation and special conformal transformations, that in the finite form read

$$x'_\mu = \rho x_\mu \qquad : \qquad \text{dilatations}$$

(A.5)

$$x'_\mu = \frac{x_\mu - c_\mu x^2}{1 - 2c \cdot x + c^2 x^2} \qquad : \qquad \text{spec. conformal transformation}$$

Note that all these transformations leave the light-cone ($x^2 = 0$) invariant.

Action of transformation on fields is related to the existence of unitary operators such that

(A.6) $$U(g)\,\phi(x)\,U^+(g) = S\,(g,x)\,\phi(x)$$

(g is a member of the group)

with multiplication law

$$S\,(gg',x) = S\,(g,x)\,S\,(g',\,g^{-1}x)$$

Here $\phi(x)$ stays for an arbitrary collection of fields.

It turns out to be more convenient to deal with infinitesimal transformations, i.e. representations of the algebra: they can be found with the help of the standard induced representations method.

For our purposes we need to consider only dilatations and special conformal transformations.

For dilatations we have

(A.7) $$\left[\phi(x), D\right] = i(x^\lambda \partial_\lambda + L)\,\phi(x)$$

where the matrix L can be put in the block-diagonal form

(A.8) $$L = \begin{bmatrix} \boxed{\lambda_1} & & \\ & \ddots\,\lambda_i & \\ & & \boxed{\lambda_n} \end{bmatrix}$$

and the matrices $\lambda$ have the Jordan form

20.

(A.9)
$$\lambda_j = \begin{bmatrix} 1_j & & & \\ & 1 & \ddots & \\ & & \ddots & \ddots \\ & & & 1 \; 1_j \end{bmatrix}$$

If the rank of the j-th block is 1, i.e. $\lambda_j = 1_j$, then the j-th field is said to have "scale dimension" $1_j$; in the other case the fields belonging to the block are said to form a "dilatation multiplet" and $1_j$ is called "diagonal dimension"[30-32].

For conformal transformations we have:

(A.10) $\quad [\phi(x), K_\mu] = i(2x_\mu \, x^\lambda \partial_\lambda - x^2 \partial_\mu - 2x^\lambda \, [g_{\mu\lambda} L + i \Sigma_{\mu\lambda}]) \phi(x) + \widetilde{K}_\mu \phi(x)$

($\Sigma_{\lambda\sim}$ is the spin part of the Lorentz generator $M_{\mu\nu}$)

where the form of $\widetilde{K}_\mu$ is roughly similar to that of L; however all the physically interesting cases are that for which $\widetilde{K}_\mu = 0$.

Finally we list the commutation relations of the conformal algebra:

(A.11)
$$[K_\mu, M_{\lambda\nu}] = i(g_{\mu\lambda} K_\nu - g_{\mu\nu} K_\lambda); \qquad [D, M_{\lambda\mu}] = 0$$
$$[K_\mu, P_\nu] = -2i(g_{\mu\nu} D + M_{\mu\nu}); \qquad [D, P_\mu] = -i P_\mu$$
$$[K_\mu, K_\nu] = 0; \qquad [K_\lambda, D] = -iK_\lambda$$

plus that of the Poincarè algebra.

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