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G. Parisi : ON SELF-CONSISTENCY CONDITIONS IN CONFORMAL COVARIANT FIELD THEORY

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On Self-Consistency Conditions in Conformal Covariant Field Theory (*).

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A very existing problem is to compute the anomalous dimensions [1] of the fundamental fields in a Lagrangian quantum field theory with scaling-invariant interaction.

An important step in this direction was done independently by MIGDAL [2] and by PELITI [3] and the author: they have shown that in a conformal invariant field theory with trilinear interaction it was possible to write a self-consistency equation for the vertex and the conformal invariant solution of this equation is determined by the knowledge of the renormalized coupling constant g and of the anomalous dimension η. In this way the integral Dyson equation was reduced to a simple numerical relation between η and g:

$$F(\eta, g) = 1 \,. \tag{1}$$

In the same paper [3] it was also shown that unitarity implies a self-consistency condition for the propagator and this condition is equivalent to a new indipendent constraint on η and g:

$$H_v(\eta, g) = 1 \,. \tag{2}$$

The solution of the system

$$\begin{cases} F(\eta, g) \;\; = 1 \,, \\ H_v(\eta, g) = 1 \,, \end{cases} \tag{3}$$

yields both the anomalous dimension of the field and the renormalized coupling costant for the zero-mass theory [4,5].

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(1) K. G. WILSON: *Phys. Rev.*, **179**, 1499 (1969); S. FERRARA, R. GATTO and A. F. GRILLO: Frascati preprint, to appear on *Springer Tracts*.

(2) A. A. MIGDAL: *Phys. Lett.*, **37** B, 386 (1971).

(3) G. PARISI and L. PELITI: *Lett. Nuovo Cimento*, **2**, 627 (1971).

(4) An excellent review of the argument and a detailed analysis of the convergence of the integrals in volved may be found in G. MACK and I. T. TODOROV: IC/71/139, Trieste, Oct. 1971.

(5) G. PARISI: *The dynamic of conformal invariant field theories*, Frascati preprint, LNF-71/80.

SYMANZIK (6) noticed that it was possible to write a second self-consistency equation for the propagator using the so-called Ward's differential equation for the propagator (7). In this way he arrives to the system

$$\left\{\begin{array}{l} F(\eta, g) = 1 \,, \\ H(\eta, g) = 1 \,. \end{array}\right. \tag{4}$$

It was recently shown by MACK and SYMANZIK (8) the complete equivalence of the two different approaches: the propagator satisfies the unitarity condition if the other two conditions are implemented and the systems (3) and (4) have the same solutions.

The aim of this letter is to find a very simple relation between the functions F and H in order to avoid unnecessary computations.

For the sake of simplicity let us consider a 6-dimensional $\lambda\varphi^3$ theory. The extension of the whole argument to a 4-dimensional $g\bar{\psi}\gamma_5\psi\pi$ is straightforward. The self-consistency equation for the existence of an operator O of dimension α is shown in Fig. 1:

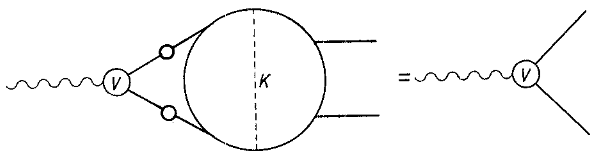

Fig. 1.

V is the vertex between O and the field φ and K is the kernel of the Bethe-Salpeter amplitude. In Fig. 2 is shown the diagrammatical expansion of the vertex in the product of the vertex Γ and of the propagator.

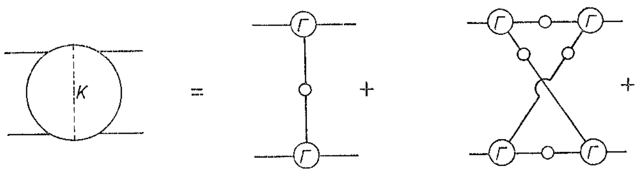

Fig. 2.

According to standard arguments (2-5) the self-consistency equation of fig, 1 is equivalent to the following equation:

$$D(\alpha, \eta, g) = 1 \,. \tag{5}$$

The result of this paper is that the system (4) is equivalent to the system

$$\left\{\begin{array}{l} D(\eta, \eta, g) = 1 \,, \\ \pi^{18} \dfrac{\Gamma^3(1+\eta/2)\,\Gamma^4(1-\eta)\,\Gamma(\frac{3}{2}\eta)\,\Gamma(-1+\eta)}{\Gamma^3(2-\eta/2)\,\Gamma^4(2+\eta)\,\Gamma(3-\eta/2)\,\Gamma(4-\eta)}\, g^2 \dfrac{\mathrm{d}}{\mathrm{d}\alpha} D(\alpha, \eta, g)\Big|_{\alpha=\eta} = 1 \,. \end{array}\right. \tag{6}$$

(6) K. SYMANZIK: *Lett. Nuovo Cimento* **3**, 734 (1972).
(7) K. SYMANZIK: in *Lectures on High-Energy Physics*, edited by B. JAKSIC (Zagreb, 1961; New York, 1965).
(8) G. MACK and K. SYMANZIK: DESY preprint 19/72 (April 1972).

The proof is simple: the functions $F(\eta, g)$ and $D(\eta, \eta, g)$ are identical by definition. The only problems arise from the second equation.

The Ward's equation for the self-energy is written in Fig. 3, where we denote by a dashed line the Green's function multiplied by the distance between the two points [9].

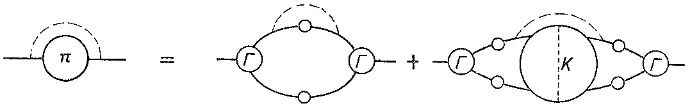

Fig. 3.

This equation can also be written as

$$\pi_\mu = \Gamma G_\mu G \Gamma + \Gamma G G K_\mu G G \Gamma \,, \tag{7}$$

where we designate the process of differentiation with respect to q_μ by the index μ, *e.g.* $\pi_\mu(q) = (\mathrm{d}/\mathrm{d}q_\mu)\pi(q^2)$.

Differentiating the Dyson equation for the vertex

$$\Gamma = \Gamma G G K \,, \tag{8}$$

we find

$$\Gamma_\mu = \Gamma_\mu G G K + \Gamma G_\mu G K + \Gamma G G K_\mu \,. \tag{9}$$

Using eqs. (7)-(9) together we arrive to the crazy result

$$\pi_\mu = 0 \,. \tag{10}$$

Equation (10) is wrong; the initial integral was convergent but it was broken in a sum of not convergent integrals ($\Gamma G_\mu G \Gamma$ is convergent but $\Gamma_\mu G G \Gamma$ is always logarithmically divergent); the right conclusion is $\pi_\mu = \infty - \infty$, an indeterminate form.

The only possibility to obtain a nonvoid formula is to make a slight modification of the initial integrals in order to have all the integrals convergent and return to the initial situation only in the final formula.

This can be realized in the following way: we compute the self-energy relative to the propagator of two operators, one of dimension η and the other of dimension $\eta + \varepsilon$, with ε small positive:

$$\pi^\varepsilon_\mu = \Gamma^\varepsilon G_\mu G \Gamma + \Gamma^\varepsilon G G K_\mu G G \Gamma \,. \tag{11}$$

All integrals in (11) are convergent and

$$\lim_{\varepsilon \to 0} \pi^\varepsilon_\mu = \pi_\mu \,. \tag{12}$$

With this modification also the other integrals are no more divergent and we find for small ε

$$\pi^\varepsilon_\mu = -\varepsilon \frac{\mathrm{d}}{\mathrm{d}\alpha} D(\alpha, \eta, g)\Big|_{\alpha=\eta} \Gamma^\varepsilon G G \Gamma_\mu \,. \tag{13}$$

([9]) K. Johnson, R. Willey and M. Baker: *Phys. Rev.*, **163**, 1699 (1967).

$\Gamma^6 GG\Gamma$ can be easily computed in the limit of small ε from the identity of ref. [10]:

$$\Gamma^6 GG\Gamma_\mu = \frac{\pi^{12}}{-\varepsilon}\left[\frac{\Gamma(1+\eta/2)\,\Gamma(1-\eta)}{\Gamma(2-\eta/2)\,\Gamma(2+\eta)}\right]^3 \frac{\Gamma(\frac{3}{2}\eta)}{\Gamma(3-\frac{3}{2}\eta)}\, g^2 (x^2)^{-4+\eta} x_\mu = \tag{14}$$

$$= \frac{\pi^{18}}{\varepsilon}\,\frac{\Gamma^3(1+\eta/2)\,\Gamma^4(1-\eta)\,\Gamma(\frac{3}{2}\eta)\,\Gamma(-1+\eta)}{\Gamma^3(2-\eta/2)\,\Gamma^4(2-\eta)\,\Gamma(3-\frac{3}{2}\eta)\,\Gamma(4-\eta)}\, g^2 \pi_\mu .$$

In the limit when ε goes to zero we find the second equation of system (6).

The main interest in formula (6) is due to its simplifying effects on the effective computation of anomalous dimensions [11].

* * *

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(10) M. D'ERAMO, G. PARISI and L. PELITI: *Lett. Nuovo Cimento*, **2**, 878 (1971).
(11) G. PARISI: in preparation.

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