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Conformal Algebra in Two Space-Time Dimensions and the Thirring Model.

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Summary. — The-infinitesimal generators of the infinite-dimensional conformal algebra in two space-time dimensions are constructed from the stress tensor. The conformal covariance of the operator product expansion in the Thirring model is verified, but the vacuum state is found to be invariant only under the $O_{2,2}$ subalgebra of the infinite conformal algebra.

In this note we shall point out some properties which emerge in the study of conformal algebra in one space and one time dimensions, and particularly the construction and implications of the algebra in the Thirring model (1). We obtain in particular that

i) in terms of the « improved » stress tensor (2) one can build up the infinitesimal conformal algebra, which is infinite dimensional and has the same structure as the « Virasoro algebra » of dual models (3,4);

(1) W. THIRRING: *Ann. of Phys.*, **3**, 91 (1958); K. JOHNSON: *Nuovo Cimento*, **20**, 773 (1961).
(2) C. G. CALLAN jr., S. COLEMAN and R. JACKIW: *Am. Journ. Phys.*, **59**, 42 (1970).
(3) See for example: G. VENEZIANO: *Progress in the theory and application of dual models*, MIT preprint (April 1971); V. ALESSANDRINI and D. AMATI: *Proc. S.I.F.*,

ii) the infinite conformal algebra (in whose definition the co-ordinate inversion (5) R plays an important role) has a model-dependent inhomogeneous term which vanishes in the commutators of the generators of the $O_{2,2}$ subgroup, and it contains besides $O_{2,2}\sim O_{2,1}\otimes O_{2,1}$, two infinite subalgebras $G_{\downarrow}$ and $G^{\uparrow}$, satisfying $G_{\downarrow}\cap G^{\uparrow}=O_{2,2}$;

iii) the operator product expansion as given by DELL'ANTONIO, FRISHMAN and ZWANZINGER (6) is explicitly covariant under the infinite algebra, but not fixed by covariance alone;

iv) the vacuum cannot be invariant except under the $O_{2,2}$ subalgebra.

In Sect. 1 below we summarize the main results on conformal covariance in two space-time dimensions.

1. – Conformal transformations in two space-time dimensions.

General conformal transformations can be defined as those transformations which multiply the infinitesimal line element $\mathrm{d}s^2$ by a factor depending on x, *i.e.* under $x_\mu \to x'_\mu$

(1) $$\mathrm{d}s^2 \to \mathrm{d}s'^2 = \lambda(x)\mathrm{d}s^2 \, .$$

The infinitesimal transformations obey the relation

(2) $$\partial_\mu \delta x_\lambda + \partial_\lambda \delta x_\mu = g_{\mu\nu}\delta\lambda \, .$$

In a two-dimensional space-time (2) implies

(3) $$\Box\, \delta\lambda = 0 \qquad \Box\, \delta x_\mu = 0 \, .$$

Whereas (2) is valid in general, (3) only holds in two dimensions.

The requirement that conformal invariance be a space-time symmetry generated (in a Langrangian framework) by a symmetric traceless stress ten-

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Course LIV (to be published), CERN preprint TH. 1925 (1971); E. DEL GIUDICE, P. DI VECCHIA and S. FUBINI: *General properties of dual-resonance models*, MIT preprint (June 1971), to appear in *Ann. of Phys.*, and references quoted therein.

(4) We are studying the implications of our result in connection with a recent proposal by E. DEL GIUDICE, P. DI VECCHIA, S. FUBINI and R. MUSTO: *Light-cone physics and duality*, MIT preprint (April 1972), to appear in *Nuovo Cimento*.

(5) S. FERRARA, R. GATTO, A. F. GRILLO and G. PARISI: *Lett. Nuovo Cimento*, **4**, 115 (1972).

(6) G. F. DELL'ANTONIO, Y. FRISHMAN and D. ZWANZIGER: preprint Weizman Institute, WIS 71/44-Ph.

sor is [7]

$$\partial_\mu(\theta^{\mu\nu}\delta x_\nu) = \theta^{\mu\nu}\partial_\mu \delta x_\nu = 0 \tag{4}$$

and it is equivalent to (2).

Equation (2) and (3) have the unique solution

$$\delta x_\mu \equiv f_\mu(x) = f^+_\mu(x^0 + x^1) + f^-_\mu(x^0 - x^1)\,, \tag{5}$$

where

$$f^+_0(x^0 + x^1) = -f^+_1(x^0 + x^1)\,, \tag{6a}$$

$$f^-_0(x^0 - x^1) = +f^-_1(x^0 - x^1)\,. \tag{6b}$$

It can be noted that a general co-ordinate transformation

$$x_\mu \to f_\mu(x)\,,$$

such that

$$u = x^0 + x^1 \to f(u)\,, \tag{7a}$$

$$v = x^0 - x^1 \to f(v)\,, \tag{7b}$$

is indeed a conformal transformation when conformal transformations are defined as those transformations which take lightlike events, $(x_1 - x_2)^2 = 0$, into lightlike events. In fact if

$$(u_1 - u_2)(v_1 - v_2) = 0$$

holds, then also

$$[f(u_1) - f(u_2)][g(v_1) - g(v_2)] = 0$$

holds.

For a given $f_\mu(x)$ satisfying (6a), (6b) we can define an operator $L(f)$ which generates the conformal transformations

$$L(f) = \int \mathrm{d}^1 x\, \theta^{0\nu} f_\nu(x)\,,$$

where $\theta^{\mu\nu}$ is an «improved» stress tensor [2].

Introducing the standard vectors

$$n^\mu \equiv (1, -1)\,, \qquad m^\mu \equiv (1, 1)\,, \tag{8}$$

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(7) D. G. BOULWARE, L. S. BROWN and R. D. PECCEI: *Phys. Rev. D*, **2**, 293 (1970).

such that

$$u = (n\cdot x)\,, \qquad v = (m\cdot x)\,,$$

one defines

(9) $$f^+(u) = n^\mu f_\mu(nx)\,, \qquad f^-(v) = m^\mu f_\mu(mx)\,;$$

one can also write

$$L(f) = \int \mathrm{d}^1x\, \theta^{0\nu}(T_{\mu\nu} + T_{\nu\mu}) f^\mu(x)\,,$$

(10) $$T_{\mu\nu} \equiv \tfrac{1}{2} n_\mu m_\nu = \tfrac{1}{2}(g_{\mu\nu} + \varepsilon_{\mu\nu}) \qquad (\varepsilon_{10} = -\varepsilon_{01} = 1)$$

and

$$f_\mu(x) = \tfrac{1}{2} m_\mu f^+(u) + \tfrac{1}{2} n_\mu f^-(v)\,.$$

For $\theta^{\mu\nu}$ satisfying

$$\partial_\mu \theta^{\mu\nu} = 0\,, \qquad \theta^{\mu\nu} = \theta^{\nu\mu}\,, \qquad \theta^\mu_\mu = 0\,,$$

one has

(11*a*) $$\theta_{00} + \theta_{01} = \theta^+(u)\,,$$

(11*b*) $$\theta_{00} - \theta_{01} = \theta^-(v)$$

and

(12) $$L(f) = \tfrac{1}{2}\int \mathrm{d}^1x[\theta^+(u) f_+(u) + \theta^-(v) f_-(v)] = L^+ + L^-.$$

Expanding $f_\mu(x)$ in eq. (7) near $x_\mu = 0$ we recover the usual generators of Poincaré, dilatations and special-conformal transformations for $n = 0, 1, 2$. For higher n one obtains additional independent generators. To this end let us assume for $f_\mu(x)$ a formal Taylor expansion

$$f_\mu(x) = \sum_{n=0}^{\infty} x^{\alpha_1}\cdot \ldots \cdot x^{\alpha_n} f_{\alpha_1,\ldots,\alpha_n,\mu}\,, \qquad f_{\alpha_1,\ldots,\alpha_n,\mu} = \frac{1}{n!}\,\partial_{\alpha_1}\ldots\partial_{\alpha_n} f_\mu(0)\,.$$

The coefficients $f_{\alpha_1,\ldots,\alpha_n,\mu}$ are symmetric and traceless in the indices $\alpha_1, \ldots, \alpha_n$, because of $\Box f_\mu(x) = 0$, and they are further restricted in their tensor structure in $\alpha_1, \ldots, \alpha_n, \mu$ from (2). For instance, up to second order in x we get

$$f_\mu = t_\mu\,, \qquad f_{\alpha\mu} = g_{\alpha\mu} d + \varepsilon_{\alpha\mu} l\,, \qquad f_{\alpha\beta\mu} = g_{\alpha\beta} c_\mu - g_{\alpha\mu} c_\beta - g_{\beta\mu} c_\alpha\,,$$

where t_μ, d, l, c_μ are constants,

$$f_\mu(x) = t_\mu + x_\mu d + x^\alpha \varepsilon_{\alpha\mu} l + \left(x^2 c_\mu - 2(xc) c_\mu\right) + \ldots$$

and we recover the six-parameter $O_{2,2}$ algebra.

The assumption of a Taylor expansion at $x=0$ only allows one, however, to build up an infinite-dimensional subalgebra $G^{\uparrow}$ of the full conformal algebra in two dimensions. To obtain the full algebra one has formally to introduce a Laurent expansion

$$f^{+}(u)=\sum_{n=-\infty}^{+\infty}c_n^{+}u^n\,,\qquad f^{-}(v)=\sum_{n=-\infty}^{+\infty}c_n^{-}v^n\,.$$

The implementation of the co-ordinate inversion R,

$$x_\mu\to-\frac{x_\mu}{x^2}\,,$$

is in fact impossible within functions satisfying a Taylor expansion for $n>2$. Following the previous discussion we expand $f_+(u)$ and $f_-(v)$ in eq. (12), in power series obtaining

$$L^{+}(f)=\sum_n c_n^{+}\int \mathrm{d}u\,u^n\theta^{+}(u)=\sum_n c_n^{+}L_{1-n}^{+}\,,\tag{13a}$$

$$L^{-}(f)=\sum_n c_n^{-}\int \mathrm{d}v\,v^n\theta^{-}(v)\;=\sum_n c_n^{-}L_{1-n}^{-}\,,\tag{13b}$$

where

$$c_n^{+}\;=\frac{1}{n!}f_{+}^{(n)}(0)\,,\qquad c_n^{-}\;=\frac{1}{n!}f_{-}^{(n)}(0)\,,\tag{14}$$

$$L_{1-n}^{+}=\int \mathrm{d}u\,u^n\theta^{+}(u)\,,\qquad L_{1-n}^{-}=\int \mathrm{d}v\,v^n\theta^{-}(v)\;;\tag{15}$$

obviously one has the relation $G^{\downarrow}=RG^{\uparrow}R^{-1}$, $G^{\uparrow}\cap G^{\downarrow}=O_{2,2}$ where the full conformal algebra is given by $G=G^{\uparrow}\cup G^{\downarrow}$.

2. – Conformal generators in the Thirring model.

After the preceding general remarks we shall now discuss the Thirring model [1] for which

$$\partial\!\!\!/\,\psi(x)=g:\gamma\cdot j(x)\psi(x):\tag{16}$$

and whose conserved currents

$$j_\mu(x)=\;:\bar\psi(x)\gamma_\mu\psi(x):\,,\tag{17a}$$

$$j_\mu^5(x)=\;:\bar\psi(x)\gamma_5\gamma_\mu\psi(x):\,=\varepsilon_\mu^{\ \nu}j_\nu(x)\tag{17b}$$

satisfy ($\Box = 4\partial_u \partial_v$)

(18) $$\Box j_\mu(x) = 0 , \qquad \Box j^5_\mu(x) = 0 .$$

One defines

(19) $$j_+(u) = j_0(x) + j_1(x) , \qquad j_-(v) = j_0(x) - j_1(x)$$

with the commutation relations

(20a) $$[j_+(u), j_+(u')] = icS'(u-u') ,$$

(20b) $$[j_-(v), j_-(v')] = icS'(v-v') ,$$

(20c) $$[j_+(u), j_-(v')] = 0 .$$

The stress tensor can be chosen as

(21) $$\theta_{\mu\nu}(x) = :j_\mu(x) j_\nu(x): - \tfrac{1}{2} g_{\mu\nu} :j_\alpha(x) j^\alpha(x): ,$$

which is conserved, symmetric and traceless. Conditions (11) are fulfilled since (c depends on the coupling constant)

(22a) $$\theta_{00} + \theta_{01} = \frac{1}{2c} :j_+(u) j_+(u): ,$$

(22b) $$\theta_{00} - \theta_{01} = \frac{1}{2c} :j_-(v) j_-(v): .$$

The operator $L^\pm(f)$ can be written as

(23) $$L^\pm(f) = \int \mathrm{d}x_\pm f_\pm(x_\pm) :j_\pm(x_\pm) j_\pm(x_\pm): ,$$

where $x_+ = u$, $x_- = v$.

From the commutation relations (20) one derives formally

(24a) $$[L^\pm_n, L^\pm_m] = (m-n) L^\pm_{n+m} + c(n)\delta_{n,-m} ,$$

(24b) $$[L^+_n, L^-_m] = 0 .$$

In eq. (24a) $c(n)$ is a c-number depending on n. Furthermore

(25) $$[L^\pm_n, j_\pm(x)] = \frac{\mathrm{d}}{\mathrm{d}x_\pm} [x_\pm^{1-n} j_\pm(x_\pm)] ,$$

expressing the well-known fact that $j_\pm$ have dimension 1. Equation (25) is to be interpreted in the sense of distributions.

The generators $L_1^\pm$, $L_0^\pm$, $L_{-1}^\pm$ satisfy the commutation relation of the algebra

$$O_{2,2} \sim O_{2,1} \otimes O_{2,1} , \tag{26}$$

which is the two-dimensional restriction of the $O_{4,2}$ conformal algebra on the Minkowski space-time. These generators coincide, of course, with those defined by DELL'ANTONIO, FRISHMAN and ZWANZIGER [6].

3. – Structure of the conformal algebra in two dimensions and its subgroups.

It is interesting to consider the general structure of the $[\theta, \theta]$-commutators in a conformally invariant theory in two dimensions:

$$[\theta^\pm(x_\pm), \theta^\pm(x'_\pm)] = c\delta'''(x_\pm - x'_\pm) + \delta'(x_\pm - x'_\pm)\Theta^\pm(x_\pm, x'_\pm) \tag{27}$$

and to observe that the homogeneous part of the algebra (24) arises from the bilocal operator $\Theta(u, u')$, whereas the c-number part comes from the disconnected c-number Schrödinger term in the $[\theta, \theta]$-commutator. This c-number term does not contribute to the Poincaré subalgebra. However the homogeneous part of the algebra is essentially model-independent, the only relevant requirement being that the lightlike restriction $\theta(u)$ of the bilocal operator $\Theta(u, u')$ coincide with the stress tensor, as implied by Poincaré invariance. On the other hand, $c(n)$ in (24a) is quite model dependent. In the Thirring model it is given by

$$c(n) = \frac{1}{12}\, n(n^2 - 1) . \tag{28}$$

If we look at the structure of the algebra (24a), (24b) we find that it contains three homogeneous subalgebras:

Subalgebra	Generators
$O_{2,2}$	$L_i^\pm$ for $i = 0, \pm 1$
$G^\uparrow$	$L_i^\pm$ for $i = +1, 0, -1, -2, \ldots, -\infty$
$G^\downarrow$	$L_i^\pm$ for $i = -1, 0, +1, +2, \ldots, +\infty$

One has

$$G^\uparrow \cap G^\downarrow = O_{2,2} . \tag{29}$$

Recalling our previous discussion, we note that only $G^\uparrow$ is obtained by considering co-ordinate transformations which can be Taylor-expanded at $x_\mu = 0$.

Enlargement to G can be obtained provided under the operation R of co-ordinate inversion

$$\theta^+(u) \to \frac{1}{|u|^4}\,\theta^+\left(\frac{1}{u}\right) \tag{30}$$

as would follow from invariance under R. Then one easily sees that

$$RL_n^\pm R^{-1} = L_{-n}^\pm\,, \tag{31}$$

recovering the property

$$RG^\uparrow R^{-1} = G^\downarrow\,, \tag{32}$$

and one obtains the full conformal algebra $G = G^\uparrow \cup G^\downarrow$.

Co-ordinate inversion can be thought of, in this context, as Hermitian conjugation. If one assumes that, as a consequence of projective invariance, the integrals defining the generators

$$L_n^+ = \int\limits_{-\infty}^{+\infty} \mathrm{d}u\, u^{1-n}\,\theta^+(u)$$

can be deformed into a closed path (unit circle) around the origin,

$$L_n^+ = \oint \mathrm{d}u\, u^{1-n}\theta^+(u) \tag{33}$$

(and the same for L_n^-), one has

$$L_n^+ = RL_nR^{-1} = L_{-n}\,,$$

since R acts on $u = \exp[i\varphi]$ according to

$$u = \exp[i\varphi] \to \exp[-i\varphi] = u^*\,.$$

However we are not able to justify the contour deformation rigorously.

4. – Conformal covariance of the operator product expansion in the Thirring model.

For the Thirring model we want to define the representation of the generators $L_n^\pm$ on the spinor field ψ. The commutation relations of $j_\mu(x)$ with $\psi(x)$

and $\bar{\psi}(x)$ are

$$[j_+(u), \psi(u', v')] = -(a+\bar{a}\gamma_5)\psi(u', v')\delta(u'-u)\,, \tag{34a}$$

$$[j_-(u), \psi(u', v')] = -(a-\bar{a}\gamma_5)\psi(u', v')\delta(v'-v)\,, \tag{34b}$$

where a, $\bar{a}$ are determined in terms of the coupling constant g from consistency with the equation of motion and from the spinor properties of ψ. From eqs. (34) and (16) we get

$$[L_n^\pm, \psi(x)] = x_\pm^{1-n}\frac{\partial}{\partial x_\pm}\psi(x) + \frac{\lambda^2}{8\pi c}(1-n)x_\pm^{-n}\psi(x) \tag{35}$$

(in the sense of distributions), where λ depends on a and $\bar{a}$ (see ref. ([6])).

We can now discuss the operator product expansion in the Thirring model. DELL'ANTONIO, FRISHMAN and ZWANZIGER have derived ([6])

$$\psi_1(u, v)\psi_1^\dagger(u', v') = (u-u_0)^\alpha(v-v_0)^\beta :\exp[\gamma\int_u^{u'} j_+(\zeta)\,\mathrm{d}\zeta + \delta\int_v^{v'} j_-(\eta)\,\mathrm{d}\eta]:\,, \tag{36}$$

where α, β, γ and δ are functions of the coupling constant g. One can check that the expansion (36) is covariant under the infinite-dimensional algebra of L_n.

We recall that for a conformal covariant operator product expansion in four-dimensional space-time we had obtained (l are the dimensions) ([8])

$$A(x)B(0) = \sum_n (x^2)^{-\frac{1}{2}(l_A+l_B-l_n+n)} x^{\alpha_1}\cdot\ldots\cdot x^{\alpha_n}\int_0^1 \mathrm{d}u\, f_n(u, x\partial, x^2\Box)\, O_{\alpha_1\ldots\alpha_n}(ux)\,, \tag{37}$$

where f_n are known transcendental functions, and $O_{\alpha_1\ldots\alpha_n}(0)$ are taken to belong to representations such that

$$[O_{\alpha_1\ldots\alpha_n}(0), K_\lambda] = 0\,. \tag{38}$$

In spite of the occurrence, in a two-dimensional space-time, of the infinite set of operators L_n which can connect different inequivalent representations of the $O_{2,2}$ algebra, the light-cone expansion for such a case is not uniquely determined, because of the appearance of an infinite number of representations of the (infinite-dimensional) conformal algebra. As we have already said, (36) is however explicitly conformal covariant.

5. – Proof of the noninvariance of the vacuum.

A final result we want to prove is that, in spite of the formal invariance of the theory under the algebra of the L_n generators, the vacuum cannot be invar-

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([8]) S. FERRARA, R. GATTO and A. F. GRILLO: *Lett. Nuovo Cimento*, **2**, 1363 (1971).

iant under L_n for $n = 2, ..., \infty$. We assume the commutators (see (25))

$$[L_n, j_+(u)] = \frac{\mathrm{d}}{\mathrm{d}u}(u^{1-n} j_+(u)) \tag{39}$$

and that j_+ satisfies the free-field equation. Let us consider the Wightman function

$$\langle 0 | j_+(u_1) j_+(u_2) | 0 \rangle , \tag{40}$$

which is necessarily of the form $(u_1 - u_2)^\alpha$.

If $L_n|0\rangle = 0$ and $L_n^+|0\rangle = 0$, one has

$$\langle 0 | [L_n, j_+(u_1) j_+(u_2)] | 0 \rangle = 0 ,$$

giving

$$(1-n)[u_1^{-n} + u_2^{-n}] + \alpha(u_1 - u_2)^{-1}(u_1^{1-n} - u_2^{1-n}) = 0 , \tag{41}$$

which is satisfied for $n = 1$ with any α, and for $n = 0$ and $n = -1$ provided $\alpha = -2$. Then necessarily $L_n|0\rangle \neq 0$ or $\langle 0|L_n \neq 0$ for $n = 2, ..., \infty$, and only the $O_{2,2}$ subalgebra can be a symmetry of the vacuum (and α is fixed $= -2$, as required by scale invariance).

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● RIASSUNTO

Si costruiscono i generatori infinitesimi dell'algebra conforme in due dimensioni spazio-temporali, algebra che risulta essere ad infiniti generatori, a partire dal tensore energia-impulso della teoria. Si verifica esplicitamente la covarianza conforme dell'espansione operatoriale nel modello di Thirring. Però si trova che lo stato di vuoto è invariante soltanto sotto la sottoalgebra $O_{2,2}$ della completa algebra conforme.

Конформная алгебра в двух пространственно-временных измерениях и модель Тирринга.

Резюме (*). — Из тензора напряжений конструируются бесконечно малые генераторы бесконечномерной конформной алгебры в двух пространственно-временных измерениях. Проверяется конформная ковариантность разложения произведения операторов в модели Тирринга. Однако, обнаружено, что вакуумное состояние является инвариантным только относительно суб'алгебры $O_{2,2}$ бесконечной конформной алгебры.

(*) *Переведено редакцией.*