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S. Ferrara and G. Parisi: CONFORMAL COVARIANT CORRELATION  
FUNCTIONS.

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## CONFORMAL COVARIANT CORRELATION FUNCTIONS

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Abstract: The general constraints of “full conformal symmetry”, on  $n$ -point correlation functions are discussed.

An identity for the three-point correlation function is derived from conformal symmetry, which allows to derive an integral representation for any “irreducible” conformal graph. The connections of these results to conformally covariant “generalized Wilson operator products expansions” are pointed out. Arguments which suggest locality property of a conformal covariant expansion for the  $n$ -point functions are discussed.

### 1. INTRODUCTION

In the recent years the importance of short distance and light-like distance behaviour of local operator products in field theories have been widely recognized.

In particular a method of investigating these behaviours has been proposed by Wilson [1] with its theory of broken scale invariance at short distances. Generalizations of Wilson’s ideas to light-like distance have already been pointed out by many authors [2-7]. The cross sections of deep inelastic electroproduction observed at SLAC emphasize the relevance of asymptotic scale invariance in the light cone region.

Recent work suggests that, in this configuration space limit, the stronger conformal invariance should be relevant too [8-11]. In fact extending Wilson analysis, the conformal algebra seems to be particularly powerful in deriving a class of properties of the so called skeleton theory and also in deriving additional properties of operator expansions generalized to the light cone. The aim of the present note is to derive and to summarize some properties of a supposed fully conformal invariant field theory. We remark that these properties should be valid in any “skeleton theory” at least from a lagrangian point of view. In fact, under quite general assumptions, scale invariance already implies conformal invariance.

The main results of this note concern:

- (i) A simple connection between conformally covariant three-point correlation functions and conformal covariant O.P.E. (Operator Product Expansion) and a simple understanding of some properties of the latter;
- (ii) An interesting set of properties and identities between conformally covariant graphs;

(iii) A demonstration of the consistency of the conformal covariant O.P.E. with the locality property of fields.

Finally we point out that some of these results may be relevant to build up a class of physically interesting models for scattering amplitudes in high-energy limits and perhaps to a more comprehensive understanding of the observed scaling law of SLAC inelastic electroproduction experiments. In fact we emphasize that, also if our results strictly apply to a fully conformal invariant (non physical) world, some of the derived properties could still hold in the real physical situation.

## 2. CONFORMAL COVARIANT $n$ -POINT FUNCTIONS

Let us start by remembering the constraints of conformal symmetry on the general  $n$ -point correlation function

$$\langle 0 | A_1(x_1) \dots A_n(x_n) | 0 \rangle , \quad (1)$$

where the operators  $A_i(x_i)$  are conformal scalars (for simplicity) i.e. they are Lorentz scalars and satisfy the equations

$$[A_i(0), D] = i l_i A_i(0) , \quad [A_i(0), K_\lambda] = 0 . \quad (2)$$

These operators transform according to irreducible representations labelled by only one non-vanishing Casimir operator whose eigenvalue is [9]

$$C = M_{\mu\nu} M^{\mu\nu} + 2 P \cdot K - 2 D^2 + 8 i D = 2 l (l - 4) , \quad (3)$$

$l$  being the scale dimension (in energy units) of the field. Note that  $C$  is invariant under the substitution  $l \rightarrow 4 - l$  so these two dimensions are associated to equivalent representations. We call  $l^* = 4 - l$  the dimension conjugate to  $l$ .

It is a simple task to prove that conformal invariance puts  $n$  additional constraints on the  $n$ -point correlation function (which depends on  $\frac{1}{2} n(n-1)$  independent variables from Poincaré invariance) so this function depends on an arbitrary function of  $N = \frac{1}{2} n(n-3)$  independent (adimensional) variables, the so called harmonic ratios. This implies that:

(i) The conformal covariant two-point function is over determined ( $N=-1$ ), in fact dilatation symmetry alone fixes its form to be

$$\langle 0 | A(x) B(y) | 0 \rangle = C_{AB} \left[ \frac{1}{(x-y)^2} \right]^{\frac{1}{2}(l_A + l_B)} , \quad (4)$$

but conformal covariance implies the additional selection rule

$$\langle 0|A(x)B(y)|0\rangle = 0 \quad \text{if} \quad l_A \neq l_B \quad . \quad (5)$$

The selection rule (5) admits the following generalization to irreducible conformal tensors of order  $n$  [12]

$$\langle 0|O_{\alpha_1 \dots \alpha_n}(x)O_{\beta_1 \dots \beta_m}(y)|0\rangle = 0 \quad \text{unless} \quad l_n = l_m, \quad n = m \quad . \quad (6)$$

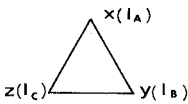
We call eq. (6) the ‘‘orthogonality property’’ of conformal irreducible operators.

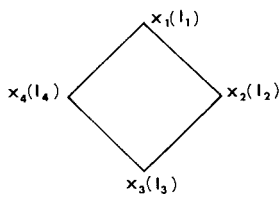
(ii) For the three-point function one has  $N=0$ ; such function is then completely determined, apart an overall constant factor, to be [12, 13]:

$$\begin{aligned} \langle 0|A(x)B(y)C(z)|0\rangle &= C_{ABC} \left[ \frac{1}{(x-y)^2} \right]^{\frac{1}{2}(l_A+l_B-l_C)} \\ &\times \left[ \frac{1}{(x-z)^2} \right]^{\frac{1}{2}(l_C+l_A-l_B)} \left[ \frac{1}{(y-z)^2} \right]^{\frac{1}{2}(l_C+l_B-l_A)} \quad . \quad (7) \end{aligned}$$

For the two, three and  $n$ -point correlation functions we use the graphical representations:

I  $\langle 0|A(x)B(y)|0\rangle :$  

II  $\langle 0|A(x)B(y)C(z)|0\rangle :$  

III  $\langle 0|A_1(x_1) \dots A_4(x_4)|0\rangle :$  

and similarly for

IV  $\langle 0|A_1(x_1) \dots A_n(x_n)|0\rangle .$

The following interesting *vertex graph identity* is a consequence of conformal symmetry:

$$\langle 0|A(x)B(y)C(z)|0\rangle = \int d^4t \langle 0|A(x)B(y)C^*(t)|0\rangle \langle 0|C(t)C(z)|0\rangle \quad (8)$$

(from now on the equality symbol is only meant apart from a multiplicative constant factor on the right-hand-side) where  $C^*(t)$  is a ‘‘conventional operator’’ carrying scale dimension  $l_C^* = 4 - l_C$  conjugate to the dimension  $l_C$  of  $C(z)$ .

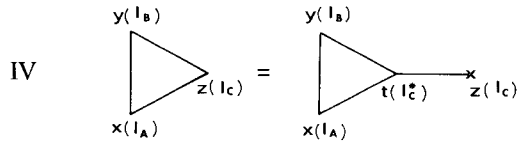
The identity (8) is a consequence of the following integral representation proven in ref. [14]

$$\begin{aligned} \int d^4t \left[ \frac{1}{(t-x)^2} \right]^\alpha \left[ \frac{1}{(t-y)^2} \right]^\beta \left[ \frac{1}{(t-z)^2} \right]^\gamma \\ = \left[ \frac{1}{(x-y)^2} \right]^{2-\gamma} \left[ \frac{1}{(y-z)^2} \right]^{2-\alpha} \left[ \frac{1}{(x-z)^2} \right]^{2-\beta} \end{aligned} \quad (9)$$

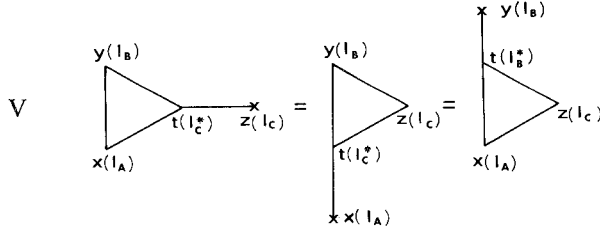
valid for  $\alpha + \beta + \gamma = 4$ . In our case  $\alpha = 2 + \frac{1}{2}(l_A - l_B - l_C)$ ,  $\beta = 2 + \frac{1}{2}(l_B - l_A - l_C)$ ,  $\gamma = l_C$  and eq. (8) stands for

$$\begin{aligned} \left[ \frac{1}{(x-y)^2} \right]^{\frac{1}{2}(l_A+l_B-l_C)} \left[ \frac{1}{(y-z)^2} \right]^{\frac{1}{2}(l_C+l_B-l_A)} \left[ \frac{1}{(x-z)^2} \right]^{\frac{1}{2}(l_C+l_A-l_B)} \\ = \left[ \frac{1}{(x-y)^2} \right]^{\frac{1}{2}(l_A+l_B-l_C^*)} \int d^4t \left[ \frac{1}{(t-x)^2} \right]^{\frac{1}{2}(l_C^*+l_A-l_B)} \\ \times \left[ \frac{1}{(t-y)^2} \right]^{\frac{1}{2}(l_C^*+l_B-l_A)} \left[ \frac{1}{(t-z)^2} \right]^{l_C} \end{aligned} \quad (10)$$

Eq. (10) is graphically represented as (integration over internal points is understood):



obviously the following identities hold



### 3. GENERALIZED CONFORMAL COVARIANT O.P.E.

We consider now a fully conformal covariant O.P.E. [15]

$$O_{\alpha_1 \dots \alpha_n}(x) O_{\beta_1 \dots \beta_m}(y) = \sum D_{\alpha_1 \dots \alpha_n \beta_1 \dots \beta_m}^{\gamma_1 \dots \gamma_J} (x-y, \frac{\partial}{\partial y}) O_{\gamma_1 \dots \gamma_J}(y) \tag{11}$$

where the  $O_s$  are irreducible conformal tensors of order  $n, m, J$  respectively and  $D_{\alpha_1 \dots \alpha_n \beta_1 \dots \beta_m}^{\gamma_1 \dots \gamma_J} (x-y, \frac{\partial}{\partial y})$  is a differential operator, which depends on  $l_n, l_m, l_J, n, m$  and  $J$ , which sums infinite towers of “derivatives” of the  $O_{\gamma_s}$ , giving the contribution of an infinite-dimensional representation of conformal algebra in the decomposition of the product  $O_{\alpha_1 \dots \alpha_n}(x) O_{\beta_1 \dots \beta_m}(y)$ . As a consequence of the selection rule of eq. (6), multiplying both sides of eq. (11) by  $O_{\delta_1 \dots \delta_J}(z)$  and taking the V.E.V. we get

$$\begin{aligned} &\langle 0 | O_{\alpha_1 \dots \alpha_n}(x) O_{\beta_1 \dots \beta_m}(y) O_{\delta_1 \dots \delta_J}(z) | 0 \rangle \\ &= D_{\alpha_1 \dots \alpha_n \beta_1 \dots \beta_m}^{\gamma_1 \dots \gamma_J} \left( (x-y), i \frac{\partial}{\partial y} \right) \langle 0 | O_{\gamma_1 \dots \gamma_J}(y) O_{\delta_1 \dots \delta_J}(z) | 0 \rangle \end{aligned} \tag{12}$$

which gives the general relation between three-point correlation functions and O.P.E. To see the connection of eq. (12) with the previously derived *vertex graph identity* (see eq. (8)) let us consider, for simplicity, conformal scalars ( $n=m=J=0$ ) so eq. (12) reads as

$$\langle 0 | A(x) B(y) C(z) | 0 \rangle = D_{AB}^C \left( (x-y), \frac{\partial}{\partial y} \right) \langle 0 | C(y) C(z) | 0 \rangle \tag{13}$$

and

$$\langle 0 | C(y) C(z) | 0 \rangle = \left[ \frac{1}{(y-z)^2} \right]^{1_C}$$

(see eqs. (4) and (5)).

From the vertex graph identity (10) we have:

$$\begin{aligned} \langle 0|A(x)B(y)C(z)|0\rangle &= \left[\frac{1}{(x-y)^2}\right]^{\frac{1}{2}(l_A+l_B-l_C^*)} \\ &\times \int d^4t \left[\frac{1}{(t-x)^2}\right]^{\frac{1}{2}(l_C^*+l_A-l_B)} \left[\frac{1}{(t-y)^2}\right]^{\frac{1}{2}(l_C^*+l_B-l_A)} \langle 0|C(t)C(z)|0\rangle \end{aligned} \quad (14)$$

so, by comparison of eqs. (13) and (14) we get

$$\begin{aligned} D_{AB}^C \left( (x-y), \frac{\partial}{\partial y} \right) &= \left[\frac{1}{(x-y)^2}\right]^{\frac{1}{2}(l_A+l_B-l_C^*)} \\ &\times \int d^4t \left[\frac{1}{(t-x)^2}\right]^{\frac{1}{2}(l_C^*+l_A-l_B)} \left[\frac{1}{(t-y)^2}\right]^{\frac{1}{2}(l_C^*+l_B-l_A)} e^{t \cdot a} \end{aligned} \quad (15)$$

which exhibits manifestly the non local structure of an irreducible representation of conformal algebra in the O.P.E. [16]

$$\begin{aligned} A(x)B(y) &= \left[\frac{1}{(x-y)^2}\right]^{\frac{1}{2}(l_A+l_B-l_C^*)} \int d^4t \left[\frac{1}{(t-x)^2}\right]^{\frac{1}{2}(l_C^*+l_A-l_B)} \\ &\times \left[\frac{1}{(t-y)^2}\right]^{\frac{1}{2}(l_C^*+l_B-l_A)} C(t) + \dots \end{aligned} \quad (16)$$

However we still remark that eq. (16) is only a formal device as it is not a true expansion à la Wilson. In fact there is not in eq. (16) an explicit factorization of the C-number light-cone singularity which, by dimensional analysis, must be

$$\left[\frac{1}{(x-y)^2}\right]^{\frac{1}{2}(l_A+l_B-l_C)}$$

and an operator part finite at  $(x-y)^2 = 0$ . The complete conformal covariant solution of this problem was found, by a general method, in ref. [15], to have the form

$$\begin{aligned} A(x)B(0) &= \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B-l_C)} \int_0^1 du u^{\frac{1}{2}(l_A-l_B+l_C)} (1-u)^{\frac{1}{2}(l_B-l_A+l_C)-1} \\ &\times {}_0F_1(l_C-1; -\frac{1}{4}u(1-u)x^2 \square) C(ux) + \dots \end{aligned} \quad (17)$$

where  ${}_0F_1(a, z)$  is an hypergeometric function and we have put  $y = 0$ .

In particular, in the light-cone limit  $x^2 \rightarrow 0$  eq. (17) reduces to [10, 17]:

$$A(x)B(0) \underset{x^2 \rightarrow 0}{\sim} \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B-l_C)} {}_1F_1\left(\frac{1}{2}(l_A+l_B-l_C); l_C; x \cdot \partial\right) C(0) + \dots \quad (18)$$

in terms of the confluent hypergeometric function

$${}_1F_1(a; c; z) = \Gamma(c) / \Gamma(a) \Gamma(c-a) \int_0^1 du u^{a-1} (1-u)^{c-a-1} e^{uz} .$$

It is a simple task to derive eq. (17) from eq. (16) i.e. to regularize à la Wilson the O.P.E. in eq. (16). For this purpose we perform on eq. (10) a Riemann-Liouville fractional transformation [15, 18] (we put  $y = 0$ )

$$\begin{aligned} & \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B-l_C^*)} \int d^4t \left[\frac{1}{(t-x)^2}\right]^{\frac{1}{2}(l_C^*+l_A-l_B)} \left(\frac{1}{t^2}\right)^{\frac{1}{2}(l_C^*+l_B-l_A)} \\ & \times \left[\frac{1}{(t-z)^2}\right]^{l_C} = \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B-l_C)} \int_0^1 du u^{\frac{1}{2}(l_A-l_B+l_C)-1} \\ & \times (1-u)^{\frac{1}{2}(l_B-l_A+l_C)-1} \int d^4t \frac{1}{[t^2+x^2u(1-u)]^{4-l_C} [u(1-u)x^2]^{l_C-2}} \\ & \times \left[\frac{1}{(t+ux-z)^2}\right]^{l_C} , \end{aligned} \quad (19)$$

which allows us to rewrite eq. (16) as:

$$\begin{aligned} A(x)B(0) &= \left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B-l_C)} \int_0^1 du u^{\frac{1}{2}(l_A-l_B+l_C)-1} \\ & \times (1-u)^{\frac{1}{2}(l_B-l_A+l_C)-1} \int d^4t \frac{1}{[t^2+x^2u(1-u)]^{4-l_C} [u(1-u)x^2]^{l_C-2}} \\ & \times e^{t \cdot \partial} C(ux) . \end{aligned} \quad (20)$$



The equivalence of eq. (20) with eq. (17) is easily achieved by noting that

$$\begin{aligned}
 \int d^4 t \frac{1}{[t^2 + x^2 u(1-u)]^{4-l_C}} e^{t \cdot \partial} &= \int_C d\alpha \alpha^{3-l_C} \\
 \times \int d^4 t e^{\alpha(t^2 + u(1-u)x^2) + t \cdot \partial} &= \int_C d\alpha \alpha^{3-l_C} e^{\alpha u(1-u)x^2} \\
 \times \int d^4 t e^{\alpha t^2 + t \cdot \partial} &= \int_C d\alpha \alpha^{1-l_C} e^{\alpha u(1-u)x^2 - \square/4\alpha} \\
 &= (u(1-u)x^2)^{\frac{1}{2}(l_C-2)} \square^{\frac{1}{2}(2-l_C)} J_{l_C-2}(u(1-u)x^2 \square^{\frac{1}{2}}) ,
 \end{aligned}
 \tag{21}$$

where the last equality is a consequence of the fundamental integral representation for the Bessel function [19].

Inserting the result of eq. (2) into eq. (20) and using the identity [20]

$$e^{-\frac{1}{2}\nu\pi i} J_\nu(z e^{\frac{1}{2}\pi i}) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)} {}_0F_1(\nu+1; \frac{1}{4}z^2)
 \tag{22}$$

we obtain the O.P.E. (17) as derived in ref. [15].

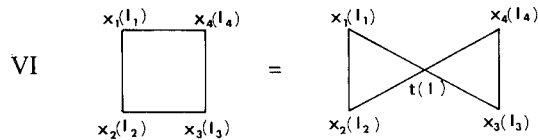
Let us now consider the more general situation concerning the conformally covariant  $n$ -point correlation function

$$\langle 0|A_1(x_1) \dots A_n(x_n)|0 \rangle .
 \tag{23}$$

We will widely use the formal device (see eqs. (8) and (16))

$$A(x)B(y) = \int d^4 t \langle 0|A(x)B(y)O^*(t)|0 \rangle O(t) ,
 \tag{24}$$

which gives the contribution of the local field  $O(t)$  to the O.P.  $A(x)B(y)$  and is a consequence of the *vertex graph identity* (8) and of the orthogonality property (6) (spin complications are not essential in the present discussion). Note that the substitution of (24) for any couple  $A_i(x_i)A_j(x_j)$  in (23) can be graphically represented as ( $n=4$  for example)



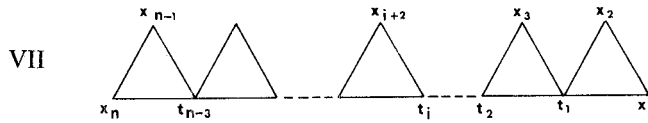
and reads as

$$\begin{aligned} &\langle 0|A_1(x_1) \dots A_i(x_i) \dots A_j(x_j) \dots A_n(x_n)|0\rangle \\ &= \int d^4 t \langle 0|A_i(x_i)A_j(x_j)O^*(t)|0\rangle \langle 0|A_1(x_1) \dots \hat{A}_i(x_i) \dots \\ &\quad \times \dots \hat{A}_j(x_j) \dots A_n(x_n)O(t)|0\rangle; \end{aligned} \tag{25}$$

we call this operation *graph contraction* as it allows to write a  $n$ -point graph in terms of a vertex and a  $n-1$  point contracted graph. By iteration of (25) to any chosen pair of operators we obtain, for each chosen configuration, the factorized expression

$$\begin{aligned} \langle 0|A_1(x_1) \dots A_n(x_n)|0\rangle &= \int d^4 t_1 \dots d^4 t_{n-3} \\ &\times \langle 0|A_1(x_1)A_2(x_2)O_1^*(t_1)|0\rangle \langle 0|O_1(t_1)A_3(x_3)O_2^*(t_2)|0\rangle \dots \\ &\times \dots \langle 0|O_{n-3}(t_{n-3})A_{n-1}(x_{n-1})A_n(x_n)|0\rangle \end{aligned} \tag{26}$$

which is graphically represented as



We call a graph of the type VII a *conformal irreducible graph*. An interesting problem is the locality property of the O.P.E., i.e. the property that each *conformal irreducible graph* satisfies the Wightman [21] axioms concerning locality.

It is clear that the two and three-point functions give no problems: they are analytic at space-like distances and with cuts for time-like distances.

Also the  $n$ -point function should satisfy the linear part of Wightman axioms because they are constructed from “good” two and three-point functions with rules which resemble the standard perturbation expansion.

It is possible to obtain a more general demonstration starting by time-ordered functions. Their Fourier transforms have standard analyticity properties. From a general theorem of Ruelle [22] it follows that the corresponding Wightman functions satisfy the Wightman axioms and in particular the locality condition.

Finally we stress that such graphs depend on the particular chosen coupling scheme (particular configuration of the external points) and on  $n-3$  arbitrarily chosen dimensions, which are the conformal quantum numbers of the operators appearing in the multiple O.P.E. Obviously the most general  $n$ -point correlation function can be obtained as a sum of (infinite) irreducible graphs of the kind VII, corresponding to all possible spins and dimensions of the internal points. Note, however, that any irre-

ducible graph is a well defined expression. It reads as a  $n-3$  multiple integral of a product of  $3n-8$  powers (it is in fact built with sequence of  $n-2$  vertices).

We are indebted with Prof. R. Gatto for many interesting discussions.

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