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Nonequivalence between Conformal Covariant Wilson Expansion in Euclidean and Minkowski Space.

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In recent time a lot of work has been done to understand the short and lightlike distance behaviour of operator products in field theories [1]

The original Wilson's idea of broken scale invariance at short distances has been later clarified and extended to the whole light-cone [2]. In this connection the role of conformal symmetry has also been investigated and it has been recognized that this stronger symmetry puts additional constraints on the underlying theory [3], and in particular severely limits the operator form of Wilson's expansion, this fact being deeply related to the uniqueness of the 3-point conformal covariant Wightman functions [4].

This last observation has been used by many authors to obtain an improved perturbation theory and to write down bootstrap equations for the anomalous dimensions [5].

In this note we emphasize that a conformal covariant Wilson expansion cannot be done in Euclidean space in the usual sense and we find the correct conformal covariant prescription which has to be used in order to obtain the operator product expansion in the pseudo-Euclidean space-time.

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(1) K. Wilson: *On products of quantum fields operators at short distances*, Cornell Report, unpublished (1964); R. Brandt: *Ann. of Phys.*, **44**, 221 (1967); C. G. Callan jr.: *Phys. Rev. D*, **2**, 1451 (1970); K. Symanzik: *Comm. Math. Phys.*, **23**, 49 (1971).

(2) W. Zimmerman: in *Lectures on Elementary Particles and Quantum Field Theory* (Cambridge, Mass., 1971); R. Brandt and G. Preparata: *Nucl. Phys.*, **27** B, 541 (1971); Y. Frishman: *Phys. Rev. Lett.*, **25**, 966 (1970).

(3) For a review see S. Ferrara, R. Gatto and A. F. Grillo: Frascati preprint LNF-71/79 (1971), to appear on *Springer Tracts*.

(4) S. Ferrara and G. Parisi: Frascati preprint LNF-72/1(1972); *Nucl. Phys.*, to be published; S. Ferrara, R. Gatto, A. F. Grillo and G. Parisi: *Lett. Nuovo Cimento*, **4**, 115 (1972).

(5) G. Parisi and L. Peliti: *Lett. Nuovo Cimento*, **2**, 627 (1971); A. A. Migdal: *Phys. Rev. Lett.*, **37** B, 98 (1971); G. Mack and I. Todorov: Trieste preprint IC/71/139 (1971); G. Mack and K. Symanzik: Desy preprint 72/20 (1972).

We recall that, in space-time, the conformal covariant contribution (of a scalar for simplicity) to the bilocal operator $A(x)B(0)$ is of the form [6]

$$(1)\qquad A(x)B(0)=\left(\frac{1}{x^2}\right)^{(l_A+l_B-l)/2}\int_0^1 \mathrm{d}u\, f(u)\,{}_0F_1\left(l-1,\ -\left(\frac{x^2}{2}\right)^2\Box\left(u(1-u)\right)\right)O(ux)=$$
$$=\left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B-l)}\int_0^1 \mathrm{d}u\, f(u)\, Z^{2-l}J_{l-2}(Z)\,O(ux)\,,$$

where

$$(2)\qquad f(u)=u^{\frac{1}{2}(l_A-l_B+l_C)-1}(1-u)^{\frac{1}{2}(l_B-l_A+l_C)-1}\,,\qquad Z=(u(1-u)\,x^2\Box)^{\frac{1}{2}}\,.$$

The generalization to general tensor operators is straightforward.

Note that A, B and O are assumed to be conformal irreducible scalar operators in the sense that

$$(3)\qquad [A(0), K_\lambda]=[B(0), K_\lambda]=[O(0), K_\lambda]=0\,.$$

The simplest way to derive this formula is to use the « vertex graph identity » [7] and the selection rule for conformal invariant two-point functions, which read in the Euclidean space

$$(4)\qquad \langle 0|O(t)O'(t')|0\rangle=0\,,\qquad \text{if } l\neq l',$$

and

$$(5)\qquad \langle 0|A(x)B(y)O(z)|0\rangle=\int \mathrm{d}t\,\langle 0|A(x)B(y)O^*(t)|0\rangle\,\langle 0|O(t)O(z)|0\rangle\,,$$

where we have

$$(6)\qquad \langle 0|O(t)O(z)|0\rangle=[-(t-z)^2+i\varepsilon]^{-l}$$

and

$$(7)\qquad \langle 0|A(x)B(y)O(t)|0\rangle=$$
$$=[-(x-y)^2+i\varepsilon]^{(l-l_A-l_B)/2}[-(x-t)^2+i\varepsilon]^{(l_B-l_A-l)/2}[-(y-t)^2+i\varepsilon]^{(l_A-l_B-l)/2}\,.$$

$O^*(t)$ is a so-called « shadow » operator, namely a conventional operator of dimension $l^*=4-l$.

By means of eq. (4), eq. (5) can be translated in operator language by

$$(8)\qquad A(x)B(y)=\int \mathrm{d}^4t\,\langle 0|A(x)B(y)\,O^*(t)|0\rangle\,O(t)\,,$$

which gives the contribution of a scalar operator $O(t)$ to the product $A(x)B(y)$.

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(6) S. FERRARA, R. GATTO and A. F. GRILLO: *Nuovo Cimento*, **2**, 1363 (1971), and we used the relation between the Bessel and the hypergeometric function.

(7) M. D'ERAMO, G. PARISI and L. PELITI: *Lett. Nuovo Cimento*, **2**, 878 (1971).

However this is not the right result in space-time, since eq. (8) is definitely different from eq. (1); in fact, after some simple calculations, we get from eq. (8)

$$A(x)B(0) = \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B-l)} \int_0^1 \mathrm{d}u\, f(u)\, Z^{2-l} K_{l-2}\left(\sqrt{-Z^2}\right) O(ux)\,, \tag{9}$$

so we see that eq. (9) is different from eq. (1) for the replacement of the Bessel function J_{l-2} by K_{l-2}.

Note that

$$K_{l-2}(z) = \frac{\pi}{2} \frac{I_{l-2}(z) - I_{2-l}(z)}{\sin \pi(l-2)} \tag{10}$$

and, for $l > 2$, formula (9) gives rise to unwanted « shadow » singularities of the form

$$\left(\frac{1}{x^2}\right)^{(l_A+l_B+l)/2-2}$$

in the operator product expansion.

The origin of the shadow singularity lies in the fact that the singularity of eq. (6) is not, for $l > 2$, the naive one

$$\left(\frac{1}{x^2}\right)^{(l_A+l_B-l)/2},$$

because of the distribution character of eq. (6), as can be easily proven by applying this distribution to a test function.

This implies that the conformal covariant Wilson expansion in the Euclidean space must have these « shadow » singularities to reproduce the three-point function.

Note that the conformal covariant operator expansion derived by BONORA, SARTORI and TONIN coincides with this last expression, so it is not correct when used in space-time (8).

The situation is quite different in Minkowsky space; in fact in this case the operator product expansion is connected to Wightman functions better than to T-ordered vacuum expectation functions.

In fact, Wightman functions do not have « shadow » singularities and therefore one can build up an operator expansion in the form of eq. (8), which, in this case, exactly reproduces the original expansion eq. (1).

For this purpose we note that the correct implementation of eq. (4) to space-time is

$$\langle 0|T(A(x)B(y)O(z))|0\rangle = \int \mathrm{d}^4 t\, \langle 0|T(A(x)B(y)O^*(t))|0\rangle \langle 0|T(O^*(t)O(z))|0\rangle \tag{11}$$

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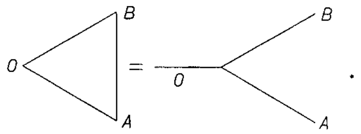
(8) L. BONORA, R. SARTORI and M. TONIN: Padova preprint, *Nuovo Cimento*, to be published.

which can be graphically represented as

(12) 

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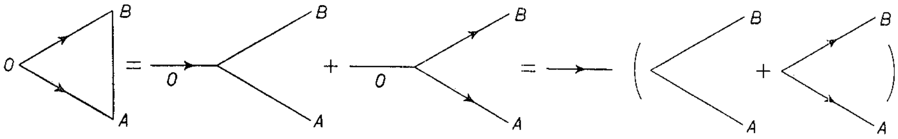
Using now the general cutting formula ([9,10]), we get

(13) 

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where arrows mean taking only the positive-frequency part of the propagators.

However the 3-point function (13) coincides with the Wightman function for $(x-y)^2<0$, which can be also written as Fourier transform with respect to the point z as (we put $y=0$)

$$W(p,x)=\operatorname*{disc}_{p}\int\exp[ipz]\langle 0|T(A(x)B(0)O(z))|0\rangle\,\mathrm{d}z=\left(\frac{1}{x^2}\right)^{(l_A+l_B-l)/2}\cdot \tag{14}$$

$$\cdot\operatorname*{disc}_{p}\int_0^1\mathrm{d}u\,\exp[iupx]f(u)\,[u(1-u)p^2x^2]^{2-l/2}(p^2)^{l-2}K_{l-2}[(u(1-u)p^2x^2)^{\frac{1}{2}}]=$$

$$=\left(\frac{1}{x^2}\right)^{(l_A+l_B-l)/2}\int_0^1\mathrm{d}u\,\exp[iupx]f(u)\,[u(1-u)p^2x^2]^{2-l/2}(p^2)^{l-2}J_{l-2}[-(u(1-u)p^2x^2)^{\frac{1}{2}}],$$

since, from the analitic structure of Bessel functions ([6,11]), one easily recognizes that « shadow » singularities have no imaginary part in p, so they do not give contribution to the Wightman functions.

Moreover, one still verifies that the Fourier transform of eq. (14) indeed coincides with the conformal covariant prescription given by eq. (1), so that we can write

$$\langle 0|A(x)B(0)=\int\mathrm{d}^4t\,\langle 0|A(x)B(0)O^*(t)|0\rangle\langle 0|O(t)\,, \tag{15}$$

where

$$\langle 0|A(x)B(0)O^*(t)|0\rangle=(-x^2)^{\frac{1}{2}(l^*-l_A-l_B)}\{[-(x-t)^2+i\varepsilon]^{\frac{1}{2}(l_B-l_A-l^*)}\cdot$$
$$\cdot[-t^2+i\varepsilon]^{\frac{1}{2}(l_A-l_B-l^*)}+[-(x-t)^2+i\varepsilon(t_0-x_0)]^{\frac{1}{2}(l_B-l_A-l^*)}[-t^2+i\varepsilon t_0]\}^{\frac{1}{2}(l_A-l_B-l^*)}$$

when $x^2<0$; analytic continuation can be then used to define the operator product in the whole space.

([9]) M. VELTMAN: *Physica*, **29**, 186 (1963).
([10]) K. SYMANZIK: private communication.
([11]) *Bateman Manuscript Project, Higher Trascendental Functions*, Vol. **2**, edited by A. ERDELYI (New York, 1953).

As a last point, note that the Wilson expansion in eq. (15) is indeed an operatorial statement, if it is intended to hold only for positive frequency (this is why eq. (15) is applied to the vacuum state).

This last statement is indeed a conformal invariant statement (under the algebra) as it is equivalent to considering a general Wightman function with all points spacelike to each other.

We then have shown that the expansion given by eq. (1) is the unique conformal covariant expansion on space-time which satisfies the right constraints due to unitarity in the sense that the corresponding (three-point) Wightman function is a discontinuity of a conformal covariant quantity.

We finally point out that the interest of having a correct Wilson expansion is connected to the program of calculating n-point conformal covariant functions by appropriate insertion of operator expansions.

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