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Conformal Symmetry at Lightlike Distances and Asymptotic Behaviour of Electromagnetic Form Factors.

S. FERRARA, A. F. GRILLO and G. PARISI (*)

Laboratori Nazionali di Frascati del CNEN - Frascati

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Summary. — Using conformal symmetry for the operator product of the hadronic electromagnetic current and the pion interpolating field, at lightlike distances, we connect the behaviour of the structure function in the scaling limit to the on-mass-shell asymptotic behaviour of the electromagnetic pion form factor. Our results are based on the analysis of the dimension spectrum of local operators appearing in the operator product expansion and on the assumption of precocious scaling. As an application of our results, predictions on the asymptotic behaviour of the e.m. pion form factor as well as for the threshold behaviour of the structure function for the singly inclusive π production in e^+e^- annihilation, are given.

1. — Introduction.

It has been recently suggested ⁽¹⁾ that there is a simple relation between the asymptotic behaviour of the on-mass-shell form factor and the dimension of the interpolating field.

This result seems at first sight to be wrong: the interpolating field ⁽²⁾ can be chosen arbitrarily, provided it has a nonzero matrix element between the vacuum and one-particle states, and on-shell quantities cannot depend on its dimension.

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⁽¹⁾ A. A. MIGDAL: *Phys. Lett.*, **37** B, 98 (1971).

⁽²⁾ H. ARAKY and R. HAAG: *Comm. Math. Phys.*, **5**, 77 (1967).

In this paper we present arguments suggesting that the asymptotic behaviour of the on-shell form factor depends on the dimension spectrum of the operators of the theory. We find that the old naive result is incorrect for too high values of the dimension.

If we assume that the spectrum of the dimensions is the one suggested by the naive quark model^(1,3), we find, as an example, a definite prediction for the asymptotic e.m. pion form factor.

To derive these results we assume that the Wilson⁽⁴⁾ expansion for the product of two operators on the light-cone⁽⁵⁾ is conformal⁽⁶⁾ covariant and moreover that the structure functions for fields of low dimensions satisfy precocious scaling⁽³⁾.

From the first assumption we derive a threshold theorem for the behaviour of the structure function near $\omega = 1$.

The second assumption enables us to saturate some mass dispersion relations with only low-mass states and derive a relation between the asymptotic behaviour of the form factor and the threshold behaviour $\omega \rightarrow 1$, as suggested by BRANDT and PREPARATA⁽⁷⁾. The arbitrariness of the interpolating field does not affect our results. We are making the assumption of precocious scaling only for low-dimension fields, not for arbitrary fields. This seems reasonable since high-dimension fields excite many more states from the vacuum than low-dimensional fields, so that the asymptotic regime starts at a much higher energy and one cannot saturate a mass dispersion relation with only a few low-mass states.

In Sect. 2 we derive the general form of the off-mass-shell form factor using conformal invariance on the light-cone.

In Sect. 3 we relate the $\omega = 1$ limit to the asymptotic behaviour of e.m. form factors.

Finally some formulae needed in the text are collected in the Appendix.

2. – Conformal algebra on the light-cone and off-mass-shell form factors: threshold theorem.

Let us consider the off-mass-shell three-point structure function related to the electromagnetic vertex of the π -field

$$(2.1) \quad W_{\mu}(p, q) = \text{Im} \int d^4x \exp [iq \cdot x] \langle 0 | T(J_{\mu}(x)\varphi(0)) | p \rangle ,$$

⁽³⁾ C. A. ORZALESI and P. D. RASKIN: *Lett. Nuovo Cimento*, **1**, 331 (1971).

⁽⁴⁾ K. WILSON: *Phys. Rev.*, **179**, 1499 (1969).

⁽⁵⁾ R. BRANDT and G. PREPARATA: *Nucl. Phys.*, **27 B**, 541 (1971); Y. FRISHMAN: *Phys. Rev. Lett.*, **25**, 966 (1970).

⁽⁶⁾ S. FERRARA, R. GATTO and A. F. GRILLO: *Phys. Lett.*, **36 B**, 124 (1971); *Nucl. Phys.*, **34 B**, 349 (1971).

⁽⁷⁾ See ref. ⁽⁵⁾ and also G. PREPARATA: *Riv. Nuovo Cimento*, **1**, 423 (1971).

where $|p\rangle$ is a one-particle state ($p^2 = m_\pi^2$) and φ a local interpolating field, *i.e.* $\langle 0|\varphi(0)|p\rangle \neq 0$.

We study now the structure function $W_\mu(p, q)$ in the scaling region which corresponds to the light-cone limit of the matrix element

$$(2.2) \quad \lim_{x^2 \rightarrow 0} \langle 0|T(J_\mu(x)\varphi(0))|p\rangle.$$

This matrix element can be evaluated by using a conformal covariant operator product expansion ⁽⁶⁾:

$$(2.3) \quad T(J_\mu(x)\varphi(0)) \underset{x^2 \rightarrow 0}{\sim} (\partial_\mu \partial^\lambda - g_\mu^\lambda \square) \sum_n \left(\frac{1}{-x^2 + i\varepsilon} \right)^{\frac{1}{2}(l-\tau_n)} \cdot x^{\alpha_1} \dots x^{\alpha_{n-1}} \int_0^1 du f_n(u) O_{\lambda\alpha_1 \dots \alpha_{n-1}}^n(ux),$$

where

$$(2.4) \quad f_n(u) = c_n u^{(d_n-l)/2} \cdot (1-u)^{(d_n-l^*)/2}$$

and $\tau_n = l_n - n$, $d_n = l_n + n$, $l^* = 4 - l$.

Note that eq. (2.3) takes automatically account of current conservation.

$O_{\alpha_1 \dots \alpha_n}^n$ are a set of local tensor operators of given dimension l_n and irreducible under the conformal algebra, *i.e.* ⁽⁶⁾

$$(2.5) \quad [O_{\alpha_1 \dots \alpha_n}^n(0), K_\lambda] = 0.$$

The advantage of the conformal covariant prescription is that the contributions of higher-order tensors such as

$$\partial_{\alpha_{n+1}} \dots \partial_{\alpha_m} O_{\alpha_1 \dots \alpha_n}(x)$$

are given from eq. (2.3) in terms of tensors of the type (2.5).

Inserting eq. (2.3) in (2.1) we get, after some computations shown in the Appendix,

$$(2.6) \quad \lim_{\substack{-q^2 \rightarrow \infty; \nu = p \cdot q \rightarrow \infty \\ \omega = 2m_\pi \nu / -q^2 \text{ fixed}}} W_\mu(p, q) = p_\mu W_2(q^2, \nu) + q_\mu W_1(q^2, \nu) = \\ = (p_\mu q^2 - p \cdot q q_\mu) \sum_n c_n (q^2)^{\frac{1}{2}(l-\tau_n)-2} (\omega)^{-\frac{1}{2}(l+\tau_n)} (\omega - 1)^{l-2} \cdot \\ \cdot \theta(\omega - 1)_2 F_1 \left(\frac{l-d_n}{2}, \frac{d_n-l^*}{2} + 1; l-1; 1 - \frac{1}{\omega} \right).$$

If we consider now the threshold behaviour $\omega \rightarrow 1$, we get

$$(2.7) \quad \lim_{\omega \rightarrow 1} W_\mu(p, q) \sim (p_\mu q^2 - p \cdot q q_\mu) (\omega - 1)^{l-2} \sum_n c_n (q^2)^{\frac{1}{2}(l-\tau_n)-2}$$

as

$$\lim_{\omega \rightarrow 1} {}_2F_1 \left(\frac{l-d_n}{2}, \frac{d_n-l^*}{2} + 1; l-1; 1 - \frac{1}{\omega} \right) = 1.$$

We note that the threshold behaviour of the structure function depends only on the dimension of the field $\varphi(x)$.

3. - Asymptotic behaviour of electromagnetic form factors.

In this Section we connect, using the hypothesis of precocious scaling, the threshold behaviour of the structure function to the asymptotic behaviour of the on-shell electromagnetic form factor.

Let us consider the threshold limit $\omega \rightarrow 1$ but $s = q^2(1 - \omega) \rightarrow \infty$: we get

$$(3.1) \quad W_2(q^2, s) \underset{\omega \rightarrow 1}{\sim} s^{l-2} \sum_n c_n (q^2)^{1-\frac{1}{2}(l+\tau_n)},$$

so the electromagnetic form factor is given by

$$(3.2) \quad F(q^2) \underset{q^2 \rightarrow \infty, \omega \rightarrow 1}{\sim} \text{Im } G^{-1}(s) W_2(q^2, s) \sim \sum_n (q^2)^{1-\frac{1}{2}(l+\tau_n)},$$

where the imaginary part of the inverse propagator $G^{-1}(s)$ of the virtual π -meson exactly cancels the s -dependence in eq. (3.1). If we now assume that eq. (3.2) still holds near $s \sim m_\pi^2$, we get that eq. (3.2) defines the asymptotic behaviour of the on-shell e.m. form factor.

This result may be obtained, in a more refined form, by using a mass dispersion relation approach as pointed out by BRANDT and PREPARATA (7).

From eq. (3.2) we see that the leading term is obtained by the contribution of the φ -field itself only if $\tau_n > 1$. However, as the twist is not less than two we get that for $1 \leq l \leq 2$ the form factor decreases as

$$(3.3) \quad F_\pi(q^2) \underset{q^2 \rightarrow \infty}{\sim} \left(\frac{1}{q^2} \right)^{l-1}.$$

However if one assumes that the dimension spectrum is the same as for the canonical quark model, one finds $l = \frac{3}{2} + \frac{3}{2} = 3$ and $\tau_n = 2$ for the minimum twist (*e.g.* the axial current), so that $F_\pi(q^2)$ becomes

$$(3.4) \quad F_\pi(q^2) \underset{q^2 \rightarrow \infty}{\sim} (q^2)^{-\frac{1}{2}} = (q^2)^{-\frac{3}{2}}.$$

A consistent check of this approach can be given in the framework of the renormalization group.

GELL-MANN and ZACHARIASEN⁽⁸⁾ found that, if $\varphi(x)$ is the fundamental field, then

$$(3.5) \quad \lim_{q^2 \rightarrow \infty} \frac{q^2}{F(q^2)G(q^2)} = 1,$$

where $F(q^2)$ and $G(q^2)$ are respectively the on-shell form factor and the propagator of the field, and we see that (3.5) is nothing but eq. (3.3).

However we agree with this result only for $1 \leq l \leq 2$. This is clear from the fact that eq. (3.5) is derived from a perturbative approach where $l-1$ is small.

As a final remark we note that the asymptotic behaviour of the π electromagnetic form factor can be related, by using duality⁽⁹⁾ arguments (or also the parton model⁽¹⁰⁾), to the threshold behaviour of the structure function $\bar{F}_2(\omega)$ for the inclusive process $e^+e^- \rightarrow \pi + \text{anything}$.

In fact we get

$$(3.6) \quad \bar{F}_2(\omega) \underset{\omega \rightarrow 1}{\sim} (\omega - 1)^{l+\tau_n-3}.$$

In particular, in the canonical quark model, eq. (3.6), becomes

$$(3.7) \quad \bar{F}_2(\omega) \underset{\omega \rightarrow 1}{\sim} (\omega - 1)^2.$$

* * *

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APPENDIX

Let us consider the T -operator product expansion of a conserved vector current with a scalar field on the light-cone. Current conservation together with the general form of the operator product expansion of any two tensors on the light-cone, derived in ref. (11), gives the unique solution

$$(A.1) \quad T(J_\mu(x)\varphi(0)) \underset{x^2 \rightarrow 0}{\sim} \sum_n (\partial_\mu \partial^\lambda - g_\mu^\lambda \square) \left(\frac{1}{-x^2 + i\varepsilon} \right)^{\frac{1}{2}(l-\tau_n)} \cdot x^{\alpha_1} \dots x^{\alpha_{n-1}} \int_0^1 du f_n(u) O_{\lambda\alpha_1 \dots \alpha_{n-1}}^n(ux),$$

when

$$f_n(u) = c_n u^{(d_n - l^*)/2} (1-u)^{(d_n - l^*)/2}$$

⁽⁸⁾ M. GELL-MANN and F. ZACHARIASEN: *Phys. Rev.*, **123**, 1065 (1961).

⁽⁹⁾ E. BLOOM and F. GILMAN: *Phys. Rev. Lett.*, **25**, 1140 (1970).

⁽¹⁰⁾ S. DRELL and T. M. YAN: *Phys. Rev. Lett.*, **24**, 181 (1970).

⁽¹¹⁾ S. FERRARA, R. GATTO and A. F. GRILLO: Frascati preprint LNF-71/70 (1971); *Phys. Rev. D*, **5**, 3103 (1972).

and

$$\tau_n = l_n - n, \quad \bar{d}_n = l_n + n, \quad l^* = 4 - l.$$

The structure function in configuration space turns out to be

$$(A.2) \quad \langle 0 | T(J_\mu(x)\varphi(0)) | p \rangle \underset{x^2 \rightarrow 0}{\sim} (p \cdot \partial \partial_\mu - p_\mu \square) \sum_n \left(\frac{1}{-x^2 + i\varepsilon} \right)^{\frac{1}{2}(l-\tau_n)} \cdot (xp)^{n-1} \int_0^1 du f_n(u) \exp[iux \cdot p].$$

As eq. (A.2) is Lorentz covariant, we can put $p = (\mathbf{t}, 0)$ and write eq. (A.2) as

$$(A.3) \quad \langle 0 | T(J_\mu(x)\varphi(0)) | p \rangle \sim \sum_n t^{n-1} \frac{\partial}{\partial t^{n-1}} (p \cdot \partial \partial_\mu - p_\mu \square) \cdot \left(\frac{1}{-x^2 + i\varepsilon} \right)^{\frac{1}{2}(l-\tau_n)} \int_0^1 du f_n(u) u^{1-n} \exp[iux \cdot p].$$

From eq. (A.3) we can easily compute the Fourier transform and the structure function in q -space, in fact we get (apart from some inessential multiplicative factors) in the limit $-q^2 \rightarrow \infty, v = p \cdot q \rightarrow \infty$ but $\omega = -2m_\pi v/q^2$ fixed ($1 \leq \omega \leq \infty$)

$$(A.4) \quad W_\mu(p, q) = \text{Im} \int d^4x \exp[iiqx] \langle 0 | T(J_\mu(x)\varphi(0)) | p \rangle \underset{x^2 \rightarrow 0}{\sim} \underset{x^2 \rightarrow 0}{\sim} (p \cdot qq_\mu - p_\mu q^2) \sum_n t^{n-1} \frac{\partial^{n-1}}{\partial t^{n-1}} \int_0^1 du f_n(u) u^{1-n} \text{disc} \left[\frac{1}{(q+up)^2} \right]^{2-\frac{1}{2}(l-\tau_n)} =$$

$$= (p \cdot qq_\mu - p_\mu q^2) \sum_n c_n(q^2)^{\frac{1}{2}(l-\tau_n)} (\omega)^{\frac{1}{2}(l-\tau_n)-2} \cdot \int_0^1 du u^{(d_n-l)/2} (1-u)^{(d_n-l^*)/2} \text{disc} \left(\frac{1}{\omega} - u \right)^{\frac{1}{2}(l-d_n)-1} =$$

$$= (p \cdot qq_\mu - p_\mu q^2) \sum_n c_n(q^2)^{\frac{1}{2}(l-\tau_n)-2} (\omega)^{\frac{1}{2}(l-\tau_n)-2} \theta(\omega - 1) \cdot \int_0^{\frac{\omega}{\omega-1}} du (1-u)^{(d_n-l)/2} u^{(d_n-l^*)/2} (x-u)^{\frac{1}{2}(l-d_n)-1} = \left(x = 1 - \frac{1}{\omega} \right)$$

$$= (p_\mu q^2 - p \cdot qq_\mu) (\omega - 1)^{l-2} \sum_n c_n(q^2)^{\frac{1}{2}(l-\tau_n)-2} (\omega)^{-\frac{1}{2}(l+\tau_n)} \theta(\omega - 1) \cdot {}_2F_1 \left(\frac{l-d_n}{2}, \frac{d_n-l^*}{2} + 1; l-1; 1 - \frac{1}{\omega} \right),$$

where the last integral has been computed as it is a Riemann-Liouville fractional integral ⁽¹²⁾.

⁽¹²⁾ Bateman Manuscript Project, Tables of Integral Transform, Vol. 2.

● RIASSUNTO

Usando la simmetria conforme per il prodotto sul cono di luce della corrente elettromagnetica degli adroni e del campo interpolante del pione, si connette l'andamento asintotico del fattore di forma elettromagnetico con la corrispondente funzione di struttura. Il nostro risultato è basato sull'analisi dello spettro delle dimensioni degli operatori locali che compaiono nell'espansione operatoriale e sull'ipotesi di precocità delle leggi di scala. Come applicazione del risultato si fanno alcune previsioni sull'andamento asintotico del fattore di forma del pione e sull'andamento di soglia della funzione di struttura relativa al processo inclusivo $e^+e^- \rightarrow \pi + \text{adroni}$.

Конформная симметрия на свето-подобных расстояниях и асимптотическое поведение электромагнитных форм-факторов.

Резюме (*). — Используя конформную симметрию для произведения операторов адронного электромагнитного тока и пионного интерполяционного поля на свето-подобных расстояниях, мы связываем поведение структурной функции в масштабном пределе с асимптотическим поведением на массовой поверхности электромагнитного форм-фактора пиона. Наши результаты основаны на анализе спектра размерности локальных операторов, появляющихся при разложении произведения операторов, и на предположении преждевременного подобия. Как применение наших результатов, приводятся предсказания для асимптотического поведения электромагнитного форм-фактора пиона, а также и для порогового поведения структурной функции для π -рождения при аннигиляции e^+e^- .

(*) *Переведено редакцией.*