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COVARIANT EXPANSION OF THE CONFORMAL FOUR -POINT FUNCTION

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Abstract: The problem of a conformally covariant expansion of the four-point function is solved both in Minkowski and in Euclidean space. The role of shadow singularities is discussed. The expansion is examined for its analyticity properties and the various light-cone limits are performed. Vertex identities are derived in momentum space and simply interpreted.

1. INTRODUCTION

The crucial role of short-distance and light-like distance behaviour in relativistic quantum field theory has recently been emphasized by many authors and its implications for physically relevant quantities have been widely recognized [1]. In this connection the importance and relevance of scale and conformal symmetry has been discussed in several directions [2].

A fundamental step towards a deep understanding of the problem was made by Wilson, with his hypothesis of an operator-product expansion at short distances [1]. The scaling behavior in electron scattering was shown to be related to the behavior near the light-cone of the current commutator [3], and the generalization of Wilson's idea to the light cone was suggested by Brandt and Preparata [4], and by Frishman [5]. Conformal symmetry puts powerful constraints on the general operator product expansion and on the form of its light-cone limit [6]. Use of conformal symmetry restrictions in the light-cone limit may be valid for the complete theory, under general hypothesis. Its use beyond that limit requires absence of symmetry breaking (in particular, massless theory).

In this work we make use of the conformally covariant form of the operator product expansion to derive an expression for the four-point correlation function. The insertion of operator-product expansions into the four-point function, after use of the two-point function selection rule [6], exhibits the correlation function as an infinite sum of irreducible conformal graphs [7], where a local tensor field, conformally irreducible, is exchanged. Using the correct prescription for the operator-product expansion we are able to overcome the difficulties due to application of conformal symmetry in Minkowsky space [8]. In fact, in contrast to what happens for the standard skeleton graph expansion [9] our irreducible conformal graphs cannot be obtained through analytic continuation from the Euclidean space. The corresponding change in the symmetry group is from O (5,1) to O(4,2).

The integral representation proposed here for the general irreducible term of the four-point correlation function has the virtue of being suitable for analytic continuation in the spin and dimension of the exchanged tensor, suggesting the possibility of reggeization.

The four-point correlation function has to satisfy an additional crossing constraint [10] besides those imposed by the space-time (and other) symmetries. Our general expression for the correlation function appears as particularly convenient to exploit the consequences of such constraint on operator-product expansions. It is also convenient to perform a general light-cone limit, which we discuss in some detail.

In the present paper we use a configuration space representation. Momentum space does not appear suitable for this problem, as evidenced for instance by the difficulties found by Bali et al. (see ref. [2]) in an investigation of conformal invariant elastic scattering leading to a representation for the amplitude in terms of a six-dimensional integral.

The paper is arranged as follows. In sect. 2 we summarize the main results of conformal symmetry in Minkowsky space. Selection rules and vertex-graph identities for T-ordered functions and Wightman functions are derived by simple procedures. In sects. 3 and 4 we calculate the irreducible four-point contributions and compare the Wightman functions to the vacuum expectation values of T-products. In sect. 5 and 6 we discuss the problem of crossing and perform the various light-cone limits. Technical points in the derivations are collected in two appendices.

2. VERTEX IDENTITIES AND OPERATOR EXPANSIONS

Let us recall the constraints that conformal symmetry puts on correlation functions [6]. For the two-point function $\langle 0|O_{\alpha_1 \dots \alpha_n}(x)\, O_{\beta_1 \dots \beta_m}(0)|0\rangle$, where $O_{\alpha_1 \dots \alpha_n}$ and $O_{\beta_1 \dots \beta_m}$ are local symmetric tensors, irreducible under the conformal algebra, i.e. satisfying $[O_{\alpha_1 \dots \alpha_n}(0), K_\lambda] = 0$ and having definite scale dimensions l_n and l_m, one has

$$\langle 0 | O_{\alpha_1 \dots \alpha_n}(x)\, O_{\beta_1 \dots \beta_m}(0) | 0 \rangle = 0 \quad \text{unless } n = m \text{ and } l_n = l_m \,. \tag{1}$$

More generally for conformally irreducible tensors $O^{(J_1, J_2)}$ and $O^{(J_1', J_2')}$ belonging to SL (2, c) representations (J_1, J_2) and (J_1', J_2') one has

$$\langle 0 | O^{(J_1,J_2)}(x)\, O^{(J_1',J_2')}(0) | 0 \rangle = 0\,, \quad \text{unless } J_1 = J_1',\ J_2 = J_2' \text{ and } l_O = l_{O'}. \tag{2}$$

For the three-point function $\langle 0 | A(x)\, B(y)\, C(z) | 0 \rangle$ where A, B, C are scalar fields, conformally irreducible, one has (γ_{ABC} is a constant and l is the scale dimension)

$$\langle 0 | A(x)\, B(y)\, C(z) | 0 \rangle = \gamma_{ABC} [(x-y)^2]^{-\frac{1}{2}(l_A + l_B - l_C)}$$
$$\times\, [(y-z)^2]^{-\frac{1}{2}(l_B + l_C - l_A)} [(x-z)^2]^{-\frac{1}{2}(l_A + l_C - l_B)}\,. \tag{3}$$

In general the n-point function $\langle 0 | A_1(x_1) \dots A_n(x_n) | 0 \rangle$ depends on N conformally invariant variables, where

$$N = \tfrac{1}{2} n(n-3) \ \text{ for } n \leqslant 6, \qquad N = 4n - 15 \ \text{ for } n \geqslant 6\,,$$

which are the so-called harmonic ratios, of the form

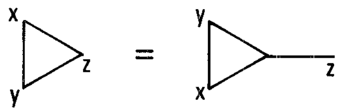
$$\frac{(x_i - x_j)^2\, (x_h - x_k)^2}{(x_i - x_h)^2\, (x_j - x_k)^2}\,. \tag{4}$$

In particular for the four-point function one has two independent harmonic ratios.

For the vacuum expectation values $\langle 0 | \mathrm{T}(A(x)\, B(y)\, C(z)) | 0 \rangle$ one has the following vertex identity [11]:

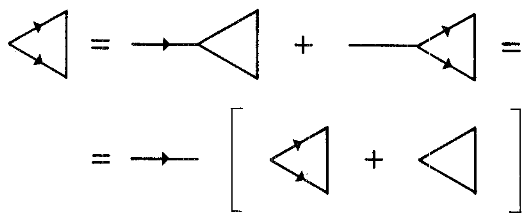
$$\langle 0 | \mathrm{T}(A(x) B(y) C(z)) | 0 \rangle = \int \mathrm{d}^4\xi\, \langle 0 | \mathrm{T}(A(x) B(y) C^*(\xi) | 0 \rangle \langle 0 | \mathrm{T}(C(\xi) C(z)) | 0 \rangle\,,$$

where $C^*(x)$ is the "shadow" of $C(x)$ (ref. [12]). The graphical representation of (5) is



(6)

However the following relation is also true [13]‡



(7)

‡ Lines with an arrow correspond to insertion of the positive frequency part of propagators (see ref. [8]); the sum of the two terms in parentheses in eq. (7) will be briefly indicated by a dashed triangle (see eq. (21)).

Both (6) and (7) are easily understood by Fourier transforming with respect to z. We have

$$T(p,x) = \int e^{ipz} \langle 0 | T(A(x)B(0)C(z)| 0\rangle d^4z$$

$$= \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B-l_C)} \int_0^1 du\, e^{iupx} f_{ABC}(u)\, [u(1-u)p^2x^2]^{\frac{1}{2}(2-l_C)}$$

$$\times (p^2)^{l_C-2} K_{l_C-2}\, [(u(1-u)p^2x^2)^{\frac{1}{2}}]\,, \tag{8}$$

where

$$f_{ABC}(u) = u^{\frac{1}{2}(l_A-l_B+l_C)-1}\,(1-u)^{\frac{1}{2}(l_B-l_A+l_C)-1}\,.$$

One can then easily verify the property $(x^2 < 0, p^2 > 0)$

$$T(p,x) = T^*(p,x)\,\Delta_C(p^2)\,, \tag{9}$$

where $T^*(p,x)$ is obtained from $T(p,x)$ by substituting l_C with $l_C^* = 4 - l_C$ and $\Delta_C(p^2) = (p^2)^{l_C-2}$. Note that eq. (9) follows from the symmetry property

$$K_\lambda(z) = K_{-\lambda}(z)\,,$$

for the Bessel function. Recalling the relation

$$K_\lambda(z) = \tfrac{1}{2}\pi(\sin \pi\lambda)^{-1}\,[I_\lambda(z) - I_{-\lambda}(z)]\,, \tag{10}$$

one obtains for the Wightman function

$$W(p,x) = \text{disc}\, T(p,x) = \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B-l_C)-1} \int_0^1 du\, e^{iupx} f_{ABC}(u)$$

$$\times\, [u(1-u)p^2x^2]^{\frac{1}{2}(2-l_C)}\,(p^2)^{l_C-2}\, I_{l_C-2}\,[(u(1-u)p^2x^2)^{\frac{1}{2}}]\,. \tag{11}$$

Eq. (11) is essentially identical to the operator-product expansion derived in ref. [6], which, is of the form

$$A(x)B(0) = \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B+l_C)} \int_0^1 du\, f_{ABC}(u)\,{}_0F_1(l_C-1; \tfrac{1}{4}x^2\, u(1-u)\Box)\, C(ux) + \ldots \tag{12}$$

(we have singled out the scalar term). We have used the well-known relation between hypergeometric and Bessel functions.

3. CONFORMAL COVARIANT FOUR-POINT FUNCTION

The problem of the determination of the conformal covariant four-point function also provides the solution of the operator-product expansion for three local operators in conformally covariant theories. In fact from an expansion (∂_{n-1} means $\partial/\partial x_{n-1}$)

$$A(x_1)\dots A(x_{n-1}) = \sum_n f^{\alpha_1\dots\alpha_m}(x_1,\dots x_{n-1};\partial_{n-1})\,O_{\alpha_1\dots\alpha_m}(x_{n-1}), \quad (13)$$

one obtains, from the selection rule for the two-point function [6]

$$\langle 0\,|A(x_1)\dots A(x_{n-1})\,O_{\beta_1\dots\beta_m}(x_n)|\,0\rangle$$
$$= f^{\alpha_1\dots\alpha_m}(x_1,x_2,\dots x_{n-1};\partial_{n-1})\,W_{\alpha_1\dots\alpha_m;\beta_1\dots\beta_m}(x_{n-1}-x_n), \quad (14)$$

where W is the two-point Wightman function. In other words $f^{\alpha_1\dots\alpha_m}$ can be determined from the Wightman function in (14). In particular let us consider

$$A(x)\,B(y)\,C(z), \quad (15)$$

and let us first extract its typical scalar contribution in the operator product expansion. From

$$A(x)B(y) = \sum_n f_n^{AB}(x,y;\partial_y)\,O_n(y), \quad (16)$$

one has

$$A(x)\,B(y)\,C(z) = \sum_n f_n^{AB}(x,y;\partial_y)\,O_n(y)\,C(z), \quad (17)$$

and, expanding again,

$$A(x)B(y)\,C(z) = \sum_{m,n} f_n^{AB}(x,y;\partial_y)\,f_m^{O_nC}(y,z;\partial_z)D_n(z). \quad (18)$$

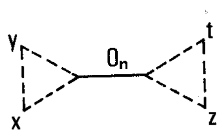
Finally

$$\langle 0 | A(x) B(y) C(z) D(t) | 0 \rangle = \sum_n f_n^{AB}(x, y, \partial_y) f_0^{OC}(y, z, \partial_z)\, W(z-t)\,, \tag{19}$$

or

$$\langle 0 | A(x) B(y) C(z) D(t) | 0 \rangle = \sum_n W_n(x, y, z, t)\,. \tag{20}$$

We call $W_n(x, y, z, t)$ an nth order irreducible contribution, whose graphical representation is



. (21)

Let us first concentrate on the $n = 0$ contribution ($n \neq 0$ will be calculated later on).

From eq. (11) and the convolution formula

$$W_0(x, y, z, t) = \left[\frac{1}{(x-y)^2}\right]^{\frac{1}{2}(l_A + l_B - l)} \left[\frac{1}{(z-t)^2}\right]^{\frac{1}{2}(l_C + l_D - l)}$$

$$\times \int d^4 p\, \theta(p^2)\, \theta(p_0) \int_0^1 du\, e^{ip(ux+(1-u)y)} f_{AB0}(u)\, [-u(1-u)(x-y)^2]^{1-\frac{1}{2}l}$$

$$\times J_{l-2}[(-p^2 u(1-u)(x-y)^2)^{\frac{1}{2}}] \int_0^1 dv\, e^{-ip(vz+(1-v)t)} f_{CD0}(v)$$

$$\times [-v(1-v)(z-t)^2]^{1-\frac{1}{2}l}\, J_{l-2}[(-p^2 v(1-v)(z-t)^2)^{\frac{1}{2}}]\,, \tag{22a}$$

when x, y, z, t are all relatively space-like.

An expression equivalent to (22a) is ($l^* = 4 - l$)

$$W_0(x, y, z, t) = \left[\frac{1}{(x-y)^2}\right]^{\frac{1}{2}(l_A + l_B - l^*)} \iint d^4\xi\, d^4\xi' \left[\frac{1}{(\xi-\xi')^2}\right]^{l}$$

$$\times \{[-(x-\xi)^2 + i\epsilon]^{\frac{1}{2}(l_B - l_A - l^*)} [-(y-\xi)^2 + i\epsilon]^{\frac{1}{2}(l_A - l_B - l^*)}$$

$$+ [-(x-\xi)^2 + i\epsilon(\xi_0 - x_0)]^{\frac{1}{2}(l_B - l_A - l^*)} [-(y-\xi)^2 + i\epsilon(\xi_0 - y_0)]^{\frac{1}{2}(l_A - l_B - l^*)}\}$$

$$\times \{[-(z-\xi')^2 + i\epsilon]^{\frac{1}{2}(l_D-l_C-l^*)} [-(t-\xi')^2 + i\epsilon]^{\frac{1}{2}(l_C-l_D-l^*)}$$

$$+ [-(z-\xi')^2 + i\epsilon(\xi_0'-z_0)]^{\frac{1}{2}(l_D-l_C-l^*)} [-(t-\xi')^2 + i\epsilon(\xi_0'-t_0)]^{\frac{1}{2}(l_C-l_D-l^*)}\} ,$$

obtained by use of the right prescription for the conformal Wilson expansion in Minkowsky space [8].

Introducing

$$-A^2 = u(1-u)(x-y)^2 < 0 , \qquad -B^2 = v(1-v)(z-t)^2 < 0 ,$$
$$-\Omega^2 = [ux + (1-u)y - vz - (1-v)t]^2 , \tag{23}$$

we can write

$$W_0(x,y,z,t) = \left[\frac{1}{(x-y)^2}\right]^{\frac{1}{2}(l_A+l_B-l)} \left[\frac{1}{(z-t)^2}\right]^{\frac{1}{2}(l_C+l_D-l)}$$
$$\times \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v\, f_{AB0}(u) f_{CD0}(v) (A^2B^2)^{\frac{1}{2}(2-l)} \int \mathrm{d}^4p\, \theta(p^2)\,\theta(p_0)$$
$$\times\, \mathrm{e}^{ip\Omega} J_{l-2}[(p^2A^2)^{\frac{1}{2}}]\, J_{l-2}[(p^2B^2)^{\frac{1}{2}}] . \tag{24}$$

We now use the result [14]

$$\int \mathrm{d}^4p\,\theta(p^2)\,\theta(p_0)\,\mathrm{e}^{ip\Omega} J_{l-2}[(p^2A^2)^{\frac{1}{2}}]\, J_{l-2}[(p^2B^2)^{\frac{1}{2}}]$$
$$= \int_0^\infty \mathrm{d}m^2 \int \mathrm{d}^4p\,\delta(p^2-m^2)\,\theta(p_0)\,\mathrm{e}^{ip\Omega} J_{l-2}[(p^2A^2)^{\frac{1}{2}}]\, J_{l-2}[(p^2B^2)^{\frac{1}{2}}]$$
$$= \int_0^\infty \mathrm{d}m^2 \frac{m}{\sqrt{\Omega^2}} K_1[m\sqrt{\Omega^2}]\, J_{l-2}[m\sqrt{A^2}]\, J_{l-2}[m\sqrt{B^2}]$$
$$= (A^2B^2)^{-1} (\lambda_+^2-1)^{-\frac{3}{4}}\, Q^{\frac{3}{2}}_{l-\frac{5}{2}}(\lambda_+) , \tag{25}$$

where $Q_\nu^\mu(z)$ are the usual Legendre functions of the second kind, and

$$\lambda_+ = \frac{A^2+B^2+\Omega^2}{2(A^2B^2)^{\frac{1}{2}}} = -\tfrac{1}{2}\,[uv(1-u)(1-v)(x-y)^2(z-t)^2]^{-\frac{1}{2}}$$

$$\times\,[(y-t)^2(1-u)(1-v)+u(1-v)(x-t)^2+v(1-u)(z-y)^2+uv(x-z)^2]\,. \qquad (26)$$

We note that (25) can be rewritten as

$$(A^2B^2)^{-1}\,\lambda_+^{-l}\;{}_2F_1(\tfrac{1}{2}(l+1),\tfrac{1}{2}l;l-1;\lambda_+^{-2})\,. \qquad (27)$$

After inserting (27) into (24) we obtain

$$\langle 0\,|A(x)B(y)\,C(z)D(t)|\,0\rangle = \left[\frac{1}{(x-y)^2}\right]^{\frac{1}{2}(l_A+l_B-l)}\left[\frac{1}{(z-t)^2}\right]^{\frac{1}{2}(l_C+l_D-l)}$$

$$\times\int_0^1 \mathrm{d}u\int_0^1 \mathrm{d}v f_{AB0}(u)\,f_{CD0}(v)\,(A^2B^2)^{-\frac{1}{2}l}\,\lambda_+^{-l}\;{}_2F_1(\tfrac{1}{2}(l+1),\tfrac{1}{2}l,\,l-1;\lambda_+^{-2})\,. \qquad (28)$$

We now make use of the Mellin-Barnes representation [15]

$${}_2F_1(\tfrac{1}{2}(l+1),\tfrac{1}{2}l;l-1;\lambda_+^{-2}) = \int_{-i\infty}^{+i\infty}\mathrm{d}\tau\,(-\lambda_+^2)^{-\tau}$$

$$\times\,\Gamma(\tfrac{1}{2}(l+1)+\tau)\,\Gamma(\tfrac{1}{2}l+\tau)\,\Gamma(-\tau)\,[\Gamma(l-1+\tau)]^{-1}\,, \qquad (29)$$

and we finally get

$$\langle 0\,|A(x)B(y)\,C(z)D(t)|\,0\rangle = \left[\frac{1}{(x-y)^2}\right]^{\frac{1}{2}(l_A+l_B-l)}\left[\frac{1}{(z-t)^2}\right]^{\frac{1}{2}(l_C+l_D-l)}$$

$$\times\left[\frac{1}{(x-t)^2}\right]^{\frac{1}{2}(l_A-l_B+l)}\left[\frac{1}{(y-z)^2}\right]^{\frac{1}{2}(l_C-l_D+l)}\left[\frac{1}{(y-t)^2}\right]^{\frac{1}{2}(l_B-l_A-l_C+l_D)}$$

$$\times\,\rho^{\frac{1}{2}(l_C-l_D+l)}\int_0^1 \mathrm{d}\sigma\,\sigma^{\frac{1}{2}(l_A-l_B-l_C+l_D)-1}\,(1-\sigma)^{\frac{1}{2}(l_B-l_A-l_C+l_D)-1}$$

$$\times\left(\frac{\rho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{\frac{1}{2}(l_D-l_C-l)}\;{}_2F_1\left[\tfrac{1}{2}(l+l_D-l_C),\tfrac{1}{2}(l+l_C-l_D);l-1;\left(\frac{\rho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{-1}\right]. \qquad (30)$$

We have introduced the harmonic ratios

$$\rho = \frac{(x-t)^2\,(z-y)^2}{(x-y)^2(z-t)^2}\,, \qquad \eta = \frac{(x-z)^2\,(y-t)^2}{(x-y)^2(z-t)^2}\,. \tag{31}$$

Eq. (30) is manifestly conformal covariant, as it depends on the harmonic ratios. However eq. (28), which can be written as

$$\left[\frac{1}{(x-y)^2}\right]^{\frac{1}{2}(l_A+l_B)}\left[\frac{1}{(z-t)^2}\right]^{\frac{1}{2}(l_C+l_D)}$$
$$\times \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v\, f_{AB}(u)\, f_{CD}(v)\, \lambda_+^{-l}\;{}_2F_1(\tfrac{1}{2}(l+1), \tfrac{1}{2}l; l-1; \lambda_+^{-2})\,, \tag{32}$$

where

$$f_{AB}(u) = u^{\frac{1}{2}(l_A-l_B)-1}\,(1-u)^{\frac{1}{2}(l_B-l_A)-1}$$

(and similarly for f_{CD}), exhibits manifest symmetry under interchange of the eternal legs.

We also note that the spectral condition is satisfied by the expression (30). This can easily be seen by observing that (30) factorizes in momentum space into a product of two vertices and one propagator, which have Fourier transforms with support at positive p^2.

4. CONNECTION TO THE WICK-ROTATED AMPLITUDE

Let us consider the four-point function in Euclidean space. We expand $A(x)B(y)$ and $C(z)D(t)$ obtaining a double integral. Inserting the vertex identity of sect. 1 we obtain

$$A(xyzt) \equiv \langle 0\,|A(x)B(y)\,C(z)D(t)|\,0\rangle = \left[\frac{1}{(x-y)^2}\right]^{\frac{1}{2}(l_A+l_B-l)}\left[\frac{1}{(z-t)^2}\right]^{\frac{1}{2}(l_C+l_D-l)}$$

$$\times \int \mathrm{d}^4\xi \left[\frac{1}{(x-\xi)^2}\right]^{\frac{1}{2}(l_A-l_B+l)}\left[\frac{1}{(x-\xi)^2}\right]^{\frac{1}{2}(l_B-l_A+l)}\left[\frac{1}{(z-\xi)^2}\right]^{\frac{1}{2}(l^*+l_C-l_D)}\left[\frac{1}{(t-\xi)^2}\right]^{\frac{1}{2}(l^*+l_D-l_C)}. \tag{33}$$

By a simple generalization of the standard Feynman representation we find

$$A(xyzt) = \left[\frac{1}{(x-y)^2}\right]^{l_B} \left[\frac{1}{(x-z)^2}\right]^{\frac{1}{2}(l_A - l_B + l_C - l_D)}$$

$$\times \left[\frac{1}{(x-t)^2}\right]^{\frac{1}{2}(l_A - l_B + l_D - l_C)} \left[\frac{1}{(z-t)^2}\right]^{\frac{1}{2}(l_B - l_A + l_D - l_C)}$$

$$\times \int_0^1 du\, u^{-1+\frac{1}{2}(l_C - l_D + l_A - l_B)} (1-u)^{-1+\frac{1}{2}(l_A - l_B + l_D - l_C)}$$

$$\times {}_2F_1\left(\tfrac{1}{2}(l_B - l_A + l^*), \tfrac{1}{2}(l_B - l_A + l); 2; 1 - \frac{\rho}{u} - \frac{\eta}{1-u}\right) . \qquad (34)$$

The derivation holds for $l < 2$; for $l > 2$ the formula is understood as an analytic continuation.

Note that the amplitude is symmetric under $l \leftrightarrow l^* = 4 - l$, due to the symmetry of ${}_2F_1$ in its first two arguments. The symmetry is evident from the method used in reducing the amplitude to the form (33) through the vertex identity. For $(x-y)^2 \to 0$ one finds the singularities

$$[(x-y)^{-2}]^{\frac{1}{2}(l_A + l_B - l)} , \qquad [(x-y)^{-2}]^{\frac{1}{2}(l_A + l_B - l) - 2} . \qquad (35)$$

The first one is the right one expected in the Wilson expansion; the second one corresponds to a contribution in the expansion of $A(x)B(y)$ of the shadow operator [12] of dimensions $l^* = 4 - l$. However, by a known decomposition of the hypergeometric function [16], one can decompose the amplitude into a contribution from exchange of dimension l and one from l^*. Such decomposition destroys the shadow symmetry [12]: it is analogous to the similar situation in Regge theory for the twin symmetry. It is interesting to observe that the contribution which corresponds to the right light-cone singularity is exactly the one calculated from the Wilson expansion in Minkowsky space, as given by (30).

5. HIGHER-SPIN CONTRIBUTIONS

Before treating the general case, we calculate the contribution of a first-order tensor (four-vector) to $W(xyzt)$. We first recall the general vertex identity derived in ref. [12].

$$\langle 0 | \mathrm{T}(A(x)B(y)\, O_{\alpha_1 \ldots \alpha_n}(0)) | 0 \rangle$$

$$= \int \mathrm{d}^4\xi \, \langle 0 | \mathrm{T}(A(x)B(y)\, O^*_{\beta_1 \ldots \beta_n}(t)) | 0 \rangle \langle 0 | \mathrm{T}(O^{\beta_1 \ldots \beta_n}(t)\, O_{\alpha_1 \ldots \alpha_n}(0)) | 0 \rangle \,, \tag{37}$$

where

$$\langle 0 | \mathrm{T}(A(x)B(y)\, O_{\alpha_1 \ldots \alpha_n}(z)) | 0 \rangle = [(x-y)^2]^{-\frac{1}{2}(l_A + l_B - \tau_n)}$$

$$\times \; [(x-z)^2]^{-\frac{1}{2}(l_A - l_B + \tau_n)} \, [(y-z)^2]^{-\frac{1}{2}(l_B - l_A + \tau_n)}$$

$$\times \left\{ \left(\frac{1}{z-x} + \frac{1}{y-z} \right)_{\alpha_1} \cdots \left(\frac{1}{z-x} + \frac{1}{y-z} \right)_{\alpha_n} - \text{traces} \right\} ,$$

$$\tau_n = l_n - n \,, \qquad \left(\frac{1}{x} \right)_\mu \equiv \frac{x_\mu}{x^2} \;. \tag{38}$$

In (38) the distances are all supposed space-like. We also recall the prescription for writting down the conformal covariant Wilson expansion as an integral over space-time leading to eq. (22b) (ref. [8]). We then realize that for an n=1 exchanged four-vector everything runs as for $n = 0$, see (22b), provided one substitutes $l \to \tau$ and adds in the integral representation for the four-point function additional terms arising from

$$\left(\frac{1}{\xi - y} + \frac{1}{x - \xi} \right)_\alpha \left(\frac{1}{\xi - z} + \frac{1}{t - \xi} \right)^\alpha . \tag{40}$$

The four terms from (40) are each related to scalar contributions of shifted dimensions (by one unity) and modified by powers of the external distances. Such contributions can be obtained from (22b) by substituting respectively

$$\begin{cases} \Delta_{AB} \to \Delta_{AB} + 1 \,, \\ \Delta_{CD} \to \Delta_{CD} + 1 \,, \end{cases} \qquad \begin{cases} \Delta_{AB} \to \Delta_{AB} - 1 \,, \\ \Delta_{CD} \to \Delta_{CD} - 1 \,, \end{cases}$$

$$\begin{cases} \Delta_{AB} \to \Delta_{AB} + 1 \,, \\ \Delta_{CD} \to \Delta_{CD} - 1 \,, \end{cases} \qquad \begin{cases} \Delta_{AB} \to \Delta_{AB} - 1 \,, \\ \Delta_{CD} \to \Delta_{CD} + 1 \,, \end{cases} \tag{41}$$

where

$$\Delta_{AB} = l_A - l_B\,, \qquad \Delta_{CD} = l_C - l_D\,. \tag{42}$$

We introduce

$$\Sigma_{AB} = l_A + l_B\,, \qquad \Sigma_{CD} = l_C + l_D\,, \tag{43}$$

and call

$$W(\Sigma_{AB}, \Sigma_{CD}, \Delta_{AB}; \Delta_{CD}; l_n, n\,|\,xyzt)\,, \tag{44}$$

the contribution to $W(xyzt)$ for the four operators $A(x), B(y), C(z), D(t)$, from a local tensor of order n and dimension l_n. For $n = 1$ we have

$$\begin{aligned} W(\Sigma_{AB}, \Sigma_{CD}, \Delta_{AB}, \Delta_{CD}; l_1, 1\,|\,xyzt) &= [(x-y)^2(z-t)^2]^{-\frac{1}{2}} \\ &\times \{(y-z)^2 W(\Sigma_{AB}, \Sigma_{CD}, \Delta_{AB}-1, \Delta_{CD}+1; l_1, 0\,|\,xyzt) + (x-t)^2 \\ &\times W(\Sigma_{AB}, \Sigma_{CD}, \Delta_{AB}+1, \Delta_{CD}-1; l_1, 0\,|\,xyzt) - (x-z)^2 \\ &\times W(\Sigma_{AB}, \Sigma_{CD}, \Delta_{AB}+1, \Delta_{CD}+1; l_1, 0\,|\,xyzt) - (y-t)^2 \\ &\times W(\Sigma_{AB}, \Sigma_{CD}, \Delta_{AB}-1, \Delta_{CD}-1; l_1, 0\,|\,xyzt)\}\,. \end{aligned}$$

We introduce formal operators $\Delta^{\pm}_{AB}, \Delta^{\pm}_{CD}$ which shift the corresponding values of Δ by ± 1. They satisfy $[\Delta^{\pm}_{AB}, \Delta^{\pm}_{CD}] = 0$ and $\Delta^{+}_{AB}\Delta^{-}_{AB} = 1 = \Delta^{+}_{CD}\Delta^{-}_{CD}$. Let us introduce

$$\begin{aligned} T = [(x-y)^2 (z-t)^2]^{-\frac{1}{2}} [&(y-z)^2\, \Delta^{-}_{AB}\Delta^{+}_{CD} + (x-t)^2\, \Delta^{+}_{AB}\, \Delta^{-}_{CD} \\ &- (x-z)^2\, \Delta^{+}_{AB}\, \Delta^{+}_{CD} - (y-t)^2\, \Delta^{-}_{AB}\, \Delta^{-}_{CD}]\,, \end{aligned} \tag{46}$$

and rewrite (45) as

$$W(\Sigma_{AB}, \Sigma_{CD}, \Delta_{AB}, \Delta_{CD}; l_1, 1\,|\,xyzt) = T\, W(\Sigma_{AB}, \Sigma_{CD}, \Delta_{AB}, \Delta_{CD}; l_1, 0\,|\,xyzt)\,. \tag{47}$$

In general, for a tensor of order n, the additional tensor structure (essentially from (38)) contains the nth power of a term as in (40) and additional lower spin contributions to ensure Lorentz irreducibility. The final result is formally

$$W(\Sigma_{AB}\Sigma_{CD}\Delta_{AB}\Delta_{CD}; l_n, n\,|\,xyzt) = C'_n(T)\, W(\Sigma_{AB}\Sigma_{CD}\Delta_{AB}\Delta_{CD}; l_n, 0\,|\,xyzt)\,, \tag{48}$$

where $C_n'(T)$ is a Gegenbauer polynomial and T is defined in (45)–(47).

The integral representation (32) gives a more useful expression. For the vector contribution one has

$$W(\Sigma_{AB},\Sigma_{CD},\Delta_{AB},\Delta_{CD};l_1,1\,|\,xyzt) = [(x-y)^2]^{-\Sigma_{AB}}[(z-t)^2]^{-\Sigma_{CD}}$$

$$\times\int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v\, f_{\Delta_{AB}}(u)\, f_{\Delta_{CD}}(v)\, \lambda_+^{-l_1}\; {}_2F_1(\tfrac{1}{2}(l_1+1), \tfrac{1}{2}l_1; l_1-1; \lambda_+^{-2})\, 2\lambda_- \,, \qquad (49)$$

where

$$f_{\Delta_{AB}}(u) = u^{\frac{1}{2}\Delta_{AB}-1}(1-u)^{-\frac{1}{2}\Delta_{AB}-1}\,, \qquad \lambda_\pm = -\tfrac{1}{2}[uv(1-u)(1-v)(x-y)^2(z-t)^2]^{-\frac{1}{2}}$$

$$\times[-v(1-u)(z-y)^2 - u(1-v)(x-t)^2 \mp (1-u)(1-v)(y-t)^2 \mp uv(x-z)^2]\,.$$

For the general tensor contribution one obtains

$$W(\Sigma_{AB},\Sigma_{CD},\Delta_{AB},\Delta_{CD};l_n,n\,|\,xyzt) = [(x-y)^2]^{-\Sigma_{AB}}[(z-t)^2]^{-\Sigma_{CD}}$$

$$\times\int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v\, f_{\Delta_{AB}}(u)\, f_{\Delta_{CD}}(v)\, C_n'(2\lambda_-)\, \lambda_+^{-l_n}\; {}_2F_1(\tfrac{1}{2}(l_n+1), \tfrac{1}{2}l_n; l_n-1; \lambda_+^{-2})\,. \qquad (50)$$

In particular for $n = 0$ we recover (32).

Note that under

$$u \leftrightarrow 1-u\,, \qquad x \leftrightarrow y\,, \qquad \Delta_{AB} \leftrightarrow \Delta_{BA} = -\Delta_{AB}\,,$$

one has

$$\lambda_\pm \leftrightarrow -\lambda_\pm\,,$$

and from

$$C_n'(-x) = (-)^n\, C_n'(x)\,, \qquad (51)$$

one finds that the amplitude is odd in $x-y$ for odd n, even for even n (and similarly for $z-t$).

We remark that (50) can be continued analytically in the spin label n (reggeization) from the analiticity of C_n' in n.

We also note that (50) depends on Σ_{AB}, Σ_{CD} only through the external line propagators, whereas the kernel depends on Δ_{AB}, Δ_{CD}. We see that an expansion

$$W(xyzt) = [(x-y)^2]^{-\Sigma_{AB}}[(z-t)^2]^{-\Sigma_{CD}} \int_0^1 du \int_0^1 dv\, f_{\Delta_{AB}}(u) f_{\Delta_{CD}}(v) F(\lambda_+, \lambda_-) \,, \tag{51}$$

(where the f's were defined after (32), and λ_+, λ_- after (49)) is conformal covariant, since it obtains from (50) after summing over the whole spectrum of contributing spins and dimensions.

6. LIGHT-CONE LIMITS

We derive here the limits $(x-y)^2 \to 0$, $(z-t)^2 \to 0$ of the general contribution (n, l_n) to (50). In the limits one has

$$\lambda_+ \to \infty \,, \qquad \lambda_- \to \infty \,, \tag{52}$$

and from the asymptotic behaviours the C_n' and of the hypergeometric functions one finds for $(x-y)^2 \to 0$ (or for $(z-t)^2 \to 0$)

$$W(\Sigma_{AB}, \Sigma_{CD}, \Delta_{AB}, \Delta_{CD}; l_n, n\,|\,xyzt)$$

$$\sim [(x-y)^2]^{-\Sigma_{AB}}[(z-t)^2]^{-\Sigma_{CD}} \int_0^1 du \int_0^1 dv\, f_{\Delta_{AB}}(u) f_{\Delta_{CD}}(v) \,, \tag{53}$$

$$\times (\lambda_-)^n (\lambda_+)^{-l_n} = [(x-y)^2]^{-\frac{1}{2}(l_A+l_B-\tau_n)} [(z-t)^2]^{-\frac{1}{2}(l_C+l_D-\tau_n)}$$

$$\times \int_0^1 du \int_0^1 dv\, u^{\frac{1}{2}(l_A-l_B+\tau_n)-1} (1-u)^{\frac{1}{2}(l_B-l_A+\tau_n)-1}$$

$$\times v^{\frac{1}{2}(l_C-l_D+\tau_n)-1} (1-v)^{\frac{1}{2}(l_D-l_C+\tau_n)-1} [-v(1-u)(z-y)^2$$

$$-u(1-v)(x-t)^2 + (1-u)(1-v)(y-t)^2 + uv(x-z)^2]^n$$

$$\times [-v(1-u)(z-y)^2 - u(1-v)(x-t)^2 - (1-u)(1-v)(y-t)^2 - uv(x-z)^2]^{-l_n} \,.$$

The integral representation in the latter expression converges for $\tau_n > |\Delta|$ $(\Delta = \Delta_{AB}, \Delta_{CD})$. We find that the integral representation does not give additional singularities for light-like distances and we recover the singularities of the light-cone expansion of ref. [6]. In particular for $n = 0$ the integral in (53) gives

$$[(x-z)^2]^{\frac{1}{2}(l_D-l_C-l)}\,[(x-t)^2]^{\frac{1}{2}(l_B-l_A+l_C-l_D)}$$

$$\times\,[(y-t)^2]^{\frac{1}{2}(l_A-l_B-l)}\;{}_2F_1\left(\tfrac{1}{2}(l_B-l_A+l),\tfrac{1}{2}(l_C-l_D+l);l;1-\frac{(x-t)^2(y-z)^2}{(x-z)^2(y-t)^2}\right). \quad (54)$$

The calculation of (54) is given in appendix A. It turns out to be particularly simple as it corresponds to putting ${}_0F_1 = 1$ in the Wilson expansion (12).

Finally let us consider the crossed light-cone limits for the contribution (32) to $W(xyzt)$,

$$\lim_{(x-t)^2\to 0} W(xyzt)\,, \qquad \lim_{(z-y)^2\to 0} W(xyzt)\,. \quad (55)$$

From (30) we get

$$\lim_{(x-t)^2\to 0} W(\Sigma_{AB},\Sigma_{CD},\Delta_{AB},\Delta_{CD};l,0\,|\,xyzt)$$

$$\sim [(x-t)^2]^{-\frac{1}{2}(l_A+l_D-l_C-l_B)}\{[(x-y)^2]^{-\frac{1}{2}(l_A+l_B-l)}\,[(z-t)^2]^{-\frac{1}{2}(l_C+l_D-l)}$$

$$\times\,[(y-t)^2]^{-\frac{1}{2}(l_B-l_A+l)}\,[(x-z)^2]^{-\frac{1}{2}(l_C-l_D+l)}\int_0^1 \mathrm{d}\sigma\,\sigma^{\frac{1}{2}(l_A-l_B-l_C+l_D)-1}$$

$$\times\,(1-\sigma)^{\frac{1}{2}(l_B-l_A+l)-1}\;{}_2F_1(\tfrac{1}{2}(l+l_D-l_C),\tfrac{1}{2}(l+l_C-l_D);l-1;(1-\sigma)\,\eta^{-1})\,, \quad (56)$$

and the same for $(z-y)^2 \to 0$. The integral converges for $l_A + l_D > l_C + l_B$ and $l > l_A - l_B$. Eq. (56) shows that the crossed-channel light-cone singularity is independent of the internal dimensions. The same happens for higher spin contributions.

7. CONCLUSIONS

We have solved the problem of the conformal covariant four-point function, or, equivalently, of the conformal covariant expansion of the product of three local operators (Wilson expansion for three operators).

The method we employed is based on insertion of the covariant Wilson expansion and use of the two-point selection rule. The four-point function is expressed (see (50)) as a sum of irreducible conformal contributions, describing the exchange of an irreducible conformal tensor.

We have discussed both Wightman functions and vacuum expectation-values of T-products. In the Euclidean space we have found the additional singularities due to the occurrence of the shadow symmetry, a symmetry similar to twin symmetry in Regge theory.

Our expression allows for an explicit verification of the various light-cone limits on the four point function, and we have studied in detail the different limiting produres.

The vertex-identities from conformal symmetry play an important algebric role. We have also given a simple momentum-space description of such identities. We have verified the analiticity and spectral properties implicit in our expressions for the irreducible contributions to the four-point functions.

The solution, obtained here, of the problem of a conformal covariant expansion for four-point functions allows for interesting prospects. Practical applications may take advantage of a conformally covariant Wilson expansion for three operators, which is now at our disposal. One will also have to examine in detail the consequences of crossing in the expressions derived here, that is of exchange in the Wightman functions of the order of operators for space-like distances [10]. Finally, the explicit dependence of the terms in the expansion from the spin label of the exchanged local tensor field suggests a possible conformal covariant procedure of Reggeization.

Some remarks by Professor Kurt Symanzik, whom we would like to thank, have been useful to us in carrying out this investigation.

APPENDIX A

We report some details of the calculation of the scalar contribution to the four-point function. From (28) and (29) we get

$$\langle 0|A(x)B(y)C(z)D(t)|0\rangle = [(x-y)^2]^{-\frac{1}{2}(l_A+l_B-l)}\,[(z-t)^2]^{-\frac{1}{2}(l_C+l_D-l)}$$

$$\times \int_{-\mathrm{j}\infty}^{+\mathrm{j}\infty} \mathrm{d}\tau \frac{\Gamma(\frac{1}{2}(l+1)+\tau)\Gamma(\frac{1}{2}l+\tau)\Gamma(-\tau)}{\Gamma(l-1+\tau)}\, 2^{2\tau}(-1)^{-\tau} \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v\, f_{AB0}(u)\, f_{CD0}(v)$$

$$\times (A^2B^2)^{-\frac{1}{2}l}(-\lambda_+)^{-2\tau-l} = [(x-y)^2]^{-\frac{1}{2}(l_A+l_B)}\,[(z-t)^2]^{-\frac{1}{2}(l_C+l_D)}$$

$$\times \int_{-\mathrm{j}\infty}^{+\mathrm{j}\infty} \mathrm{d}\tau\,\Gamma(\tfrac{1}{2}(l+1)+\tau)\Gamma(\tfrac{1}{2}l+\tau)\,\Gamma(-\tau)(-1)^{-2\tau}\,2^{2\tau}(-1)^{\tau} \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v\, u^{\frac{1}{2}(l_A-l_B)-1}$$

$$\times (1-u)^{\frac{1}{2}(l_B-l_A)-1}\, v^{\frac{1}{2}(l_C-l_D)-1}\,(1-v)^{\frac{1}{2}(l_D-l_C)-1}\,[uv(1-u)(1-v)(x-y)^2(z-t)^2]^{\frac{1}{2}l+\tau}$$

$$\times \{-(y-t)^2(1-u)(1-v)-u(1-v)(x-t)^2-v(1-u)(z-y)^2-uv(x-z)^2\}^{-2\tau-l}$$

$$= [(x-y)^2]^{-\frac{1}{2}(l_A+l_B-l)} [(z-t)^2]^{-\frac{1}{2}(l_C+l_D-l)} \int_{-i\infty}^{+i\infty} d\tau \Gamma(\tfrac{1}{2}(l+1)+\tau)\Gamma(\tfrac{1}{2}l+\tau)\Gamma(-\tau)2^{2\tau}$$

$$\times(-1)^{-3\tau}\int_0^1 du \int_0^1 dv\, u^{\frac{1}{2}(l_A-l_B+l)-1+\tau} (1-u)^{\frac{1}{2}(l_B-l_A+l)-1+\tau}$$

$$\times v^{\frac{1}{2}(l_C-l_D+l)-1+\tau} (1-v)^{\frac{1}{2}(l_D-l_C+l)-1+\tau} [(x-y)^2(z-t)^2]^\tau$$

$$\times \{(y-t)^2(1-u)(1-v) + u(1-v)(x-t)^2 + v(1-u)(z-y)^2 + uv(x-z)^2\}^{-2\tau-l} . \tag{A.1}$$

We integrate over u obtaining

$$\int_0^1 dv\, v^{\frac{1}{2}(l_C-l_D+l)+\tau-1} (1-v)^{\frac{1}{2}(l_D-l_C+l)+\tau-1} 2^{2\tau}(-1)^\tau$$

$$\times \{(y-t)^2 + v[(z-y)^2-(y-t)^2]\}^{\frac{1}{2}(l_A-l_B-l)-\tau}$$

$$\times \{(x-t)^2 + v[(x-z)^2 - (x-t)^2]\}^{\frac{1}{2}(l_B-l_A-l)-\tau} B(\tfrac{1}{2}(l_A-l_B+l)+\tau, \tfrac{1}{2}(l_B-l_A+l)+\tau) . \tag{A.2}$$

By use of the integral representation for the double hypergeometric function (in ref. [16]) we integrate over v. The double integral in du, dv in (A.1) becomes

$$B(\tfrac{1}{2}(l_A-l_B+l)+\tau, \tfrac{1}{2}(l_B-l_A+l)+\tau)\, B(\tfrac{1}{2}(l_C-l_D+l)+\tau, \tfrac{1}{2}(l_D-l_C+l)+\tau)$$

$$\times 2^{2\tau}(-1)^\tau [(x-t)^2]^{-\frac{1}{2}(l_A-l_B+l)-\tau} [(z-y)^2]^{-\frac{1}{2}(l_C-l_D+l)-\tau}$$

$$\times [(y-t)^2]^{\frac{1}{2}(l_A-l_B+l_C-l_D)}$$

$$\times {}_2F_1\left(\tfrac{1}{2}(l_C-l_D+l)+\tau, \tfrac{1}{2}(l_A-l_B+l)+\tau; l+2\tau; 1-\frac{(x-z)^2(y-t)^2}{(x-t)^2(z-y)^2}\right) . \tag{A.3}$$

Inserting (A.3) into (A.1) we obtain

$$\langle 0|A(x)B(y)C(z)D(t)|0\rangle = [(x-y)^2]^{-\frac{1}{2}(l_A+l_B-l)} [(z-t)^2]^{-\frac{1}{2}(l_C+l_D-l)}$$

$$\times [(x-t)^2]^{-\frac{1}{2}(l_A-l_B+l)} [(y-z)^2]^{-\frac{1}{2}(l_C-l_D+l)} [(y-t)^2]^{-\frac{1}{2}(l_B-l_A-l_C+l_D)}$$

$$\times \int_{-i\infty}^{+i\infty} d\tau \frac{\Gamma(\frac{1}{2}(l+1)+\tau)\,\Gamma(\frac{1}{2}l+\tau)\,\Gamma(-\tau)}{\Gamma(l-1+\tau)} \; [(x-t)^2(z-y)^2]^{-\tau}[(x-y)^2(z-t)^2]^{\tau}(-1)^{\tau}$$

$$\times B(\tfrac{1}{2}(l_A-l_B+l)+\tau, \tfrac{1}{2}(l_B-l_A+l)+\tau)\, B(\tfrac{1}{2}(l_C-l_D+l)+\tau, \tfrac{1}{2}(l_D-l_C+l)+\tau)$$

$$\times\, {}_2F_1\left(\tfrac{1}{2}(l_C-l_D+l)+\tau, \tfrac{1}{2}(l_A-l_B+l)+\tau; l+2\tau; 1-\frac{(x-z)^2(y-t)^2}{(x-t)^2(z-y)^2}\right) .$$

We define the harmonic ratios

$$\zeta = \frac{(x-z)^2(y-t)^2}{(x-t)^2(z-y)^2}, \qquad \rho = \frac{(x-t)^2(z-y)^2}{(x-y)^2(z-t)^2}, \qquad \eta = \frac{(x-z)^2(y-t)^2}{(x-y)^2(z-t)^2} = \rho\zeta .$$

Using the fundamental representation for the hypergeometric function we can evaluate the Mellin-Barnes integral obtaining

$$\int_0^1 d\sigma\, \sigma^{\frac{1}{2}(l_A-l_B+l)-1}\, (1-\sigma)^{\frac{1}{2}(l_B-l_A+l)-1}\, [(1-\sigma)(1-\zeta)]^{\frac{1}{2}(l_D-l_C-l)}$$

$$\times \int_{-i\infty}^{+i\infty} d\tau\, \frac{\Gamma(-\tau)\Gamma(\frac{1}{2}(l+l_D-l_C)+\tau)\,\Gamma(\frac{1}{2}(l+l_C-l_D)+\tau)}{\Gamma(l-1+\tau)} \left[-\frac{\sigma(1-\sigma)}{\rho\,[1-\sigma(1-\zeta)]}\right]^{\tau}$$

$$= \int_0^1 d\sigma\, \sigma^{\frac{1}{2}(l_A-l_B+l)-1}\, (1-\sigma)^{\frac{1}{2}(l_B-l_A+l)-1}\, [1-\sigma(1-\zeta)]^{\frac{1}{2}(l_D-l_C-l)}$$

$$\times\, {}_2F_1\left(\tfrac{1}{2}(l+l_D-l_C), \tfrac{1}{2}(l+l_C-l_D); l-1; \left[\frac{\rho}{\sigma}+\frac{\eta}{1-\sigma}\right]^{-1}\right). \qquad \text{(A.4)}$$

Insertion of (A.4) into (A.1) leads to (30).

We next derive the light-cone limit of the scalar contribution as given by (53) with insertion of (54). We first observe from the general structure

$$\langle 0|A(x)B(y)\,C(z)D(t)|0\rangle = [(x-y)^2]^{-l_B}\, [(x-z)^2]^{\frac{1}{2}(l_B-l_A+l_D-l_C)}$$

$$\times\, [(x-t)^2]^{\frac{1}{2}(l_B-l_A+l_C-l_D)}\, [(z-t)^2]^{\frac{1}{2}(l_A-l_B-l_C-l_D)}\, f(\rho,\eta)\,, \qquad \text{(A.5)}$$

that the limits $(x-y)^2 \to 0$ with $(z-t)^2$ fixed, or $(z-t)^2 \to 0$ with $(x-y)^2$ fixed are already equivalent to the limit $(x-y)^2 \to 0$, $(z-t)^2 \to 0$, as in each case $\rho \to \infty$, $\eta \to \infty$, with ρ/η finite. We thus use the light-cone restriction [6] of the expansions for $A(x)B(y)$ and $C(z)D(t)$ respectively obtaining for $(x-y)^2 \to 0$ and (or) $(z-t)^2 \to 0$

$$\langle 0|A(x)B(y)C(z)D(t)|0\rangle \sim [(x-y)^2]^{-\frac{1}{2}(l_A+l_B-l)}$$

$$\times\ [(z-t)^2]^{-\frac{1}{2}(l_C+l_D-l)} \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v\, f_{AB0}(u)\, f_{CD0}(v)$$

$$\times\ [ux+(1-u)y-vz-(1-v)\,t)^2]^{-l} = [(x-y)^2]^{-\frac{1}{2}(l_A+l_B-l)}$$

$$\times\ [(x-t)^2]^{\frac{1}{2}(l_B-l_A-l)}[(y-t)^2]^{\frac{1}{2}(l_A-l_B-l)}$$

$$\times [(z-t)^2]^{-\frac{1}{2}(l_C+l_D-l)} \int_0^1 \mathrm{d}v\, v^{\frac{1}{2}(l_C-l_D+l)-1}\,(1-v)^{\frac{1}{2}(l_D-l_C+l)-1}$$

$$\times \left[1-v\left(1-\frac{(x-z)^2}{(x-t)^2}\right)\right]^{\frac{1}{2}(l_B-l_A-l)} \left[1-v\left(1-\frac{(y-z)^2}{(y-t)^2}\right)\right]^{\frac{1}{2}(l_A-l_B-l)} . \tag{A.6}$$

Eq. (A.6) can be written in terms of a double hypergeometric function as

$$[(x-y)^2]^{-\frac{1}{2}(l_A+l_B-l)}\ [(z-t)^2]^{-\frac{1}{2}(l_C+l_D-l)}\ [(y-t)^2]^{\frac{1}{2}(l_A-l_B-l)}$$

$$\times [(x-t)^2]^{\frac{1}{2}(l_B-l_A-l)}\, F_1\left(\tfrac{1}{2}(l_C-l_D+l), \tfrac{1}{2}(l_A-l_B+l), \tfrac{1}{2}(l_B-l_A+l), l,\ 1-\frac{(x-z)^2}{(x-t)^2}, 1-\frac{(y-z)^2}{(y-t)^2}\right), \tag{A.7}$$

which in turn (as a consequence of conformal symmetry) reduces to an expression containing a simple hypergeometric function of a harmonic ratio, according to (54).

APPENDIX B

We derive the conformal covariant four-point function as obtained by use of the vertex identities for the T-ordered products (eq. (37)). Eq. (33) is obviously symmetric under $l \leftrightarrow l^* = 4-l$. The space-time integral can be rewritten as

$$\int \mathrm{d}^4\xi \int \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma\, \mathrm{d}\delta\, \alpha^{\frac{1}{2}(\Delta_{AB}+l)-1}\, \beta^{\frac{1}{2}(-\Delta_{AB}+l)-1}\, \gamma^{1+\frac{1}{2}(\Delta_{CD}-l)}\, \delta^{1-\frac{1}{2}(\Delta_{CD}+l)}$$

$$\times\ \mathrm{e}^{\alpha(x-\xi)^2+\beta(y-\xi)^2+\delta(z-\xi)^2+\gamma(t-\xi)^2}\ ; \tag{B:1}$$

Integrating (17) over $\mathrm{d}^4\xi$ we get the expression

$$\int \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma\, \mathrm{d}\delta\, \alpha^{\frac{1}{2}(\Delta_{AB}+l)-1}\, \beta^{\frac{1}{2}(-\Delta_{AB}+l)-1}\, \gamma^{1+\frac{1}{2}(\Delta_{CD}-l)}\, \delta^{1-\frac{1}{2}(\Delta_{CD}+l)}$$

$$\mathrm{e}^{-\{\alpha\beta(x-y)^2+\alpha\gamma(x-z)^2} + \alpha\delta(x-t)^2 + \beta\gamma(y-z)^2 + \beta\delta(y-t)^2 + \gamma\delta(z-y)^2\}\ . \qquad \text{(B.2)}$$

We have made the substitution: $\alpha \to \alpha(\alpha+\beta+\gamma+\delta)^{-\frac{1}{2}}$, etc.

We can take, because of manifest conformal symmetry, $(x-y)^2 = (z-t)^2 = (x-t)^2$. Integrating over α we find

$$\int \mathrm{d}\beta\, \mathrm{d}\gamma\, \mathrm{d}\delta\, (\beta+\gamma+\delta)^{-\frac{1}{2}(\Delta_{AB}+l)}\, \beta^{-\frac{1}{2}(\Delta_{AB}-l)-1}\, \gamma^{1+\frac{1}{2}(\Delta_{CD}-l)}$$

$$\times\, \delta^{1-\frac{1}{2}(\Delta_{CD}-l)}\, \mathrm{e}^{-\{\beta\gamma(y-z)^2+\beta\delta(y-t)^2+\gamma\delta(z-t)^2\}}\, [(x-y)^2]^{-\frac{1}{2}(\Delta_{AB}+l)}\ . \qquad \text{(B.3)}$$

Using as new variable β, $\chi = \gamma/\beta$, $Y = \delta/\beta$, one has

$$\int \beta^2\, \mathrm{d}\beta\, \mathrm{d}\chi\, \mathrm{d}Y\, \beta^{-\frac{1}{2}(\Delta_{AB}+l)}\, (1+\chi+Y)^{-\frac{1}{2}(\Delta_{AB}+l)}$$

$$\times\, \beta^{1-\frac{1}{2}(\Delta_{AB}+l)}\, \chi^{1+\frac{1}{2}(\Delta_{CD}-l)}\, Y^{1-\frac{1}{2}(\Delta_{CD}+l)}$$

$$\times\, \mathrm{e}^{-\beta^2(\chi(y-z)^2+Y(y-t)^2+\chi Y(z-t)^2)}\, [(x-y)^2]^{-\frac{1}{2}(\Delta_{AB}-l)}$$

$$= \int \mathrm{d}\chi\, \mathrm{d}Y\, \chi^{1+\frac{1}{2}(\Delta_{CD}-l)}\, Y^{1-\frac{1}{2}(\Delta_{CD}+l)}\, (1+\chi+Y)^{-\frac{1}{2}(\Delta_{AB}+l)}$$

$$\times\, [\chi Y(z-t)^2 + \chi(y-z)^2 + Y(y-t)^2]^{\frac{1}{2}(\Delta_{AB}+l)-2}\, [(x-y)^2]^{-\frac{1}{2}(\Delta_{AB}-l)}\ . \qquad \text{(B.4)}$$

We introduce $\sigma = \chi + Y$, $\chi = \sigma u$, $(Y = \sigma(1-\mu))$, and find the expression

$$\int_0^1 \mathrm{d}u \int_0^\infty \mathrm{d}\sigma\, \sigma(1+\sigma)^{-\frac{1}{2}(\Delta_{AB}+l)}\, \sigma^{2-l}\, u^{1+\frac{1}{2}(\Delta_{CD}-l)}$$

$$\times\, (1-u)^{1-\frac{1}{2}(\Delta_{CD}+l)}\, \sigma^{\frac{1}{2}(\Delta_{AB}+l)-2}$$

$$\times\, [\sigma u(1-u)(z-t)^2 + u(y-z)^2 + (1-u)(y-t)^2]^{\frac{1}{2}(\Delta_{AB}+l)-2}$$

$$\times\, [(x-y)^2]^{-\frac{1}{2}(\Delta_{AB}+l)} = [(x-y)^2]^{-\frac{1}{2}(\Delta_{AB}+l)} \int_0^1 \mathrm{d}u\, u^{1+\frac{1}{2}(\Delta_{CD}-l)}$$

$$\times\, (1-u)^{1-\frac{1}{2}(\Delta_{CD}+l)}\, [u(y-z)^2 + (1-u)(y-t)^2]^{\frac{1}{2}(\Delta_{AB}+l)-2}$$

$$\times\, {}_2F_1\left(2-\tfrac{1}{2}(\Delta_{AB}+l), 2+\tfrac{1}{2}(\Delta_{AB}-l); 2; 1-\frac{u(1-u)(t-z)^2}{u(y-z)^2+(1-u)(z-t)^2}\right)$$

$$= [(x-y)^2]^{-\frac{1}{2}(\Delta_{AB}+l)}\,[(z-t)^2]^{\frac{1}{2}(\Delta_{AB}+l)-2}\int_0^1 \mathrm{d}u\, u^{-1+\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}$$

$$\times (1-u)^{-1+\frac{1}{2}(\Delta_{AB}-\Delta_{CD})}\, {}_2F_1\left(\tfrac{1}{2}(l^*-\Delta_{AB}), \tfrac{1}{2}(l-\Delta_{AB}), 2; 1-\frac{\eta}{u}-\frac{\rho}{1-u}\right) \tag{B.5}$$

giving for (33) the result

$$[(x-y)^2]^{-l_B}\,[(x-z)^2]^{\frac{1}{2}(l_B-l_A+l_D-l_C)}\,[(x-t)^2]^{\frac{1}{2}(l_B-l_A+l_C-l_D)}$$

$$\times\,[(z-t)^2]^{\frac{1}{2}(l_A-l_B+l_C-l_D)}\int_0^1 \mathrm{d}u\, u^{-1+\frac{1}{2}(l_C-l_D+l_A-l_B)}(1-u)^{-1+\frac{1}{2}(l_A-l_B+l_D-l_C)}$$

$$\times\, {}_2F_1\left(\tfrac{1}{2}(l_B-l_A+l^*), \tfrac{1}{2}(l_B-l_A+l); 2; 1-\frac{\eta}{u}-\frac{\rho}{1-u}\right). \tag{B.6}$$

Note the manifest $l \leftrightarrow l^*$ symmetry in (B.6). Using the decomposition (36), we find, for the piece having the right light-cone singularity, that the integral over u in (B.6) gets substituted by

$$\int_0^1 \mathrm{d}u\, u^{-1+\frac{1}{2}(l_C-l_D+l_A-l_B)}\,(1-u)^{-1+\frac{1}{2}(l_A-l_B+l_D-l_C)}$$

$$\times\left(\frac{\eta}{u}+\frac{\rho}{1-u}\right)^{\frac{1}{2}(l_A-l_B-l)} {}_2F_1\left(\tfrac{1}{2}(l_B-l_A+l), \tfrac{1}{2}(l_A-l_B+l); l-1; \left(\frac{\eta}{u}+\frac{\rho}{1-u}\right)^{-1}\right). \tag{B.7}$$

Finally, observing that (30) is invariant under the substitution $u \leftrightarrow 1-v$, $v \leftrightarrow 1-u$, $x \leftrightarrow t$, $z \leftrightarrow y$, $\Delta_{AB} \leftrightarrow -\Delta_{CD}$, we realize that (30) exactly coincides with (B.6) with the substitution of the expression (B.7).

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