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S. Ferrara: REDUCIBLE SCALE INVARIANCE AT SHORT-DISTANCES.

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ABSTRACT. -

It is proven that, using reducible scale invariance at short distances, conformal symmetry implies canonical (Bjorken) scaling provided diagonal dimensions of dilatation multiplets occuring in the operator product expansion of two electromagnetic currents have the canonical value $l_n = 2 + n$. If the electromagnetic current itself belongs to such multiplets then the hadron production cross section in e^+e^- annihilation falls off faster than $1/s$ at asymptotic energy.

Very recently some authors discussed[1] compatibility of logarithmic singularities in operator product expansions under the assumption of approximate scale (conformal) invariance at short-distances.

In this connection it has been argued[2] that, provided the local fields of the theory belong the reducible (indecomposable) representations of the dilatation group, canonical dimensions $l_n = 2 + n$ for the symmetric traceless (observable) operators $O_{\alpha_1 \ldots \alpha_n}(x)$ appearing in the operator expansion of two currents $J_\mu(x) J_\nu(0)$ are in agreement with an interacting scale invariant theory.

In fact, if a selection rule is operating[2] (as a consequence of a new symmetry sometimes called R invariance[2]) in such a way that the product of two currents only couples to lowest components of such representations, the Wilson[3] dimensional rule still holds and naive canonical scaling can be restored.

In this note it is proved that, as a consequence of conformal symmetry at short-distances, a selection rule holds which avoids logarithmic singularities to be present in the operator product expansion of two operators of definite dimension so that reducible operators do not occur in the short-distance expansion. As a consequence, it is possible to prove, without additional assumptions, that exact Bjorken scaling is compatible which such representations provided the tensor fields $O_{\alpha_1 \ldots \alpha_n}(x)$ are identified with the lowest components of (finite dimensional) indecomposable representations of the dilatation group. On the other hand, if the electromagnetic current itself belongs to such multiplets the disconnected piece in the expansion[4] vanishes in the scale invariant limit so $\lim_{s \to \infty} s\, \sigma_{e^+e^- \to H}(s) = 0$.

Let us consider a multiplet of local fields $O_i(x)$ (for semplicity taken to be Lorentz scalars at the beginning) which belong to a representation of the dilatation group[4].

(1) $$U_\lambda O_i(x) U_\lambda^{-1} = T_{ij}(\lambda) O_j(\lambda^{-1} x) \qquad (\lambda \geq 0)$$

where, without any loss of generality the matrix can be taken as NxN of the form

(2) $$\lambda^{l} \begin{pmatrix} 1 & \log\lambda & \cdots\cdots & \frac{1}{(N-1)!} & \log^{N-1}\lambda \\ 0 & 1 & & & \vdots \\ & & & 1 & \log\lambda \\ & & & 0 & 1 \end{pmatrix}$$

It is well known that any representation of the dilatation group is equivalent to a direct sum of such matrices. The representation is complitely specified by l, to be called the diagonal dimension and by its dimensionality N. These representations according to Ref.(4) can be enlarged to conformal group representations acting on SU(2, 2) spinors $\psi_i^\alpha(\eta)$ defined on the hypercone $\eta^A \eta_A = 0$ in a six dimensional space O(4, 2) which satisfy the covariant equations

(3) $$\eta^A \partial_A \psi_i^\alpha(\eta) = -I_{ij} \psi_j^\alpha(\eta) \qquad i = 1, \ldots N$$

where

$$I_{ij} = \frac{d}{d\lambda} T_{ij}(\lambda) \Big|_{\lambda=1}$$

Local fields on space-time are given at x=0 by

(4) $$O_i^\alpha(0) = T_{ij}(K) \psi_j^\alpha(\eta)_{x=0}$$

where

$$K = \eta_5 + \eta_6, \quad x_\mu = \frac{1}{K} \eta_\mu .$$

Constraints from conformal symmetry on general n-point functions are given by the set of equations

$$\eta_i^A \partial_A^i \langle 0|\psi_{i1}^{\alpha_1}(\eta_1)\ldots\psi_{in}^{\alpha_n}(\eta_n)|0\rangle =$$

(5)

$$=(-1)^n I_{i_1 j_1}\ldots I_{i_n j_n}\langle 0|\psi_{J_1}^{\alpha_1}(\eta_1)\ldots\psi_{J_n}^{\alpha_n}(\eta_n)|0\rangle$$

where, as a consequence of SU(2,2) symmetry the spinor functions on (5) depend only on invariant functions of variables $\eta_i \ldots \eta_n$.

For n=2 (propagator) the most general SU(2,2) invariant solution is the following (Lorentz scalars):

(6) $$F_N(x^2) = \left(\frac{1}{x^2}\right)^l \sum_{h=0}^{N-1} \frac{(-1)^h}{h!} \log^h x^2 \, C_{N-h} \qquad (l \geq 1)$$

where C_{N-h} is the common value of coefficients C_{ij} such that $i+j = 2N-h$ which gives the homogeneous contribution to

(7) $$\langle 0|O_i(x)O_j(0)|0\rangle = C_{ij}\left(\frac{1}{x^2}\right)^l + \ldots\ldots$$

and

$$C_{ij} = 0 \qquad \text{if} \qquad i+j < 2N-h\,.$$

Two-point functions of representations with different diagonal dimension vanish. Note that the V.E.V. of the diagonal component always vanishes ($C_{11}=0$) unless N=1 (irreducible representations). This implies that the V.E.V. in the short-distance limit is less singular than it would be expected by the Wilson dimensional rule (its scale invariant contribution vanishes). As a consequence the skeleton theory cannot have positive metric Hilbert-space (like Q.E.D.)[5] because if this would happen representations of this kind ($N > 1$) could not exist as a consequence of Federbush-Johnson theorem.

Generalization to spin-labels is straight-forward due to the structure of the group. The rule is the following: if the structure of the V.E.V. of the $N = 1$ (irreducible) representation is

$$\langle 0 | O_1^\alpha (x) O_1^\beta (0) | 0 \rangle = C_{11} \left(\frac{1}{x^2} \right)^{l} S^{\alpha\beta}(x) \tag{8}$$

(we already assumed $[O^\alpha (0), K_\lambda] = 0$) where $S^{\alpha\beta}(x)$ is an adimensional spin matrix (e.g. $S^{\alpha\beta}(x) = g^{\alpha\beta} - 2 x^\alpha x^\beta / x^2$ for vectors, $x \cdot \gamma^{\alpha\beta} / \sqrt{x^2}$ for spinors and so on), for $N > 1$ one has

$$\langle 0 | O_N^\alpha (x) O_N^\beta (0) | 0 \rangle = F_N (x^2) S^{\alpha\beta}(x) \tag{9}$$

Consider now three-point functions (vertices). It is better to work in six-dimensions, then

$$\begin{aligned} \langle 0 | \psi_{i_1}(\eta_1) \psi_{i_2}(\eta_2) \psi_{i_3}(\eta_3) | 0 \rangle &= F_{i_1 i_2 i_3} (\eta_i \eta_2, \eta_i \eta_3, \eta_2 \eta_3) = \\ &= F_{i_1 i_2 i_3} (K_1 K_2 K_3; (x_1 - x_2)^2, (x_1 - x_3)^2, (x_2 - x_3)^2) \end{aligned} \tag{10}$$

one can drop the space-time indices and formally write $F_{i_1 i_2 i_3}(K_1 K_2 K_3)$ instead of (10). We are already interested in $i_1 = i_2 = 1$ and general $i_3 = N (l_1 = l_2, l_3 = 1)$. One gets from (5)

$$\begin{aligned} K_1 \frac{\partial}{\partial K_1} F_{11N}(K_1 K_2 K_3) &= - l_1 F_{11N}(K_1 K_2 K_3) \\ K_2 \frac{\partial}{\partial K_2} F_{11N}(K_1 K_2 K_3) &= - l_1 F_{11N}(K_1 K_2 K_3) \end{aligned} \tag{11}$$

(11) $$K_3 \frac{\partial}{\partial K_3} F_{11N}(K_1 K_2 K_3) = -l_1 F_{11N}(K_1 K_2 K_3) - F_{11N-1}(K_1 K_2 K_3)$$

These equations can be easily solved by iteration

(12) $$F_{11I} = \dots F_{11N-1} = 0$$

and

(13) $$F_{11N} = K_1^{-l_1} K_2^{-l_1} K_3^{-1} f_{11N}((x_1-x_2)^2, (x_1-x_3)^2, (x_2-x_3)^2)$$

In fact, as a consequence of SU(2,2) symmetry $F_{11N}(K_1 K_2 K_3)$ must depend only on $K_1 K_2$, $K_1 K_3$, $K_2 K_3$ respectively ($\eta_i \eta_j = -1/2 \, K_i K_j \times (x_i - x_j)^2$) Then

(14) $$\langle 0|O_1(x) O_1(0) \tilde{O}_N(z)|0\rangle = C_{11N}\left(\frac{1}{x^2}\right)^{\frac{2l_1-1}{2}} \left[\frac{1}{z^2(x-z)^2}\right]^{1/2}$$

and this is all one needs.

Consider now the operator product expansion in the short--distance limit (q number terms)

(15) $$O_1(x) O_1(0) \underset{x \to 0}{=} C_1(x^2)\tilde{O}_1(0) + C_2(x^2)\tilde{O}_2(0) + \dots C_N(x^2)\tilde{O}_N(0) + \dots$$

(inequivalent representations)

and the C-number term in the short distance expansion

(16) $$\tilde{O}_j(0)\tilde{O}_N(z) \underset{z \to 0}{=} f_{jN}(z^2) I + \dots \text{ (q number terms)}$$

we have that in the sequence of limits $x \to 0$, $z \to 0$ $x/z \to 0$

(17) $$O_1(x)O_1(0)O_J(z) = \gamma (\frac{1}{x})^{2l_1-1} (\frac{1}{z})^{2l} \delta_{jN} I + \ldots\ldots$$

consistency(6) of this sequence of limits implies $C_2(x^2)\ldots C_N(x^2) = 0$ $\gamma = C_1 C_{1N}$ where γ, C_1, C_{IN} are the normalization of the three-point function, the coefficient of the o.p.e. and the normalization of the two-point function $\langle 0|\tilde{O}_1(0)\tilde{O}_N(z)|0\rangle$ respectively.

Note that the previous analysis does not prevent in general to have SU(2,2) symmetric logarithmic singularities. For example the vertex associated to higher components is

(18) $$\langle 0|O_2(x)\,O_2(0)\,\tilde{O}_1(2)|0\rangle = C_{122}\left(\frac{1}{x^2}\right)^{\frac{2l_1-1}{2}}\left[\frac{1}{z^2(x-z)^2}\right]^{1/2} - C_{112}\log x^2\left(\frac{1}{x^2}\right)^{\frac{2l_1-1}{2}}\left[\frac{1}{z^2(x-z)^2}\right]^{1/2}$$

In general vertex of the form $\langle 0|O_N(x)\,O_N(0)\,\tilde{O}_1(z)|0\rangle$ contains terms like $\log^{N-1} x^2 \left(\frac{1}{x^2}\right)^{\frac{2l_1-1}{2}}\left[\frac{1}{z^2(x-z)^2}\right]^{1/2}$

As a final point one must stress that if such dilatation multiplets occur in nature then

(19) $$\langle 0|O_{\alpha_1\ldots\alpha_n}(x)\,O_{\beta_1\ldots\beta_n}(0)|0\rangle_{x\to 0} = 0 \qquad (n=2,4\ldots\infty)$$

in the short-distance region, i.e. the V.E.V. vanishes in the scale invariant limit (as defined by Wilson dimensional rule). In particular if the electromagnetic current $J_\mu(x)$ belongs to such dilatation multiplets then $\langle 0|J_\mu(x)\,J_\nu(0)|0\rangle_{x\to 0} = 0$ as it happens in finite Q.E.D.(5).

This would imply that

$$\lim_{s \to \infty} \sigma_{e^+e^- \to H}(s) / \sigma_{e^+e^- \to \mu^+\mu^-}(s) = 0$$

at very high energy. This situation would, as expected, invalidate parton model results[7].

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