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LOGARITHMIC SCALING AND SPONTANEOUS BREAKING

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Indecomposable representations of dilatations allow for logarithms in scale invariant operator product expansion. We prove that in absence of spontaneous breaking, they are incompatible with conformal invariance and positivity.

It has been pointed out that logarithmic singularities in Wilson expansion may occur in dilatation invariant theories under spontaneous breaking [1], or through occurrence of indecomposable representations of dilatation [2]. Here we prove a theorem which states the incompatibility of such representations with exact invariance under conformal algebra and positivity. The theorem applies to the skeleton theory, in Wilson's sense [3], assumed to be invariant under conformal algebra. If indecomposable representations of dilatation occur in the skeleton theory (and therefore logarithms appear in the light-cone expansion) spontaneous breaking must take place.

We shall first describe indecomposable representations and the appearance of logarithms; then enlarge to conformal invariance and prove the theorem stated. We shall also discuss the inclusion of such representations in the conformal covariant operator product expansion [4] also in connection to causality. The manifestly conformal covariant description in six dimension will also be reported.

Indecomposable representations of the dilatation algebra. The study of the representations of the dilatation group is elementary. Under dilatation $x \to \sigma x$ the local operators $O_i^{\{\mu\}}(x)$ (i runs within the representation and $\{\mu\}$ summarizes the Lorentz indices) transform into $\mathcal{D}_{ij}(\sigma)\, O_j(\sigma x)$. The infinitesimal law is therefore ($\boldsymbol{O}(x)$ is one-column vector)

$$[\boldsymbol{O}^{\{\mu\}}(x), D] = \mathrm{i}(L + x^{\lambda}\partial_{\lambda})\, \boldsymbol{O}^{\{\mu\}}(x) \qquad (1)$$

where D is the generator and L is a numerical matrix. A finite dimensional L can be reduced to diagonal and Jordan matrices, corresponding respectively to (one-dimensional) irreducible- and indecomposable-representations. The Jordan matrix of order n is the $n \times n$ matrix

$$L = \begin{bmatrix} l & & & 0 \\ 1 & l & & \\ & \cdot & \cdot & \\ 0 & & 1 & l \end{bmatrix} \qquad (2)$$

where l is a real parameter ("diagonal scale dimension"). Indecomposable representations (n, l) are spanned by "dilatation multiplets" $\boldsymbol{O}^{\{\mu\}}$ $(1 \leqslant i \leqslant n)$ satisfying (1) and (2). Irreducible representations may then be considered as particular cases $(n = 1)$.

Dilatation invariance at short distance is conveniently studied à la Wilson. Suppose $A_i(x)$ $(i = 1, \ldots, n_A)$ belongs to (n_A, l_A), $B_j(x)$ to (n_B, l_B), and $O_k(x)$ to (n_O, l_O). The leading contribution from $O(x)$ at short distance is

$$A_i(x) B_j(0) = \qquad (3)$$
$$= (\text{const.})\, (x^2)^{-\frac{1}{2}(l_A + l_B - l_O)} (\log x^2)^{i+j+n_0-3}\, O_1(0) + \ldots$$

The remarkable aspect is the appearance of the logarithm whenever any of i, j, n_0 is > 1, i.e. whenever a truly indecomposable $(n > 1)$ representation contributes. Eq. (3) follows from commuting both sides with D, according to eq. (1). Logarithms also appear in the Wightman functions $W_{ij}^{\{\mu,\nu\}}(x) = \langle O_i^{\{\mu\}}(x)\, O_j^{\{\nu\}}(0)\rangle_0$. One finds (we abrreviate $x^2 - \mathrm{i}\epsilon x^0$ as x^2), e.g. for scalar fields,

$$W_{11}(x^2) = c_{11}(x^2)^{-l} \quad (4a)$$

$$W_{12}(x^2) = (-\tfrac{1}{2}c_{11}\log(x^2)+c_{12})(x^2)^{-l} \quad (4b)$$

$$W_{21}(x^2) = (-\tfrac{1}{2}c_{11}\log(x^2)+c_{21})(x^2)^{-l} \quad (4c)$$

$$W_{22}(x^2) = = [\tfrac{1}{4}c_{11}\log^2(x^2) - \tfrac{1}{2}(c_{12}+c_{21})\log(x^2)+c_{22}](x^2)^{-l} \quad (4d)$$

etc. where c_{mn} are constants. This follows by inserting eq. (1) into $\langle [D, O_i(x)\,O_j(0)]\rangle_0 = 0 = \langle [D, O_i(x)]\, O_j(0) + O_j(x)[D, O_j(0)]\rangle_0$ and solving the resulting differential equations, where we have assumed

$$D|0\rangle = 0 \quad (5)$$

(dilatationally invariant vacuum).

Enlargement to conformal algebra. For Lorentz-irreducible multiplets $[O^{\{\mu\}}(0), K_\lambda] = 0$ (K_λ is the generator of special conformal transformations). Besides eq. (1) one now has

$$[O^{\{\mu\}}(x), K_\lambda] \quad (6)$$
$$= i(2x_\lambda x\cdot\partial - x^2\partial_\lambda - 2ix^\rho\Sigma_{\rho\lambda} + 2Lx_\lambda)\,O^{\{\mu\}}(x)\,.$$

Indecomposable representations with respect to K_λ (for which $[O, K_\lambda] \neq 0$) can similarly be treated. We now show the following theorem,"Conformally invariant theories, with invariant vacuum state, and containing indecomposable dilatation multiplets are incompatible with positivity."

Proving this we first consider scalar fields for simplicity. Eqs. (4) follow from eqs. (1) and (5). Suppose $[O(0), K_\lambda] = 0$. From $\langle [K_{\lambda 1}\, O_i(x)\, O_j(0)]\rangle_0 = 0$, using eq. (5) one obtains $(2x_\lambda x\cdot\partial - x^2\partial_\lambda + 2lx_\lambda)\, W_{12}(x^2) = 0$ and, comparing with eq. (4b), one finds $c_{11} = 0$. Therefore $W_{11}(x^2) = 0$ and from positivity one then concludes $O_1(x) \equiv 0$. But then $[O_2(x), D] = i(l+x\partial)O_2(x)$ behaves either irreducibly or as the lowest component of an indecomposable multiplet. In the latter case however the proof can be repeated.

Generalization to higher Lorentz tensors (or spinors) is immediate: it is enough to commute with the longitudinal component $(x\cdot K)$. Generalization to indecomposable representations of K_μ is also obvious, it is enough to consider the irreducible component of lowest scale dimension.

Note that, in addition to (5), we have now used

$$K_\lambda|0\rangle = 0. \quad (7)$$

Therefore non-validity of eq. (5), or of eq. (7), or of both would invalidate the proof (and the vacuum would not be invariant).

A conformal covariant light-cone expansion can be derived for multiplets satisfying eqs. (1) and (6), extending preceding results [4]. It must be stressed that, according to the theorem we just have proved, the occurrence of "dilatation multiplets" in the skeleton theory (i.e. on the Light-cone) requires either absence of positivity or, more physically, spontaneous breaking of conformal symmetry. Operator product expansions may loose any sense in such cases. Nevertheless one can formally obtain a conformal covariant operator product expansion on the light-cone, with dilatation multiplets, and which exhibits manifest causality.

A manifestly conformal covariant formalism can conveniently be set up to describe dilatation multiplets. This is realized with operator-valued representations of the spinor group SU(2,2) on the 6-dim. hypercone. To have a well-defined correspondence with space-time it is convenient to enlarge to representations ψ_i^α of $SU(2,2)\otimes\mathcal{D}$ where $\mathcal{D}$ is the group of dilatations on the hypercone. The general correspondence with operators in Minkowski space O_i^α is written in the form (at $x = 0$)

$$\psi_i^\alpha(0,\kappa) = T_{ij}(\kappa^{-1})\,O_j^\alpha(0). \quad (8)$$

We recall that [4] $\kappa = \eta_5 + \eta_6$, $x_\mu = \eta_\mu/\kappa$ where η^A ($A = 0,1,2,3,5,6$) are the 6-dim. coordinates taken along the hypercone $\eta_i^A\eta_A = 0$. The indices (α, i) in eq. (8) belong to representations of SU(2,2) and $\mathcal{D}$ respectively. Irreducible representations of dilatation are obtained for $T_{ij}(\kappa^{-1}) = \delta_{ij}\kappa^l$, where l is the dimension. In general, under dilatation $\psi_i^\alpha(0,\kappa)$ goes into $\Delta_\alpha^\beta(\sigma)\,\psi_\beta(\sigma^{-1}\kappa)$ where $\Delta_\alpha^\beta(\sigma)$ represents the SU(2,2) transformation corresponding to dilatation. Therefore $O_i^\alpha(0)$ goes into $T_{ij}(\sigma)\Delta_\beta^\alpha(\sigma)\,O_j^\beta(0)$. Indecomposible representations of D in Minkowski space are thus derived from the corresponding representation of $\mathcal{D}$ on the 6-dim. light-cone. The direct product form $SU(2,2)\otimes\mathcal{D}$ thus shows that any representations of Poincaré ⊗ dilatations, reducible irreducible or indecomposable under dilatation, can straightforwardly be enlarged to the whole conformal algebra.

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