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V. Montelatici: USEFUL FORMULAE AND CALCULATED
FUNCTIONS FOR THE "LUMINOUS" ELECTRON. -

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The theory of the emission of electromagnetic waves from an accelerated point electric charge is based on well established physical concepts. A general and consistent theory of the electromagnetic field generated by a point charge moving with relativistic velocity implies the use of the theory of special relativity. It is obvious that the properties of the emitted energy are completely different when the values of velocities are low or are comparable with the light velocity.

In this report we do not write anything new, but we want to have in hands useful formulae and calculated functions applicable to that particular "lamp" which is a synchrotron or a storage ring. In order to recall the necessary theoretical background, we shall first repeat concepts which can be easily found in classical books, such as: "the field theory" by L. Landau and E. Lifchitz. The expressions taken from this book and used to derive the formulae obtained in this report shall be indicated with the notation: (L. L.; formula number).

The study of the radiation emitted by an accelerated point charge, was initiated by Liénard in 1898 and by G. A. Schott in 1907; but only in the years ranging from 1940 to 1950 a renewal interest arose in this field. It is also interesting to note that the Schott's formula, derived in the first decade of this century, is applicable either to the ultrarelativistic case or to non relativistic one.

In the fourth decade the classical theory of the radiation acquires a well refined theoretical bases with the works by D. Ivanenko, I. Ya. Pomeranchouk (1944), D. Ivanenko, A. A. Sokolov (1948), J. Schwinger (1949), and with a large number of experimental works such as those performed by F. Elder, R. V. Langmuir, H. C. Pollock (1948), D. Tomboulian, P. Hartman (1956), F. A. Korolev, O. F. Kulikov, A. G. Ersov (1960).

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1. - INTRODUCTION. -

A point electric charge, with initial velocity, \bar{v} , in a constant magnetic field, H, describes a circular path of radius R, when $\bar{v} \cdot \bar{H} = 0$. The radius, R, is related to the field, H, and to the energy, $E = m_0 c^2 (1 - \beta^2)^{-1/2}$, of the charge, by the relation:

$$(1.1) \quad R = \frac{\beta}{e_0 H} E$$

where $\beta = v/c$ and $e_0 = -e_0$ is the electronic charge. In the case of a relativistic electron, the component of the acceleration along the radius R and directed towards the circle center, is so large that the electron becomes a powerful source of electromagnetic radiation.

The loss of energy in unit time can be determined by using the motion equation of the electron by including the losses due to the radiation.

This calculation, done by Liénard in 1898, gives the result:

$$(1.2) \quad W_0 = -\frac{dE}{dt} = \frac{2}{3} \frac{c e_0^2}{R^2} \beta^4 (1 - \beta^2)^{-2}$$

where the physical constants are expressed in C.G.S. electrostatic unites.

In the ultrarelativistic case, $\beta \cong 1$, than (1.2) becomes:

$$(1.3) \quad W_0 (\beta \cong 1) = \frac{2}{3} \frac{c e_0^2}{R^2} \left(\frac{E}{m_0 c^2} \right)^4$$

while in the non relativistic case, $\beta \ll 1$, one has:

$$(1.4) \quad W_0 (\beta \ll 1) = \frac{2}{3} \frac{c e_0^2}{R^2} \beta^4$$

From these classical results, one has that the radiated energy rises rapidly with the electron energy when $\beta = 1$.

Then one is forced to pay attention to the higher harmonics of the fundamental frequency, given by the rotation frequency of the orbiting electron:

$$(1.5) \quad \frac{v}{2\pi R} \xrightarrow{\beta \cong 1} \frac{c}{2\pi R} = \frac{\omega_0}{2\pi}$$

Thus the emitted radiation has wave length:

$$(1.6) \quad \lambda = \frac{2\pi c}{\nu \omega_0}$$

where ν is the harmonic number.

The radiated energy in the interval of harmonics between 10^7 and 10^{11} gives wave length ranging from infrared radiation (18800 Å) to X rays (1.88 Å), for a 3 m radius of the electron orbit.

With the Frascati synchrotron, at orbital energy of 1 GeV, one observes a glow almost white which corresponds, in the chromaticity diagram, to the point c in Fig. 1; this fact implies that all colours are present in the luminous spectrum (their addi

tive composition gives white light in the cromacity diagram).

But one also observes the presence of some sky blue-violet cromacity: this fact implies the presence of some more intense fundamental wave length in the range of shorter wave lengths, Fig. 1.

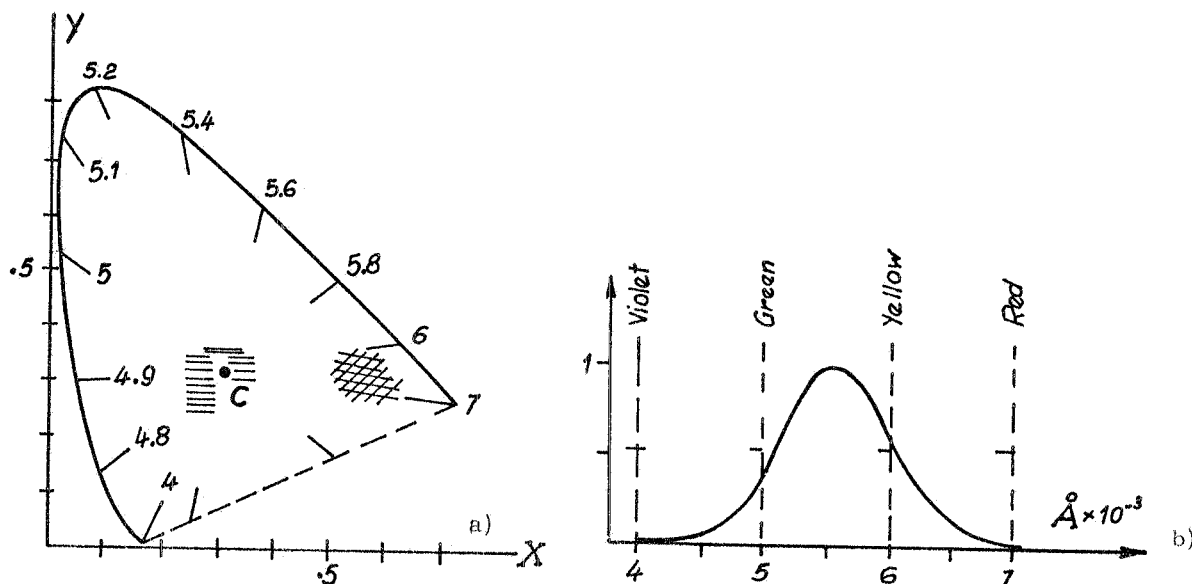


FIG. 1a) - Cromaticity diagram of visible radiation. Point C corresponds to white light. The opposite pure colours on cromacity curve which are on a straight line are complementary, their addition gives white light; b) - relative sensivity of human eye to colours.

Similarly, in other synchrotrons with lower orbital energies one sees a glow displaced toward longer waver lenglths, for example a yellow-red glow. An accurate cromatic evaluation of the glow requires an experimental study of the photon spectrum intensity. Then one has to compare the experimental values with the results of the theory which predicts the intensity of the emitted radiation as a function of the basic parameters of a synchrotron; such as the radius of the electron orbit, the electron energy, and the electron total number. We shall deduce the practical formulae and give a graphical method to obtain all the emitted intensities of energy at all wave length for any value of the radius R and orbital energy.

Incidentally it is to note that the observation of luminous waves from accelerated electrons in vacuum is the most direct proof of the existance of the electron itself.

2. - INSTANTANEOUS ANGULAR DISTRIBUTION OF THE RADIATED ENERGY. -

The Liénard-Wichert expression (L. L. ; 63.8; 63.9) for the electric field \vec{E} and for the magnetic field \vec{H} are used to determine the instantaneous distribution of the radiation intensity of a moving electric point charge with velocity \vec{v} and acceleration $\vec{\dot{v}} = -(\dot{v}^2/R^2)\vec{R}$. The Lienard-Wichert formulas are:

$$(2.1) \quad \vec{E} = e_0 \frac{1 - \beta^2}{(R' - \frac{\vec{R}' \cdot \vec{v}}{c})^3} (\vec{R}' - \frac{\vec{v}}{c} R') + e_0 \frac{\vec{R}' \Delta \left[(\vec{R}' - \frac{\vec{v}}{c} R') \Delta \vec{v} \right]}{c^2 (R' - \frac{\vec{R}' \cdot \vec{v}}{c})^3}, \quad \vec{H} = \frac{1}{R'} \vec{R}' \Delta \vec{E}$$

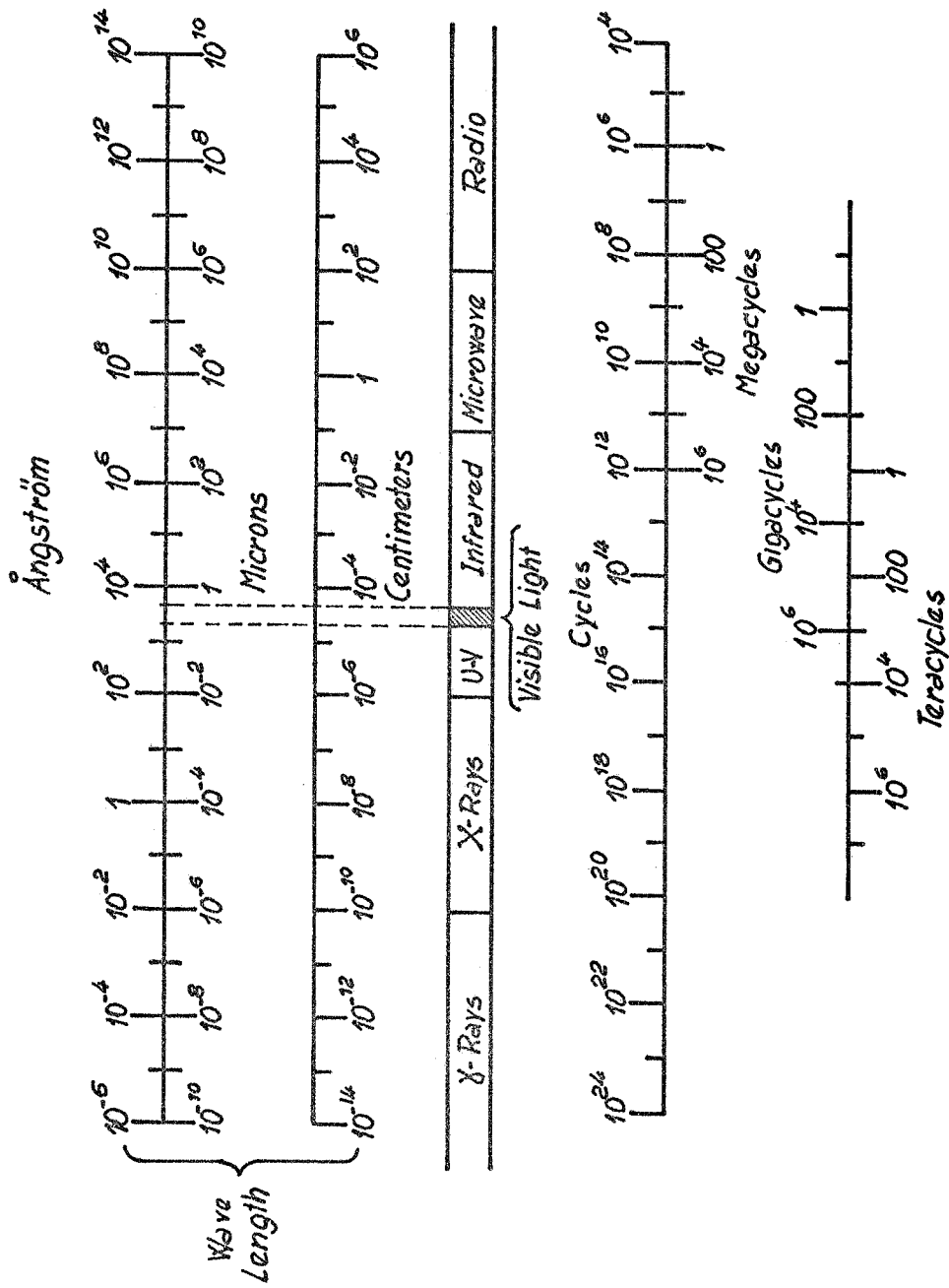


FIG. 2 - The electromagnetic spectrum.

They give the field \vec{E} and \vec{H} in the point B of the observer at a distance \bar{R}' from the point charge at P; and they are calculated at the retarded time $\tau = t - R'(\tau)/c$ (L. L. ; 63.1). The electric field is constituted by two parts: one depends only on the velocity and does not depend on the acceleration, but it depends on the distance as $(R')^{-2}$; the other part depends on the velocity and acceleration, and also on the distance as $(R')^{-1}$.

While the part depending on $(R')^{-2}$ determines the spatial limit of the "wave zone", the part depending on $(R')^{-1}$ determines the emission of the radiation. Indeed the first part, in (2.1), corresponds to a field created by a moving charge with uniform motion; then when the charge is seen in a Galileian reference frame, it is at rest, and cannot therefore radiate. Let us now study the angular properties of the electric field. Since the scalar terms do not change the direction of the field \vec{E} , but change only its modulus, it suffices to analyze the angular behaviour of the vector $\vec{\varphi} \equiv \bar{R}' \wedge \left[(\bar{R}' - (\bar{v}/c)R') \wedge \frac{\dot{\bar{v}}}{v} \right]$ to visualize this property. This vector, see Fig. 3, is zero when $\overline{CM} = \overline{CN} = R' - (\bar{v}/c)R'$ is parallel to $\overline{PD} = \bar{v} \perp \dot{\bar{v}}$, in the plane formed with the direction of the vector \bar{v} and $\dot{\bar{v}}$.

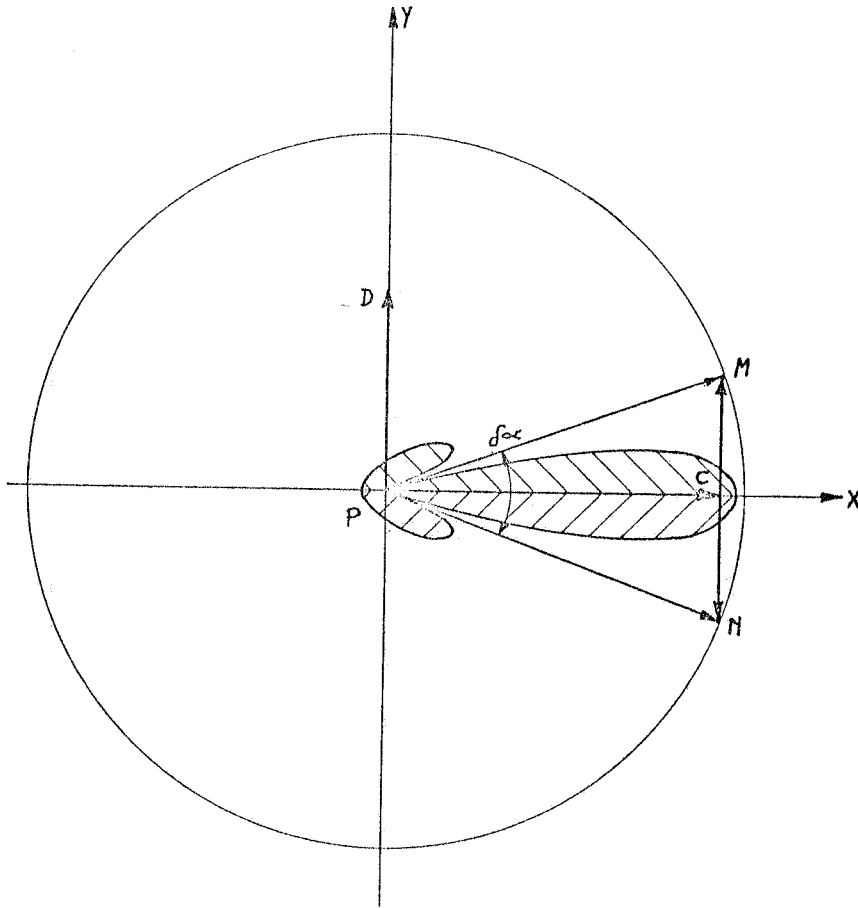


FIG. 3 - Orbital plane $x+y$ formed by the vector $\overline{PD} = \dot{\bar{v}}$
 $\overline{PC} = (\bar{v}/c)R'$. The nodal curve represents the function (2.6)
 when $\gamma = 0$. The intensity value of the back radiation is

$$\frac{9}{32} \left(\frac{m_0 c^2}{E} \right)^4 = \frac{2 \times 10^{-14}}{[E(\text{GeV})]^4}$$

of the total radiation intensity emitted by one electron.

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Thus, $\bar{\theta}$ is zero along the direction \overline{PM} and \overline{PN} , where $|\overline{PM}| = |\overline{PN}| = |\overline{R}'|$.

The angle between the \overline{PM} and \overline{PC} vectors gives the direction along which the electric field is zero, that is when $\cos^2 \alpha = \beta^2$. Then, for $\beta \cong 1$, one has $\delta \alpha \cong 2\sqrt{1-\beta^2}$. This angle represents the smallest angular interval around the direction of the velocity \bar{v} in which the radiation is not zero.

Formula (2.1) depends on the angle between the vector \overline{PB} and the velocity \bar{v} ; then we can use an axis system with its origin in the point charge P and with direction defined as follows:

$$\overline{0x} \parallel \bar{v}; \quad \overline{0y} \parallel (-\overline{0P}) \equiv \bar{R}; \quad \overline{0z} = \overline{0x} \wedge \overline{0y}.$$

The angles (α, γ) and the angles (θ, φ) , shown in Fig. 4, are therefore related by the relation:

$$(2.2) \quad \cos \alpha = \cos \varphi \cdot \sin \theta, \quad \cos \theta = \sin \alpha \cdot \sin \gamma$$

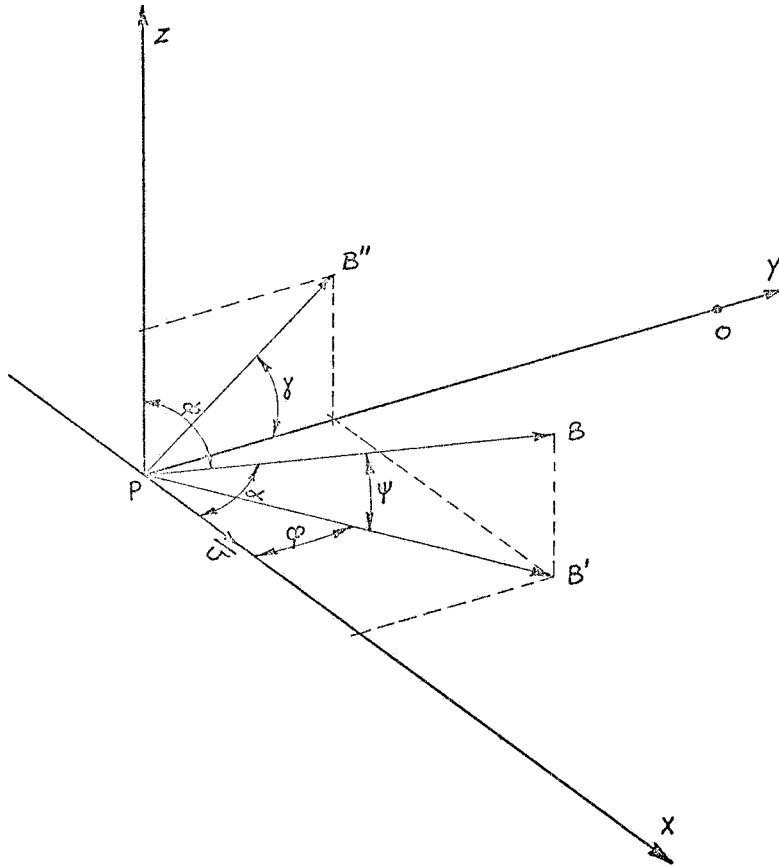


FIG. 4 - Coordinate system used $\overline{PB} = \bar{R}'$; $\overline{OP} = -\bar{R}$

The vectors \overline{PB}'' and \overline{PB}' are the projection of the vector \overline{PB} on the plane $x+z$ and $x+y$ respectively. One can deduce the following relation:

$$(2.3) \quad \frac{\bar{R} \cdot \bar{R}'}{RR'} = -\sin \alpha \cos \gamma, \quad \frac{\bar{R}' \cdot \bar{v}}{R' v} = \cos \alpha, \quad d\Omega = \sin \theta d\theta d\varphi = \sin \alpha d\alpha d\gamma$$

The instantaneous intensity, $dW^{\mathbf{x}}$, of the radiated energy in the solid angle, $d\Omega$, is the quantity of energy which crosses the elementary spherical surface, $\overline{ds} = \overline{R}' R' d\Omega$, with its center at P and with radius $\overline{PB} = \overline{R}'$.

The energy flux, given by the Poyntig vector, $\overline{S} = \frac{c}{4\pi} \overline{E} \wedge \overline{H}$ (L. L. 47) is:

$$(2.4) \quad dW^{\mathbf{x}} = \overline{S} \cdot \overline{ds} = \frac{c}{4\pi} \left[R'^2 E^2 - (\overline{R}' \cdot \overline{E})^2 \right] d\Omega$$

where $\overline{E} \cdot \overline{R}'$ can be zero for some conditions to be determined.

Formula (2.4) can be written in the form:

$$(2.5) \quad dW^{\mathbf{x}} = \frac{c e_o^2}{32 \pi R'^2} \beta^4 \frac{8 R'^2}{e_o^2 \beta^4} \left[R'^2 E^2 - (\overline{R}' \cdot \overline{E})^2 \right] d\Omega = \frac{c e_o^2}{32 \pi R'^2} \beta^4 \psi(\alpha, \gamma)$$

By using (2.1) and Keeping in mind the angular relation (2.3) one has:

$$(2.6) \quad \frac{1}{8} \psi = \frac{(\beta - \cos \alpha)^2 + (1 - \beta^2) \sin^2 \alpha \sin^2 \gamma}{(1 - \beta \cos \alpha)^6} + \frac{2 \sin \alpha (\beta - \cos \alpha) \cos \gamma}{(1 - \beta \cos \alpha)^6} \frac{1 - \beta^2}{\beta} \frac{R}{R'} +$$

$$+ \frac{\sin^2 \alpha}{(1 - \beta \cos \alpha)^6} \left(\frac{1 - \beta^2}{\beta} \frac{R}{R'} \right)^2$$

This expression is useful to analyze the meaning of the "wave zone" both in the ultrarelativistic case and in the non relativistic case. Indeed, when one takes $R' \gg (1 - \beta^2 / \beta) R$, the terms in the first and second order in R/R' are neglected, and the linear dimensions of R' with respect to R can be seen to depend on $\beta = v/c$. So, in the case $\beta = 0$, one has $R' \gg R$, whereas in the case $\beta = 1$ the condition $R' \gg R$ is sufficient to perform the "wave zone" approximation in both cases.

One deduces that in the ultrarelativistic case a "wave zone" is obtained of the order of the orbit radius: i. e. the "wave zone" is much smaller than the "wave zone" of the non relativistic case; this fact is just due to the factor $1 - \beta^2$ which compresses the "wave zone" with respect to the non relativistic case.

This situation is useful in the experiments, since measurements can be performed relatively near the orbit of the electron, for example at a distance of one radius. Now, in the "wave zone" approximation, one can drop the terms depending on $(R')^{-2}$ and $\overline{R} \cdot \overline{E}$ in the formulae (2.1) and (2.5) respectively.

By using $\beta = 0$ in (2.6) and by using the result in (2.5), the instantaneous angular distribution of the emitted radiation in the solid angle $d\Omega$ is given as:

$$(2.7) \quad dW^{\mathbf{x}}(\beta=0) = \frac{c e_o^2}{4 \pi R'^2} \left[\beta^4 \cos^2 \alpha + \sin^2 \alpha \sin^2 \gamma \right] d\Omega$$

It represents the angular distribution of the dipole shown skematically in Fig. 5. In the ultrarelativistic case one has (Fig. 6):

$$(2.8) \quad dW^{\mathbf{x}}(\beta \cong 1) = \frac{c e_o^2}{4 \pi R'^2} \beta^4 \frac{(\beta - \cos \alpha)^2 + (1 - \beta^2) \sin^2 \alpha \sin^2 \gamma}{(1 - \beta \cos \alpha)^6} d\Omega$$

By virtue of the high power difference $1 - \beta(\overline{R}' \cdot \overline{v}/R'v)$ in the denominator of (2.8) one has a large intensity in the small angular interval $\Delta\alpha$; that is when: $(1 - \beta) + (\Delta\alpha/2)^2 \cong (1 - \beta)$

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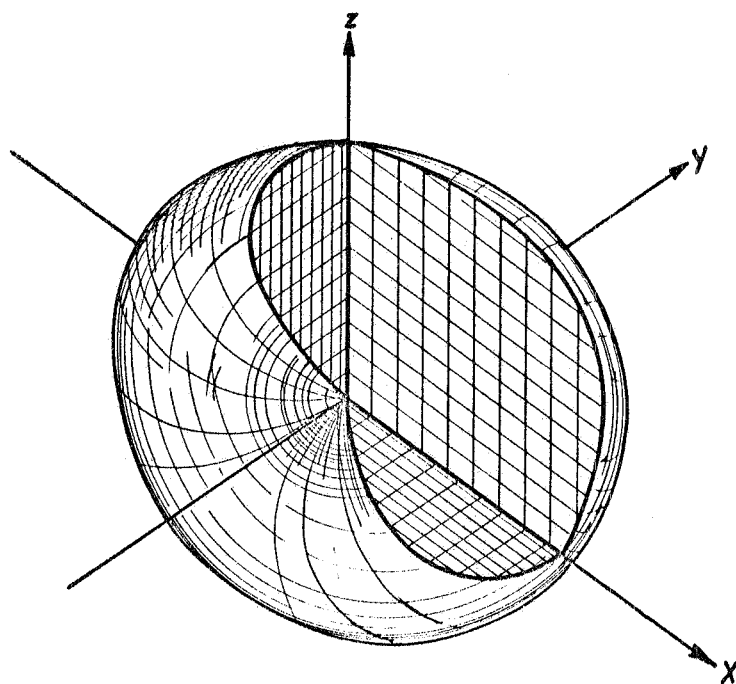


FIG. 5 - Schematic of the spatial dipolar distribution of the emitted dipolar radiation at a fixed time.

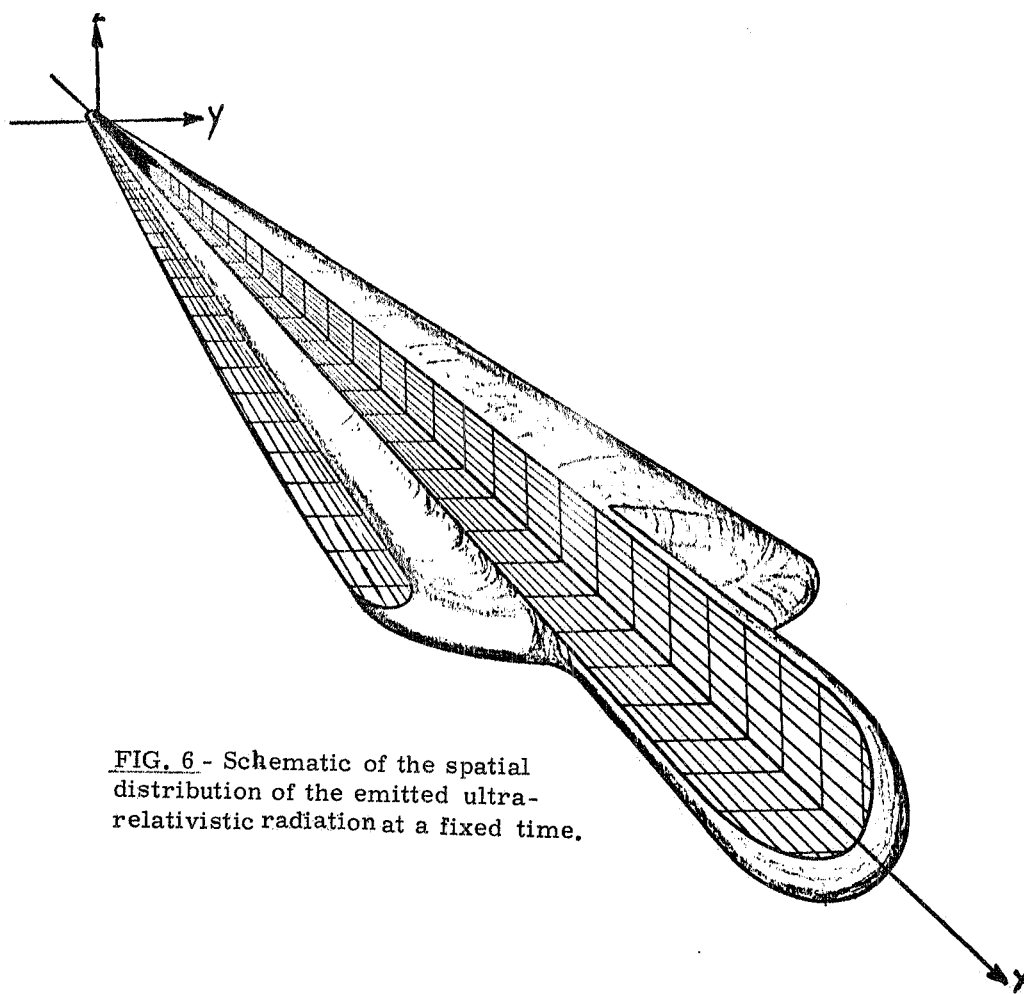


FIG. 6 - Schematic of the spatial distribution of the emitted ultra-relativistic radiation at a fixed time.

or when $\Delta\alpha = (1-\beta)^{1/2} \simeq (1-\beta^2)^{1/2}$ for $\beta \rightarrow 1$. One deduces that the maximum value of the radiated energy in along the direction of the velocity of the electron.

Now we can introduce the value (2.6) in (2.5), (using the "wave zone" approximation), and we can calculate the average taken over a revolution of the electron:

$$(2.9) \quad dW = \frac{c e_o^2}{32 \pi R^2} \langle \psi \rangle d\Omega$$

The average value is taken at the time t at the observer point B:

$$(2.10) \quad \langle \psi \rangle = \frac{\omega_o}{2\pi} \int_0^{\Gamma = \text{period}} \psi(\alpha, \gamma) dt$$

Since the time interval, dt , of the observer is related to the retarded time interval, $d\tau$, by the relation (L. L. ; 73, 10):

$$dt = (1 - \beta \cos \alpha) d\tau = (1 - \beta \cos \varphi \sin \theta) d\tau \quad \text{and} \quad \varphi = \omega_o \tau$$

one obtain

$$(2.11) \quad \begin{aligned} \frac{1}{8} \langle \psi(\theta, \varphi) \rangle &= \frac{1}{2\pi} \int_0^{2\pi} \psi(\theta, \varphi) (1 - \beta \cos \varphi \sin \theta) d\varphi = \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{(\beta - \cos \varphi \sin \theta)^2 + (1 - \beta^2) \cos^2 \theta}{(1 - \beta \cos \varphi \sin \theta)^5} d\varphi = \\ &= \frac{1}{2} \left[\frac{2 + \beta^2 \sin^2 \theta}{(1 - \beta^2 \sin^2 \theta)^{5/2}} - \frac{(1 - \beta^2)(4 + \beta^2 \sin^2 \theta) \sin^2 \theta}{4(1 - \beta^2 \sin^2 \theta)^{7/2}} \right] \end{aligned}$$

Substituting this expression in (2.9) one has:

$$(2.12) \quad \begin{aligned} dW(\theta) &= \frac{c e_o^2}{8\pi R^2} \beta^4 \left[\frac{2 + \beta^2 \sin^2 \theta}{(1 - \beta^2 \sin^2 \theta)^{5/2}} - \frac{(1 - \beta^2)(4 + \beta^2 \sin^2 \theta) \sin^2 \theta}{4(1 - \beta^2 \sin^2 \theta)^{7/2}} \right] d\Omega \\ &= \frac{c e_o^2}{8\pi R^2} \beta^4 \frac{1 + \cos^2 \theta - \frac{\beta^2}{4} (1 + 3\beta^2) \sin^4 \theta}{(1 - \beta^2 \sin^2 \theta)^{7/2}} d\Omega \end{aligned}$$

One has, from (2.12) the average intensity ratio along the directions, $\theta = \pi/2$ and $\theta = 0$:

$$(2.13) \quad \frac{dW(\theta = \pi/2)}{dW(\theta = 0)} = \frac{4 + 3\beta^2}{8(1 - \beta^2)^{5/2}}$$

This shows that the ratio between the intensity value along the orbital plane to that along its normal direction becomes extremely large for $\beta \rightarrow 1$, or for:

$$1 - \beta^2 \sin^2 \theta \simeq 1 - \beta^2$$

10.

or, for:

$$(2.14) \quad \Delta\theta \simeq (1-\beta^2)^{1/2} \xrightarrow{\beta \rightarrow 1} 0$$

3. - AVERAGE SPECTRAL DISTRIBUTION OF THE RADIATED ENERGY. -

As the motion of the charge is periodic we can use a Fourier series analysis of the average vector potential over a revolution period (L. L. ; 74) to obtain the spectral distribution, which gives the single harmonics of the components of the e. m. field, (L. L. ; 74). Introducing these components in the expression of the radiation intensity (L. L. ; 66) and summing over all harmonics we obtain the total average radiation intensity:

$$(3.1) \quad dW(\theta)_{\text{tot}} = \sum_{\nu=1}^{\infty} W(\nu, \theta) = \frac{c e_o^2}{2\pi R^2} \beta^2 \left\{ \sum_{\nu=1}^{\infty} \nu^2 \cot^2 \theta J_{\nu}^2(x) + \beta^2 \sum_{\nu=1}^{\infty} \nu^2 J_{\nu}'^2(x) \right\} d\Omega$$

where $x = \nu \beta \sin\theta$ and $J_{\nu}(x)$, $J_{\nu}'(x)$ are the Bessel function and its derivative of ν order.

This formula was deduced the first time by Scott on 1907 and it is applicable to the relativistic and non relativistic cases. From (3.1) all information are deduced from the radiated energy. By Keeping in mind the relation: (Watson, theory of Bessel function page 573)

$$(3.2) \quad \sum_{\nu=1}^{\infty} \nu^2 J_{\nu}^2(\nu \eta) = \frac{\eta^2(4+\eta^2)}{16(1-\eta^2)^{7/2}}$$

$$\sum_{\nu=1}^{\infty} \nu^2 J_{\nu}'^2(\nu \eta) = \frac{4+3\eta^2}{16(1-\eta^2)^{5/2}}$$

formula (3.1) becomes:

$$dW(\theta) = \frac{c e_o^2}{32\pi R^2} \beta^4 \left[\frac{\cos^2 \theta (4 + \beta^2 \sin^2 \theta)}{(1 - \beta^2 \sin^2 \theta)^{7/2}} + \frac{4 + 3\beta^2 \sin^2 \theta}{(1 - \beta^2 \sin^2 \theta)^{5/2}} \right] d\Omega =$$

$$= \frac{c e_o^2}{8\pi R^2} \beta^4 \frac{1 + \cos^2 \theta - \frac{\beta^2}{4} (1 + 3\beta^2) \sin^4 \theta}{(1 - \beta^2 \sin^2 \theta)^{7/2}} d\Omega$$

the last expression is coincident with (2.12). The first form of this relationship is constituted by two parts: one is the Bessel function series, the second is the derivative of the Bessel function both combined with the square of the harmonic number ν .

The first term has a zero value when $\theta = \pi/2$, and the second is a maximum. That is this last term represents the polarized component of the radiation with the electric field vector lying on the orbital plane; this component is:

$$(3.3) \quad dW(\theta)_{\sigma} = \frac{c e_o^2}{8\pi R^2} \beta^4 \frac{4 + 3\beta^2 \sin^2 \theta}{(1 - \beta^2 \sin^2 \theta)^{5/2}} d\Omega$$

Then the polarized component with the electric vector lying on a plane normal to the orbit is:

$$(3.4) \quad dW(\theta)_{\pi} = \frac{c e_o^2}{8 \pi R^2} \beta^4 \frac{\cos \theta (4 + \beta^2 \sin^2 \theta)}{(1 - \beta^2 \sin^2 \theta)^{7/2}}$$

This derivation of the polarized components is rigorously valid for $\theta = \pi/2$ and $\theta = 0$, but one can show the validity of (3.3) and (3.4) for any values of θ .

Since each rectilinear vibration (σ or π) is a sum of all harmonics, each harmonic has the direction of the vibration; and the single harmonic σ_{ν} (π_{ν}) is associated to the derivative of the Bessel function (to the Bessel function), that is:

$$(3.5) \quad dW(\nu, \theta)_{\sigma} = \frac{c e_o^2}{2 \pi R^2} \beta^4 \nu^2 J_{\nu}'^2(\nu \beta \sin \theta) d\Omega$$

$$dW(\nu, \theta)_{\pi} = \frac{c e_o^2}{2 \pi R^2} \beta^2 \cot^2 \theta \nu^2 J_{\nu}^2(\nu \beta \sin \theta) d\Omega$$

Integrating over all angles from (3.3) and (3.4) one obtains:

$$(3.6) \quad W_{\sigma} = \frac{2}{3} \frac{c e_o^2 \beta^4}{(1 - \beta^2)^2 R^2} \frac{6 + \beta^2}{8}, \quad W_{\pi} = \frac{2}{3} \frac{c e_o^2 \beta^4}{(1 - \beta^2)^2 R^2} \frac{2 - \beta^2}{8}$$

The sum of (3.6) is

$$W_o = W_{\sigma} + W_{\pi} = \frac{2}{3} \frac{c e_o^2}{(1 - \beta^2)^2} \frac{\beta^4}{R^2}$$

which is the classical formula given by Liénard. One may note that $W_{\sigma} = (7/8)W_o$ and $W_{\pi} = (1/8)W_o$, while, for $\beta = 0$ dipolar case, one has:

$$(3.7) \quad W_{\sigma} = \frac{3}{4} \frac{2}{3} \frac{c e_o^2}{R^2} \beta^4, \quad W_{\pi} = \frac{1}{4} \frac{2}{3} \frac{c e_o^2}{R^2} \beta^4$$

and one sees that the emitted radiation is always polarized mainly along the orbital plane and that the intensity of the σ component rises as the energy of the electron. From (3.5) each harmonics appears having a strong dependence on the angle θ , that is the harmonics have different angular distribution.

4. - ASYMPTOTIC RELATION OF THE SPECTRAL DISTRIBUTION. -

To study the spectral distribution intensity of the radiation in the ultrarelativistic case, it is necessary to modify the Schott's formula. Ivanenko and Sokolov (1948) were able to derive asymptotic formulae of the Bessel functions which are applicable in the ultrarelativistic case and for very high harmonic numbers, Schwinger (1949) obtained analogous formulae. Here we shall report only the results.

The argument of the Bessel functions is the variable: $x = \nu \beta \sin \theta$, see (3.1); for high harmonic numbers, ν , the asymptotic relations are:

$$(4.1) \quad J_{\nu}(x) = \frac{1}{\pi \sqrt{3}} \left(1 - \frac{x^2}{\nu^2}\right)^{1/2} K_{1/3} \left[\left(1 - \frac{x^2}{\nu^2}\right)^{3/2} \frac{\nu}{3} \right]$$

$$J_{\nu}'(x) = \frac{1}{\pi \sqrt{3}} \left(1 - \frac{x^2}{\nu^2}\right) K_{2/3} \left[\left(1 - \frac{x^2}{\nu^2}\right)^{3/2} \frac{\nu}{3} \right]$$

12.

where $K_{1/3}(x)$ and $K_{2/3}(x)$ are functions related to the Bessel functions which have imaginary arguments:

$$(4.2) \quad K_{1/3}(x) = \frac{\pi}{\sqrt{3}} \left[i^{1/3} J_{-1/3}(ix) - i^{-1/3} J_{1/3}(ix) \right]; \quad K_{2/3}(x) = \frac{\pi}{\sqrt{3}} \left[i^{2/3} J_{-2/3}(ix) - i^{-2/3} J_{2/3}(ix) \right];$$

and from the properties of the Bessel functions:

$$(4.3) \quad K_{5/3}(x) = \frac{4}{3} \frac{1}{x} K_{2/3}(x) + K_{1/3}(x)$$

Once the (4.1) are substituted into the Scott's formula, which is taken with a single harmonic number, approximate expressions in the asymptotic form are obtained: (the approximations made are: $\beta^2 (\cos^2\theta/\sin^2\theta) = \cos^2\theta$; $\beta^4 = 1$):

$$(4.4) \quad dW(\nu, \theta) = \frac{c e_o^2}{R^2 6 \pi^3} \nu^2 \left\{ \varepsilon^2 K_{2/3}^2\left(\frac{\nu}{3} \varepsilon^{3/2}\right) + \varepsilon \cos^2\theta K_{1/3}^2\left(\frac{\nu}{3} \varepsilon^{3/2}\right) \right\} d\Omega$$

where

$$\varepsilon = 1 - \beta^2 \sin^2\theta$$

This formula gives the radiated energy at a selected wave length $\lambda = 2\pi c/\omega = 2\pi c/\nu \omega_o = (2\pi c/\nu)(R/c)$ and at a selected angle, ψ , between the propagation direction of the radiated energy and the orbital plane.

Integrating (4.4) over all angles θ and over all harmonics one has the total spectral and angular behaviour. This integration implies these integrals (Ivanenko, Sokodov; Classical Electrodynamics)

$$(4.5) \quad \int_0^\pi \varepsilon^2 K_{2/3}^2\left(\frac{\nu}{3} \varepsilon^{3/2}\right) \sin\theta d\theta = \frac{\pi \varepsilon_o^{5/2}}{y \sqrt{3}} \left[\int_y^\infty K_{5/3}(x) dx + K_{2/3}(y) \right]$$

$$\int_0^\pi \varepsilon K_{1/3}^2\left(\frac{\nu}{3} \varepsilon^{3/2}\right) \sin\theta \cos^2\theta d\theta = \frac{\pi \varepsilon_o^{5/2}}{y \sqrt{3}} \left[\int_y^\infty K_{5/3}(x) dx - K_{2/3}(y) \right]$$

where

$$\varepsilon_o = \left(\frac{m_o c^2}{E} \right)^2$$

Definitively one obtains:

$$(4.6) \quad W(\nu) = \frac{2 c e_o^2}{\pi R^2} \frac{\sqrt{3}}{4} \left(\frac{E}{m_o c^2} \right) y \int_y^\infty K_{5/3}(x) dx$$

where

$$y = \frac{\omega}{\omega_c} = \nu \frac{\omega_o}{\omega_c} = \nu \frac{2}{3} \left(\frac{m_o c^2}{E} \right)^3; \quad \omega = \nu \omega_o = \nu \frac{c}{R}; \quad \omega_c = \frac{3}{2} \omega_o \left(\frac{E}{m_o c^2} \right)^3$$

Since

$$\nu = \frac{\omega}{\omega_0} (\approx \frac{10^{16}}{10^8}, R = 1 \text{ m}) \gg 1$$

the discrete distribution over the harmonics is transformed to a continuous distribution; and the energy emitted at the harmonic number, ν , can be transformed into a differential energy, $dW(\nu)$, emitted in the harmonic interval $d\nu = d\omega/\omega_0$; then (4.6) takes the differential form:

$$(4.7) \quad dW(\nu) = \frac{2}{3} \frac{c e_0^2}{R^2} \left(\frac{E}{m_0 c^2} \right)^4 \frac{9\sqrt{3}}{8\pi} y \int_y^\infty K_{5/3}(x) dx \cdot dy$$

Once (4.7) is integrated over all continuous spectrum one has the total radiated energy. This calculation is performed by using the integral (Gradshten, Ryzkik, 6.561-16):

$$(4.8) \quad \int_0^\infty x^\mu K_\nu(x) dx = 2^{\mu-1} \Gamma\left(\frac{1+\mu-\nu}{2}\right) \Gamma\left(\frac{1+\mu+\nu}{2}\right); \quad \mu+1 \pm \nu > 0$$

that is:

$$(4.9) \quad \int_0^\infty y^2 K_{5/3}(y) dy = 2 \Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{7}{3}\right) = \frac{8\pi}{9\sqrt{3}}$$

Then from (4.7) one obtains, integrating by parts:

$$(4.10) \quad \begin{aligned} W &= W_0 (\beta \simeq 1) \frac{9\sqrt{3}}{8\pi} \int_y^\infty y dy \int_0^\infty K_{5/3}(x) dx = \\ &= W_0 (\beta \simeq 1) \frac{9\sqrt{3}}{8\pi} \int_0^\infty y^2 K_{5/3}(y) dy = \frac{2}{3} \frac{c e_0^2}{R^2} \left(\frac{E}{m_0 c^2} \right)^4, \end{aligned}$$

Formula (4.10) is written more correctly in the form:

$$W = \sum_{\nu=0}^\infty W(\nu) = \int_0^\infty dW(\nu) = \frac{W_0}{\beta^4} = W_0 \left[1 + 2(1-\beta^2) + 3(1-\beta^2)^2 \right];$$

that is the asymptotic formula differs from the exact one by high orders terms in $(m_0 c^2/E)$. Formula (4.7) can be expressed in terms of the radiation wave length:

$$(4.11) \quad dW(\lambda) = \frac{1}{2\pi} \frac{c e_0^2}{R^3} \left(\frac{E}{m_0 c^2} \right)^7 y^2 \varphi(y) d\lambda$$

where

$$(4.12) \quad \varphi(y) = \frac{9\sqrt{3}}{8\pi} y \int_y^\infty K_{5/3}(x) dx; \quad y = \frac{\lambda_c}{\lambda}, \quad \lambda_c = \frac{4\pi R}{3} \left(\frac{m_0 c^2}{E} \right)^3$$

14.

$$(4.13) \quad \lambda_c (\text{\AA}) = \frac{5.59 R(m)}{[E(\text{GeV})]^3}$$

The useful expression (4.13) gives the displacement towards the wave lengths shorter than the critical wave length, λ_c , as the energy rises. It is similar to the Wien law for the thermal radiation. Formula (4.11) depends on the parameter y via the universal function:

$$G(y) = y^3 \int_y^{\infty} K_{5/3}(x) dx$$

which was calculated and represented in Fig. (7). The maximum is for $\lambda_m/\lambda_c = 0.42$, and the band width at half maximum is $\Delta\lambda = 0.84\lambda_c$. This band width is asymmetric, being more narrow at wave lengths smaller than the critical one, and more broad at wave lengths higher than the critical one.

The differential photon number over all angles can be deduced from (4.6) as:

$$(4.14) \quad dn(\nu) = \frac{dW(\nu)}{\hbar\omega} = \frac{4}{9\hbar} \frac{e_o^2}{R} \left(\frac{E}{m_o c^2}\right) y^{-1} \varphi(y) dy = \frac{5\sqrt{3}}{6\hbar} \frac{e_o^2}{R} \left(\frac{E}{m_o c^2}\right) \frac{3}{5} \int_y^{\infty} K_{5/3}(x) dx dy$$

After integrating over all harmonics one obtains the total number of photons:

$$(4.15) \quad n = \frac{5\sqrt{3}}{6\hbar} \frac{e_o^2}{R} \frac{E}{m_o c^2}$$

During a complete revolution of the electron along its orbit, that is during the time interval $2\pi R/c$, one obtains the number of photons per cycle:

$$(4.16) \quad n(\text{cycle}) = \frac{2\pi R}{c} n = \frac{1}{137} \frac{\pi 5\sqrt{3}}{3} \frac{E}{m_o c^2} = 1.295 \cdot 10^2 E(\text{GeV})$$

the formula (4.14), written in terms of wavelengths, becomes:

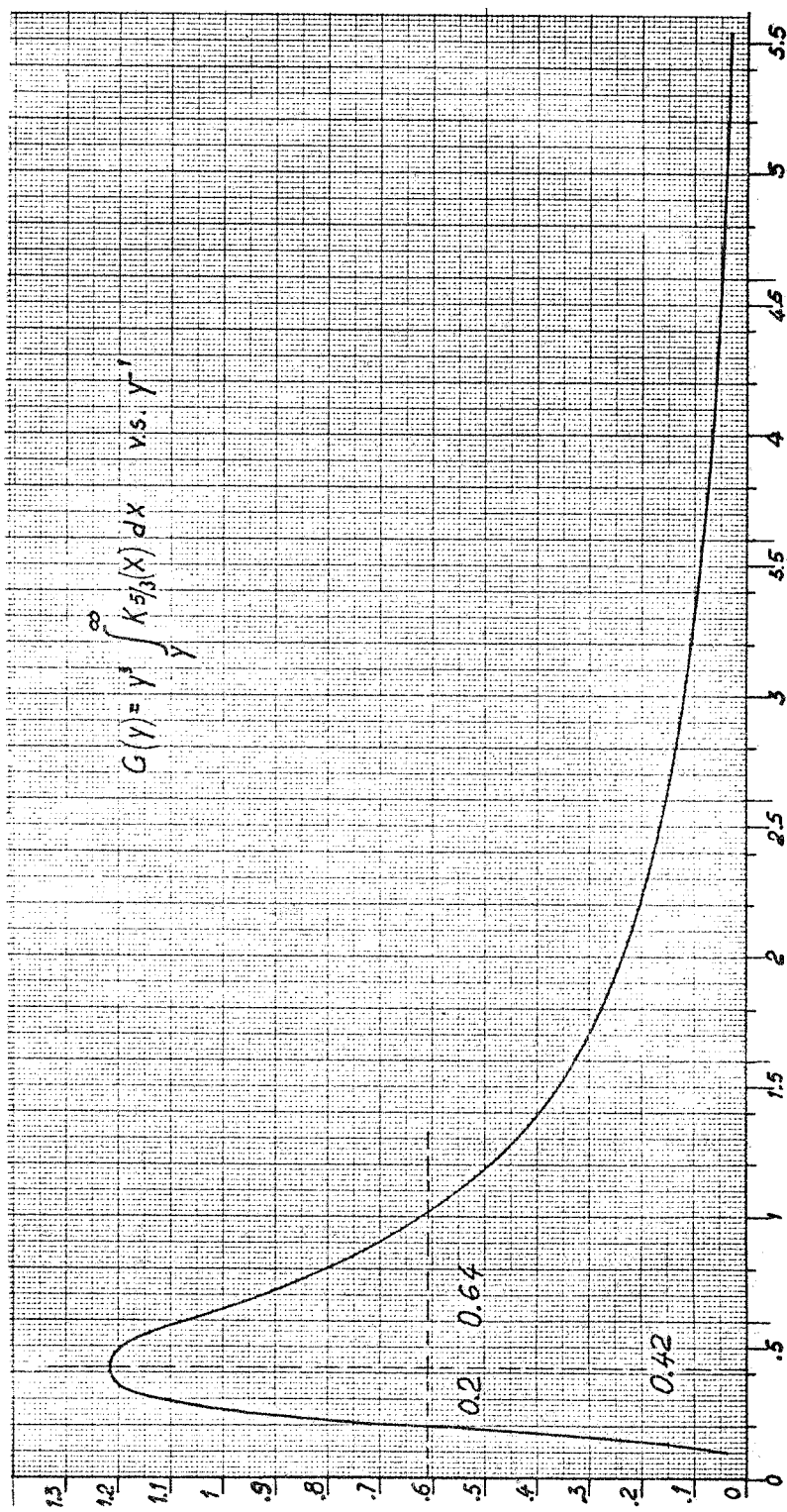
$$(4.17) \quad dn(\lambda) = \frac{1}{3\pi} \frac{e_o^2}{\hbar R^2} \left(\frac{E}{m_o c^2}\right)^4 y \varphi(y) d\lambda$$

The adimensional function $\varphi(y)$, normalized to 1 over all the spectrum, determines the behaviour of the radiation. Its limit behaviour can be visualized by using certain limit formulas for the two cases:

$$y \ll 1; \quad y \gg 1$$

(see for example Schwinger 1949)

$$(4.18) \quad \varphi(y) = \frac{9\sqrt{3}}{8\pi} y \int_y^{\infty} K_{5/3}(x) dx = \begin{cases} 2^{2/3} \Gamma(\frac{2}{3}) y^{1/3} & \text{for } y \ll 1 \\ \sqrt{\frac{\pi}{2}} y^{1/2} \exp(-y) & \text{for } y \gg 1 \end{cases}$$

FIG. 7 - The function $G(y)$.

16.

from which one can deduce the limit expression of the spectral distribution as:

$$(4.19) \quad dW(\nu, y \ll 1) = \frac{3\sqrt{3}}{2^{4/3}\pi} \Gamma\left(\frac{2}{3}\right) \frac{c e_o^2}{R^2} \left(\frac{E}{m_o c^2}\right)^4 y^{1/3} dy$$

$$dW(\nu, y \gg 1) = \frac{3\sqrt{3}}{2^{5/2}\pi^{1/2}} \frac{c e_o^2}{R^2} \left(\frac{E}{m_o c^2}\right)^4 y^{1/2} \exp(-y) dy$$

The radiated energy vanishes with an exponential law for frequencies higher than the critical one, while it goes to zero for frequencies going to zero.

The functions (4.11) and (4.19) are written as follow:

$$(4.20) \quad dW(\lambda) = \left\{ \frac{1}{2} \frac{9\sqrt{3}}{8\pi^2} \frac{c e_o^2}{R^3} \left(\frac{E}{m_o c^2}\right)^7 \right\} y^3 \int_y^\infty K_{5/3}(x) dx \quad d\lambda$$

$$(4.21) \quad dW(\lambda, y \ll 1) = \left\{ \frac{9\sqrt{3}}{16\pi^2} \frac{c e_o^2}{R^3} \left(\frac{E}{m_o c^2}\right)^7 \right\} \frac{2\pi^3 \sqrt{4}}{\Gamma\left(\frac{1}{3}\right) \sqrt{3}} y^{7/3} \quad d\lambda =$$

$$= \left\{ \frac{9\sqrt{3}}{16\pi^2} \frac{c e_o^2}{R^3} \left(\frac{E}{m_o c^2}\right)^7 \right\} 2.12 x y^{7/3} \quad d\lambda$$

$$(4.22) \quad dW(\lambda, y \gg 1) = \left\{ \frac{9\sqrt{3}}{16\pi^2} \frac{c e_o^2}{R^3} \left(\frac{E}{m_o c^2}\right)^7 \right\} \sqrt{\frac{\pi}{2}} y^{5/2} \exp(-y) d\lambda$$

$$= \left\{ \frac{9\sqrt{3}}{16\pi^2} \frac{c e_o^2}{R^3} \left(\frac{E}{m_o c^2}\right)^7 \right\} 1.255 x y^{5/2} \exp(-y) d\lambda$$

The three functions, by making their common, factor equal to 1, are shown in Fig. 8 as a function of $y^{-1} = \lambda / \lambda_c$.

Now we can notice that formula (4.21) is independent of the orbital energy, E, and the maximum value of (4.20) depends on the wave length, λ_m , both with the same law. This fact can be used to develop the following graphical method.

Formula (4.21), written in terms of wave length, is:

$$(4.23) \quad \frac{dW(\lambda, y \ll 1)}{d\lambda} = (4\pi)^{1/3} \frac{c e_o^2}{R^{2/3}} 2.12 \lambda^{-7/3};$$

and the maximum of formula (4.20) is given as:

$$(4.24) \quad \frac{dW(\lambda_m)}{d\lambda} = (4\pi)^{1/3} \frac{c e_o^2}{R^{2/3}} \left(\frac{0.42}{\lambda_m}\right)^{7/3} \times 1.22$$

from which $dW(\lambda, y \ll 1) = 13.1 dW(\lambda_m)$.

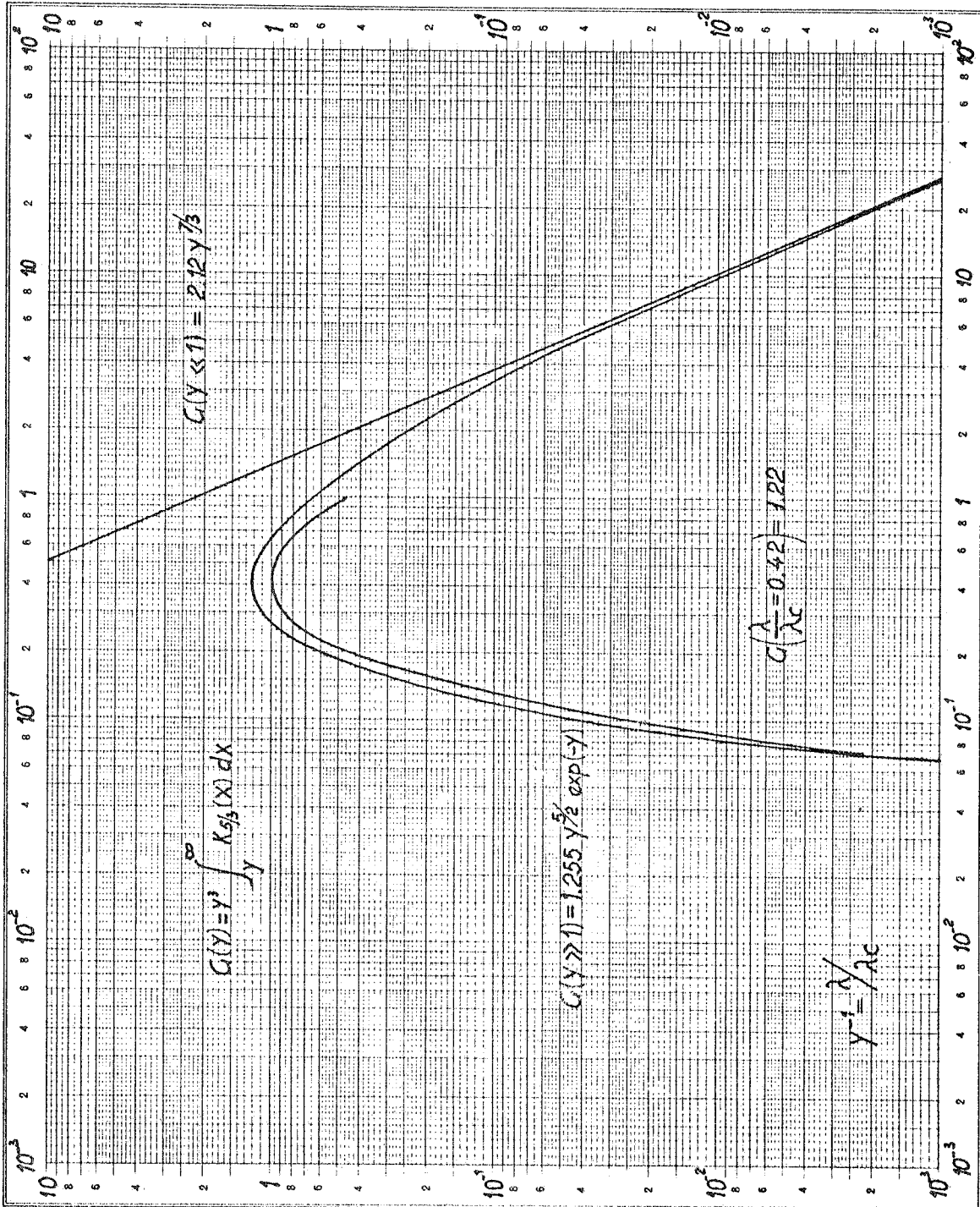


FIG. 8 - The functions $G(y)$, $G(y \ll 1)$, $G(y \gg 1)$.

Furthermore Fig. 9 gives the function $\lambda_m = 0.42 (4\pi R/3) (m_0 c^2/E)^3$, that is it gives the correspondence $E(\text{GeV}) \leftrightarrow \lambda_m$ for any radius R ; for the moment let us take the curve for which $R = 1(\text{m})$.

By drawing the functions:

$$\frac{dW(\lambda_m)}{d\lambda} \text{ v. s. } \lambda_m \quad \text{and} \quad \frac{dW(\lambda, y \ll 1)}{d\lambda} \text{ v. s. } \lambda$$

on the same abscissa, Fig. 10, 11, we obtain the points corresponding to the maximum of the radiated energy for an orbital energy $E(\text{GeV})$ corresponding to a fixed wave length λ_m , curve (1) while the curve (2) gives the limit behaviour of the $G(y)$ function.

By superimposing the curves (1) and (2) drawn on a transparent sheet to curve (1) and (2) of Fig. 10, 11, point (A) determines the radiated energy at the wavelength λ_m , while the curve $G(y)$ gives, in its position, the radiated energy at the other wave lengths.

By shifting the transparent paper, and by maintaining the coincidence (1) \leftrightarrow (1) and (2) \leftrightarrow (2) one can find the radiated energies for all the orbital energies and for all the wavelengths when $R = 1\text{m}$.

When $R \neq 1\text{m}$ one has to do as shown by this example: $R = 3.6\text{m}$, $E = 1\text{ GeV}$.

- From Fig. 9 the correspondence $E(1\text{ GeV}, 3.6\text{m}) \leftrightarrow \lambda_{m_0} = 8.7 \text{ \AA}$ is determined.
- In Fig. 10 point A is selected at the abscissa $\lambda_m = 8.7 \text{ \AA}$.
- The point ordinate of A is 4×10^{-9} Watt.
- The value of the radiated energy is

$$\begin{aligned} \frac{dW(\lambda_m)}{d\lambda} &= R^{-2/3} \times 4 \times 10^{-9} \text{ Watt.} \\ &= 0.426 \times 4 \times 10^{-9} \text{ Watt.} \end{aligned}$$

By this graphical method one finds all the values of the radiated energies for all the orbital radii after one orbital energy has been chosen. This possibility is due to the property of the emitted radiation that has a limit curve the same for all the orbital energies higher than the critical one. From formula (4.17) the differential photon number is deduced for various energies as a function of the wave length. The analogous expression to (4.20) and (4.21) are:

$$\frac{dn(\lambda)}{d\lambda} = \left\{ \frac{9\sqrt{3}}{24\pi^2} \frac{e_0^2}{\hbar R^2} \left(\frac{E}{m_0 c^2} \right)^4 \right\} y^2 \int_y^\infty K_{5/3}(x) dx \quad (4.25)$$

$$\frac{dn(\lambda, y \ll 1)}{d\lambda} = \left\{ \frac{9\sqrt{3}}{24\pi^2} \frac{e_0^2}{\hbar R^2} \left(\frac{E}{m_0 c^2} \right)^4 \right\} \frac{2\pi^3 \sqrt{4}}{\sqrt{3} \Gamma(\frac{1}{3})} y^{4/3}$$

The function, $G(y)y^{-1}$ is represented in Fig. 12a, 12b. One can notice that the photon spectrum has a maximum for $\lambda_m^F = 0.70 \lambda_c$; the curve λ_m^F v. s. E is represented in Fig. 13 for some orbital radii. Repeating the same procedure just exposed one finds the photon number value at different energies and for different orbital radii by using the curve of Figure 14, 15 and the transparent sheet.

The formulae giving the maximum photon number and the limit curve as a function of λ_m^F and λ are:

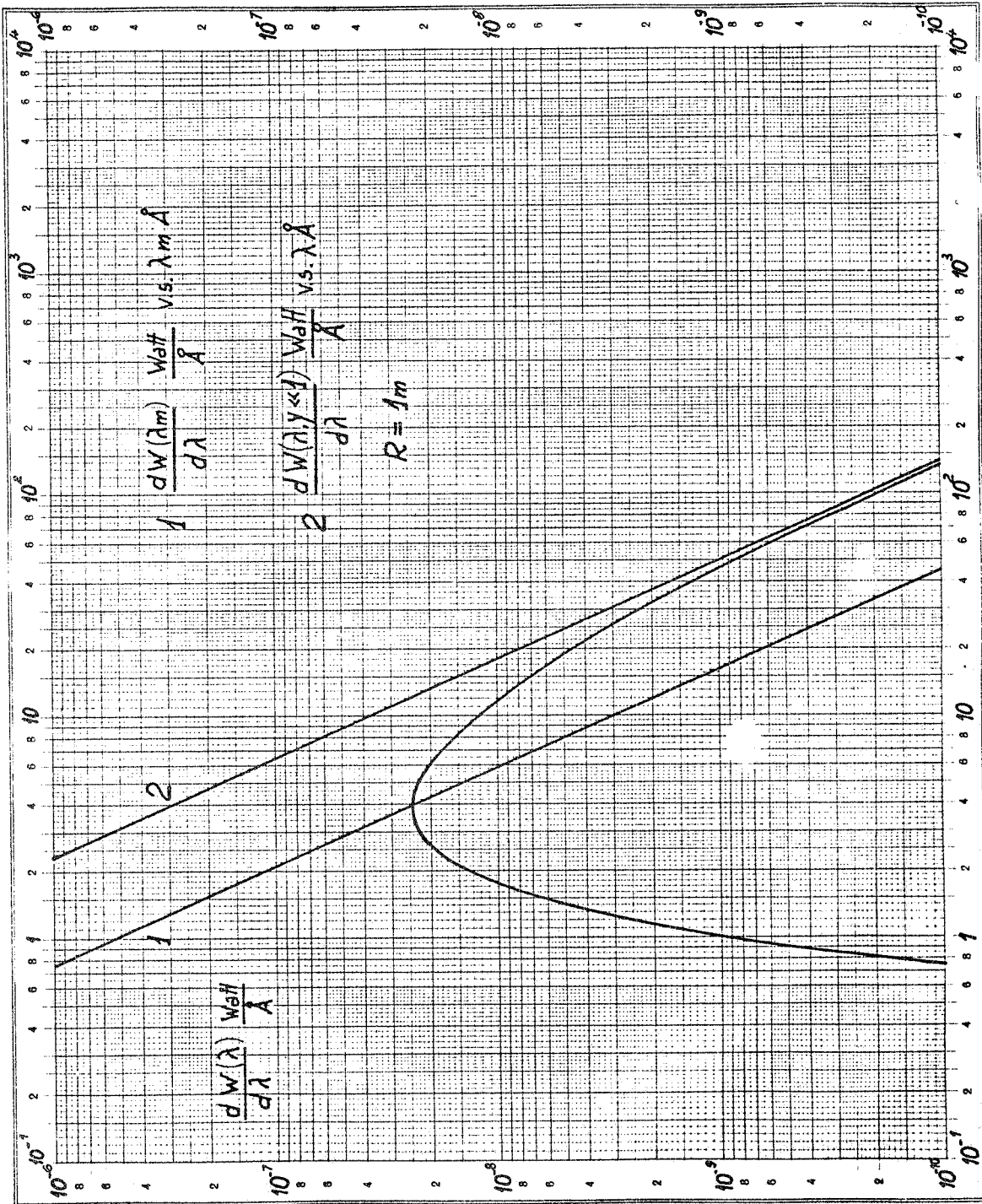


FIG. 10 - Curve (1) is the function $\frac{dW(\lambda m)}{d\lambda}$ v. s. λm . Curve (2) is the function

$$\frac{dW(\lambda, y \ll 1)}{d\lambda} \text{ v. s. } \lambda$$

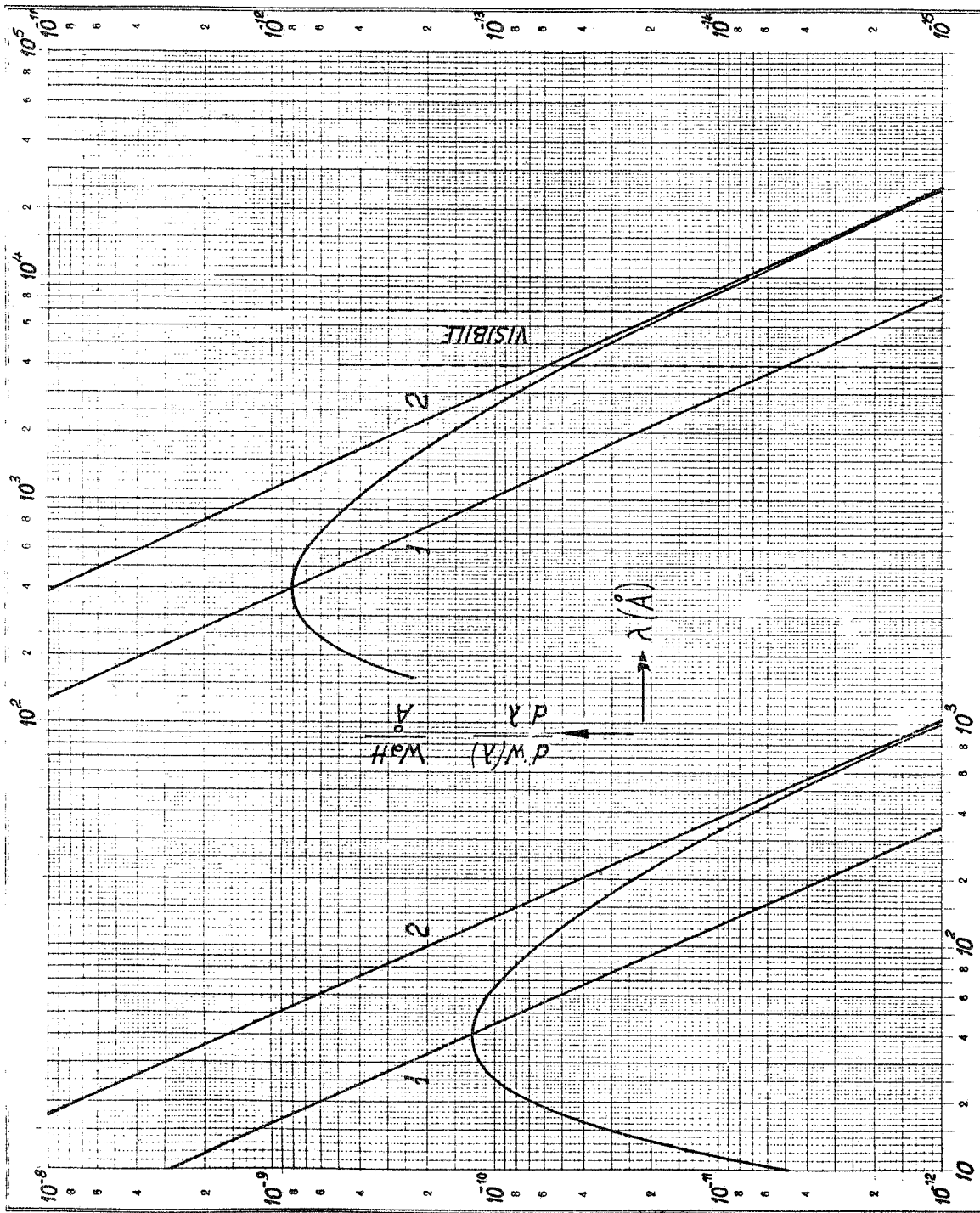


FIG. 11 - Curve (1) is the function $\frac{dW(\lambda, \gamma \ll 1)}{d\lambda}$ v. s. λ . Curve (2) is the function $\frac{dW(\lambda, m)}{d\lambda}$ v. s. λ .

$$\frac{dW(\lambda, \gamma \ll 1)}{d\lambda} \text{ v. s. } \lambda$$

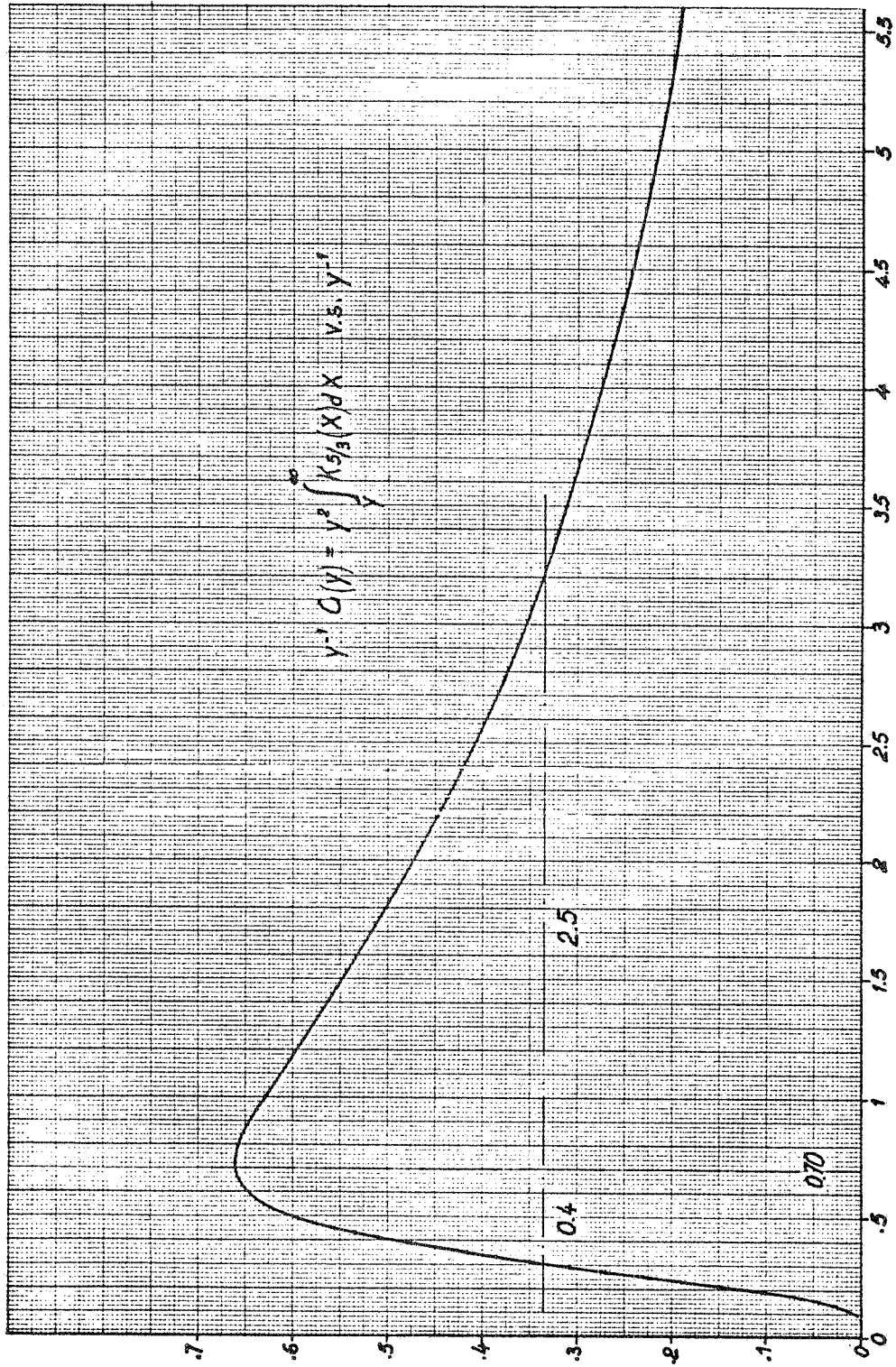


FIG. 12 - a) The function $G(y) y^{-1}$.

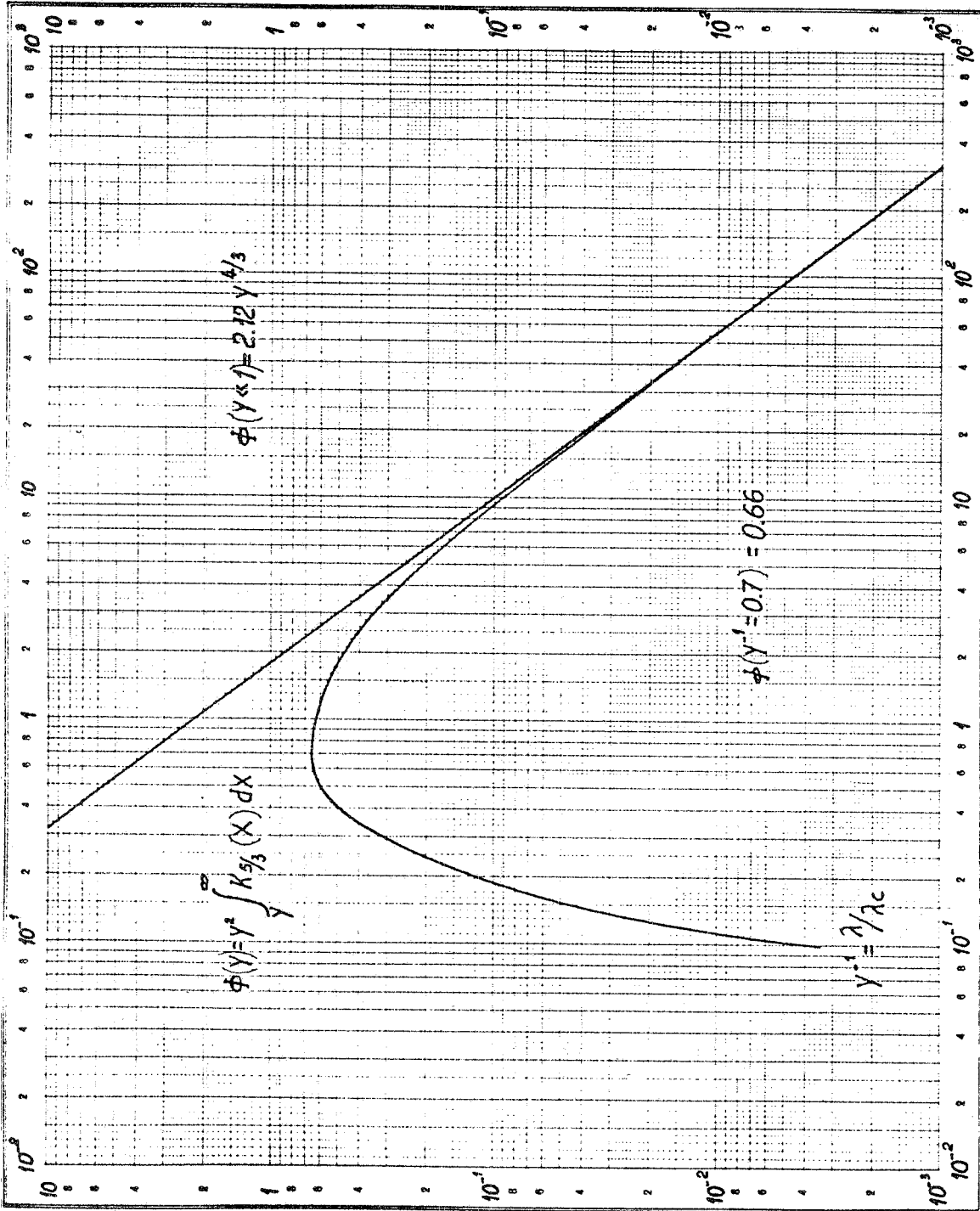


FIG. 12 - b) The function $G(y)y^{-1}$.

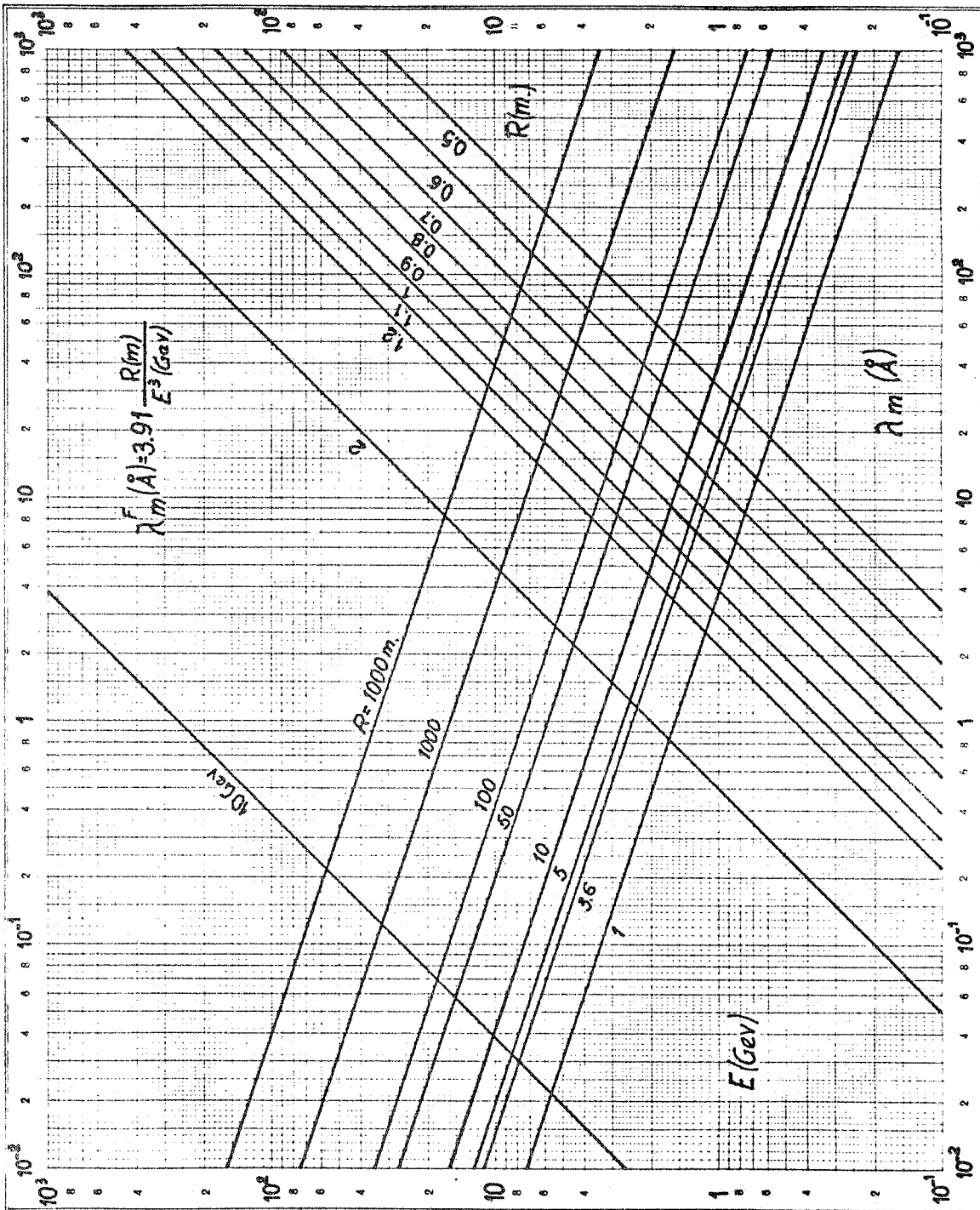


FIG. 13 - The function $\lambda_m = 0.70 \frac{4\pi R}{3} \left(\frac{m_0 c^2}{E} \right)^3$

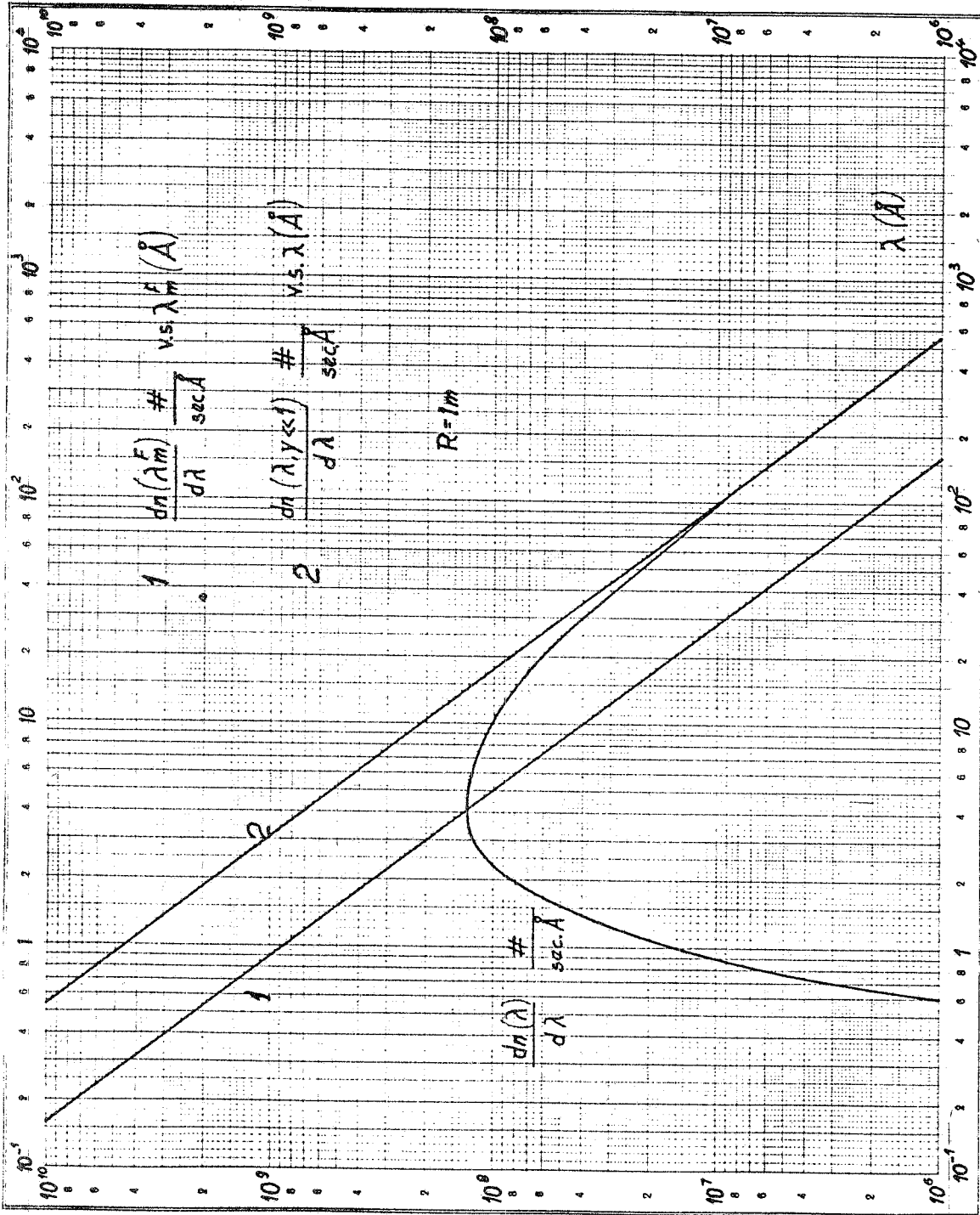


FIG. 14 - Curve (1) is the function $\frac{dn(\lambda, y \ll 1)}{d\lambda}$ v.s. λ . Curve (2) is the function $\frac{dn(\lambda, m)}{d\lambda}$ v.s. λ .

$$\frac{dn(\lambda, y \ll 1)}{d\lambda} \text{ v.s. } \lambda$$

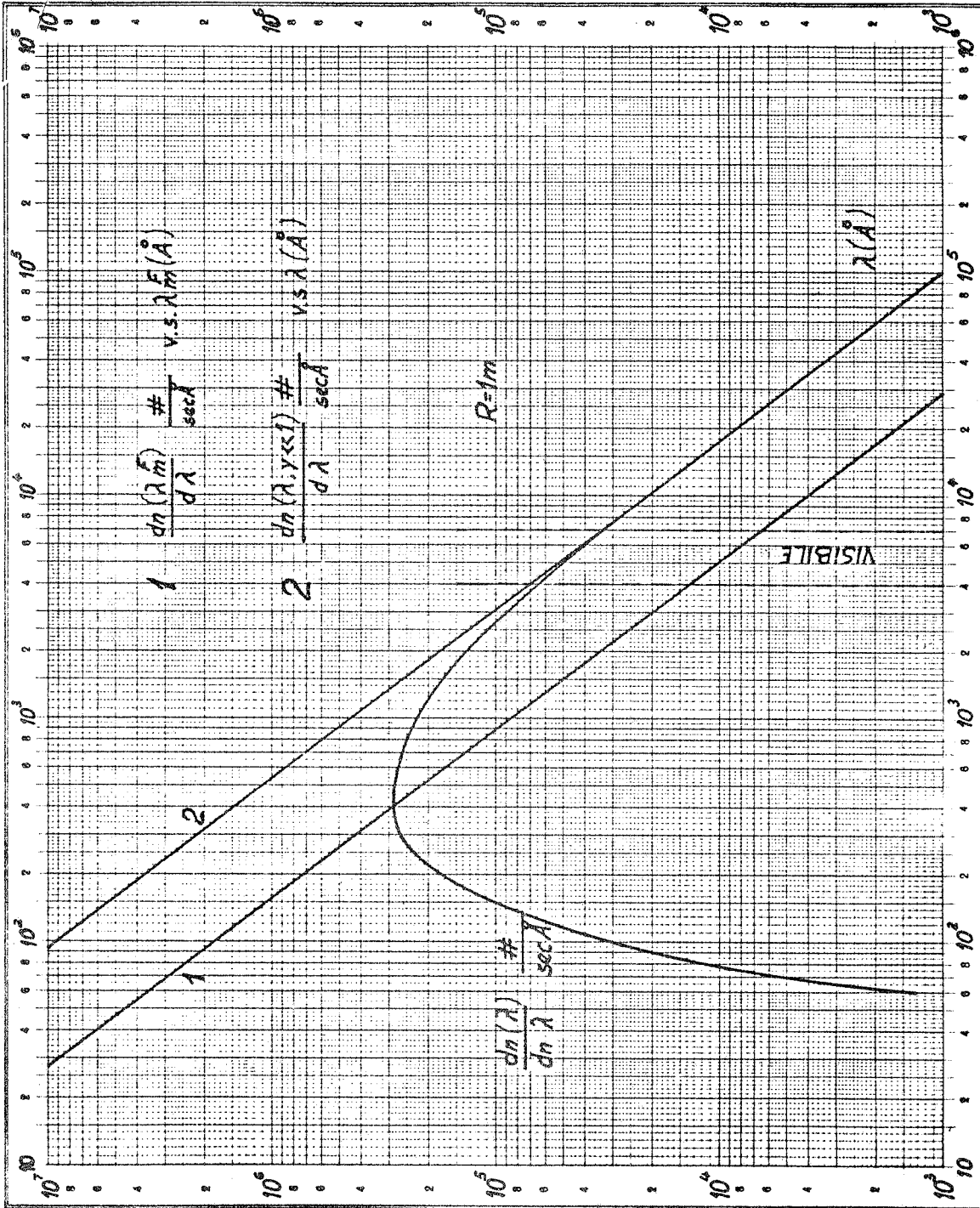


FIG. 15- Curve (1) is the function $\frac{dn(\lambda_m)}{d\lambda}$ v.s. λ_m . Curve (2) is the function $\frac{dn(\lambda, \gamma_{cs1})}{d\lambda}$ v.s. λ .

$$\frac{dn(\lambda, \gamma_{cs1})}{d\lambda} \text{ v.s. } \lambda$$

$$\begin{aligned} \frac{dn(\lambda_m^{ph})}{d\lambda} &= \frac{3^{1/6}}{2^{1/3} \pi^{2/3}} \frac{e_o^2}{\hbar R^{2/3}} \left(\frac{0.70}{\lambda_m^{ph}}\right)^{4/3} \phi(y^{-1} = 0.70) = \\ &= \frac{3^{1/6}}{2^{1/3} \pi^{2/3}} \frac{e_o^2}{\hbar R^{2/3}} \left(\frac{0.70}{\lambda_m^{ph}}\right)^{4/3} 0.66 \end{aligned}$$

(4.26)

$$\begin{aligned} \frac{dn(\lambda, y \ll 1)}{d\lambda} &= \frac{3^{1/6}}{2^{1/3} \pi^{2/3}} \frac{e_o^2}{\hbar R^{2/3}} \frac{2\pi \sqrt[3]{4}}{\sqrt{3} \Gamma(\frac{1}{3})} \lambda^{-4/3} = \\ &= \frac{3^{1/6}}{2^{1/3} \pi^{2/3}} \frac{e_o^2}{\hbar R^{2/3}} 2.12 \lambda^{-4/3} \end{aligned}$$

from which one has

$$\frac{dn(\lambda_m, y \ll 1)}{dn(\lambda_m)} = \frac{2.12}{(0.70)^{4/3} 0.66} = 5.25$$

The formula which gives the displacement of the wave length, at which there is the maximum radiated energy as a function of the orbital energy, is similar to the Wien law of the displacement of the thermal radiation. But the behaviour of the thermal radiation is very different from the behaviour of the non thermal radiation. The non thermal radiation is polarized while the other is not. The thermal radiation, as the blackbody temperature changes, is represented by a family of curves which don't have a common limit curve as in the non thermal case, Fig. 16.

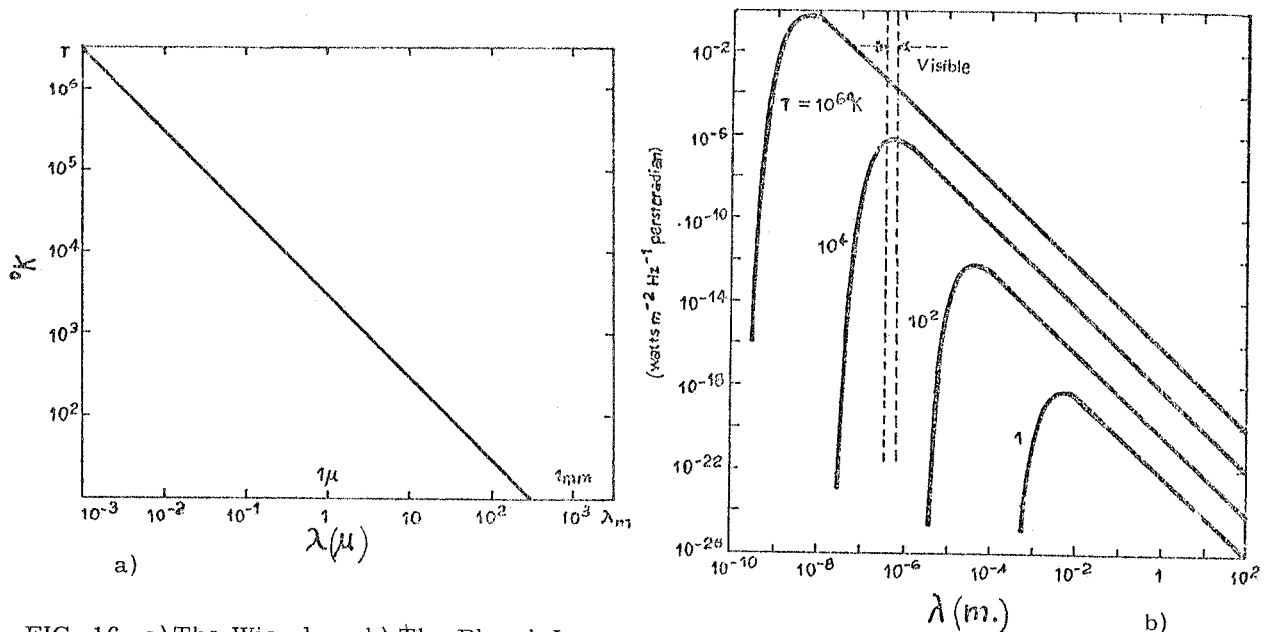


FIG. 16 - a) The Wien law; b) The Planck Law.

The particular property of the non thermal radiation of having a limit curve has practical importance because shows that the photon number emitted in the visible and in infrared spectrum is independent (it depending only on the wave length) of the orbital energies above 0.5 GeV.

It is clear that for these wave lengths the orbital energy fluctuations do not

affect the photon emission, which is affected only by the photon number.

For this reason a non destructive accelerator can be an absolute standard source of radiation, since the number of orbiting electrons are constant. This is true, obviously within a time interval during which electrons are not lost.

Finally, following Scott (1912), one can do some consideration on the coherence of the emitted radiation. Let us assume that on the orbit N electrons are randomly distributed. Assuming an origin axis, each electron determines an arc on the orbit which subtends an angle, φ_K , where K takes values from 1 to N .

Then the Fourier components of the vector potential of the ν^{th} harmonic has a phase factor: $\exp(i \nu \varphi_K)$. In order to calculate the radiation intensity one must calculate the squared modulus of the product between the vector potential and the electron velocity direction. Then the formula giving the radiation intensity depends on the squared modulus, F_ν , of the sum of the phase of each electron:

$$(4.27) \quad F_\nu = \left| \sum_{K=1}^N \exp(i \nu \varphi_K) \right|^2$$

then, (3.1) for one harmonic, becomes:

$$(4.28) \quad dW(\nu, \theta) = \frac{c e^2}{2 \pi R^2} \beta^2 \left\{ \nu^2 \cot^2 \theta J_\nu^2(x) + \nu^2 \beta_\nu^2 J_\nu'^2(x) \right\} \left| \sum e^{i \nu \varphi_K} \right|^2 d\Omega$$

The factor F is written in the form:

$$(4.29) \quad \left| \sum_{K=1}^N e^{i \nu \varphi_K} \right|^2 = N + \sum_{K=1}^N \sum_{\substack{m=1 \\ K \neq m}}^N \cos \nu (\varphi_K - \varphi_m)$$

now the average double sum is zero when the electron positions are randomly distributed on the orbit. The intensity of N electrons is just the sum of the single electron intensity. On the contrary, one can postulate a uniform distribution of the electron on the orbit; then

$$\varphi_K - \varphi_{K-1} = \varphi_m - \varphi_{m-1} = \frac{2\pi}{N}$$

and the factor F_ν becomes:

$$(4.30) \quad F_\nu = N + \sum_{K=1}^N \sum_{\substack{m=1 \\ K \neq m}}^N \cos \nu |K-m| \frac{2\pi}{N} = \sum_{K=1}^N \sum_{m=1}^N \cos \nu |K-m| \frac{2\pi}{N} = \\ = N \sum_{j=1}^N \cos j \frac{\nu}{N} 2\pi$$

If ν/N is an integer number, F_ν is equal to N^2 ; if ν/N is a non integer number, F_ν is equal to zero; then there are certain wavelength for which $\nu/N \neq$ integer, where no radiation is emitted.

Finally, for an uniform distribution, one has: (G. H. ; 1.342-1,2)

$$(4.31) \quad \left| \sum_{K=1}^N e^{i\nu\varphi} \right|^2 = \left| \sum_{K=0}^{N-1} e^{i\nu K\varphi} \right|^2 = N^2 \left[\frac{\sin \frac{N\nu\varphi}{2} / \frac{N\nu\varphi}{2}}{\sin \frac{\nu\varphi}{2} / \frac{\nu\varphi}{2}} \right]^2 \xrightarrow{\nu\varphi \rightarrow 0} 1.$$

This formula shows that there is coherence for frequencies $\omega \ll (\omega_0/\varphi)$ or $\lambda \gg 2\pi d$ where d is the arc distance between two adjacent electrons; for example, for $N=10^{10}$, $R=3$ m and $\lambda=10^6 2\pi d$, one has: $\lambda \simeq 1$ cm.

It is clear that F_ν strongly depends on the spatial distribution of the electrons; for a detailed discussion see: Ndvick Saxon (1954).

5. - APPROXIMATE RELATIONS OF THE SCHOTT'S FORMULA IN THE ULTRARELATIVISTIC CASE. -

The Schott's formula (4.4) is more useful written as a function of the elevation angle, which is the angle between the propagation direction of the emitted radiation and the orbital plane, rather than the elementary solid angle.

As $d\Omega = \sin\theta \, d\theta \, d\varphi = \cos\psi \, d\psi \, d\varphi$, and since $\psi \simeq \sqrt{1-\beta^2} \simeq 0$, one can make the following approximations:

$$\cos\psi \simeq 1; \quad \sin\psi \simeq \psi; \quad \varepsilon = 1 - \beta^2 \sin^2\theta = 1 - \beta^2 \cos^2\psi \simeq 1 - \beta^2 + \psi^2$$

and the relation:

$$\frac{\nu}{3} (1 - \beta^2 \sin^2\theta)^{3/2} = \frac{\nu}{3} (1 - \beta^2 \cos^2\psi)^{3/2} = \frac{1}{2} \frac{\omega}{\omega_c} \left[1 + \left(\frac{E}{m_0 c^2} \psi \right)^2 \right]^{3/2}$$

After integration on φ angle (4.4) becomes:

$$(5.1) \quad dW(\nu, \psi) = 2\pi \frac{e^2}{R^2} \frac{e_0^2}{\pi^3} \left(\frac{\omega}{\omega_c} \right)^2 \frac{1}{\omega_0} \left(1 + \frac{\psi^2}{1-\beta^2} \right) (1-\beta^2)^2 \times$$

$$\times \left\{ K_{2/3}^2(x) + \frac{\psi^2}{\varepsilon} K_{1/3}^2(x) \right\} d\psi d\omega = \frac{3}{4} \frac{e_0^2}{R \pi^2} \left(\frac{E}{m_0 c^2} \right)^2 \left(\frac{\omega}{\omega_c} \right)^2 \left[1 + \left(\frac{E}{m_0 c^2} \psi \right)^2 \right]^2 \times$$

$$\times \left\{ K_{2/3}^2(x) + \frac{\left(\frac{E}{m_0 c^2} \psi \right)^2}{1 + \left(\frac{E}{m_0 c^2} \psi \right)^2} K_{1/3}^2(x) \right\} d\psi d\omega$$

where

$$x = \frac{1}{2} \frac{\lambda_c}{\lambda} \left[1 + \left(\frac{E}{m_0 c^2} \psi \right)^2 \right]^{3/2}$$

The term $K_{2/3}^2(x)$ in (5.1) corresponds to the σ component and the term $K_{1/3}^2(x)$ corresponds to the π component, as one can see by comparison with (4.1) and (3.5).

In order to obtain the angular distribution independently of the frequencies one must integrate over all harmonics, and then one has to use the integral (G.R.; 6.576-4):

$$(5.2) \quad \int_0^{\infty} K_{\rho}(x) K_{\nu}(x) x^{\mu-1} dx = \frac{2^{\mu-3}}{\Gamma(\mu)} \Gamma\left(\frac{\mu+\nu+\rho}{2}\right) \Gamma\left(\frac{\mu+\nu-\rho}{2}\right) \Gamma\left(\frac{\mu-\nu+\rho}{2}\right) \Gamma\left(\frac{\mu-\nu-\rho}{2}\right)$$

$$\mu > \nu + \rho$$

So one has:

$$(5.3) \quad dW(\psi) = \int_0^{\infty} dW(\omega, \psi) d\omega \cdot d\psi =$$

$$= \frac{c e_o^2}{16 R^2} \left(\frac{E}{m_o c^2}\right)^5 \left[1 + \left(\frac{E}{m_o c^2}\right)^2 \psi^2\right]^{-5/2} \left\{ 7 + 5 \frac{\left(\frac{E}{m_o c^2}\right)^2 \psi^2}{1 + \left(\frac{E}{m_o c^2}\right)^2 \psi^2} \right\} d\psi$$

By separating the two polarized components, one has:

$$(5.4) \quad dW(\psi)_{\sigma} = \frac{c e_o^2}{16 R^2} \left(\frac{E}{m_o c^2}\right)^5 \frac{7}{(1+\eta^2)^{5/2}}; \quad dW(\psi)_{\pi} = \frac{c e_o^2}{16 R^2} \left(\frac{E}{m_o c^2}\right)^5 \frac{5 \eta^2}{(1+\eta^2)^{7/2}}$$

where the two functions of the parameter $\eta = (E/m_o c^2) \psi$:

$$(5.5) \quad f(\eta)_{\sigma} = \frac{7}{(1+\eta^2)^{5/2}}, \quad f(\eta)_{\pi} = \frac{5 \eta^2}{(1+\eta^2)^{7/2}}$$

are represented in Fig. 17. The $f(\eta)_{\sigma}$ function has a maximum for $\eta=0$ and the $f(\eta)_{\pi}$ function has two maxima for $\eta = \pm(2/5)^{1/2}$ (that is at the angles $\psi = 0 \pm \sqrt{2/5} (m_o c^2/E)$; further more $f_{\sigma}^{\text{Max}}/f_{\pi}^{\text{Max}} = 7/2 (7/5)^{7/2} = 11.33$.

The total emitted radiation is

$$(5.6) \quad f(\eta) = f(\eta)_{\sigma} + f(\eta)_{\pi} = \frac{7+12 \eta^2}{(1+\eta^2)^{7/2}}$$

From (5.1), expressed in wave length terms, one has:

$$(5.7) \quad dW(\lambda, \psi) =$$

$$= \frac{27}{32 \pi^3} \frac{c e_o^2}{R^3} \left(\frac{E}{m_o c^2}\right)^8 \left[1 + \left(\frac{E}{m_o c^2}\right)^2 \psi^2\right]^2 \left(\frac{\lambda c}{\lambda}\right)^4 \left\{ K_{2/3}^2(x) + \frac{\left(\frac{E}{m_o c^2}\right)^2 \psi^2}{1 + \left(\frac{E}{m_o c^2}\right)^2 \psi^2} K_{1/3}^2(x) \right\} d\psi d\lambda$$

which is written as a function of the parameters:

$$(5.8) \quad \eta = \frac{E}{m_o c^2} \psi = \frac{E(\text{GeV})}{5.12 \times 10^{-4}} \psi; \quad x = 1.19 \frac{\lambda m}{\lambda} (1+\eta^2)^{3/2}$$

Then:

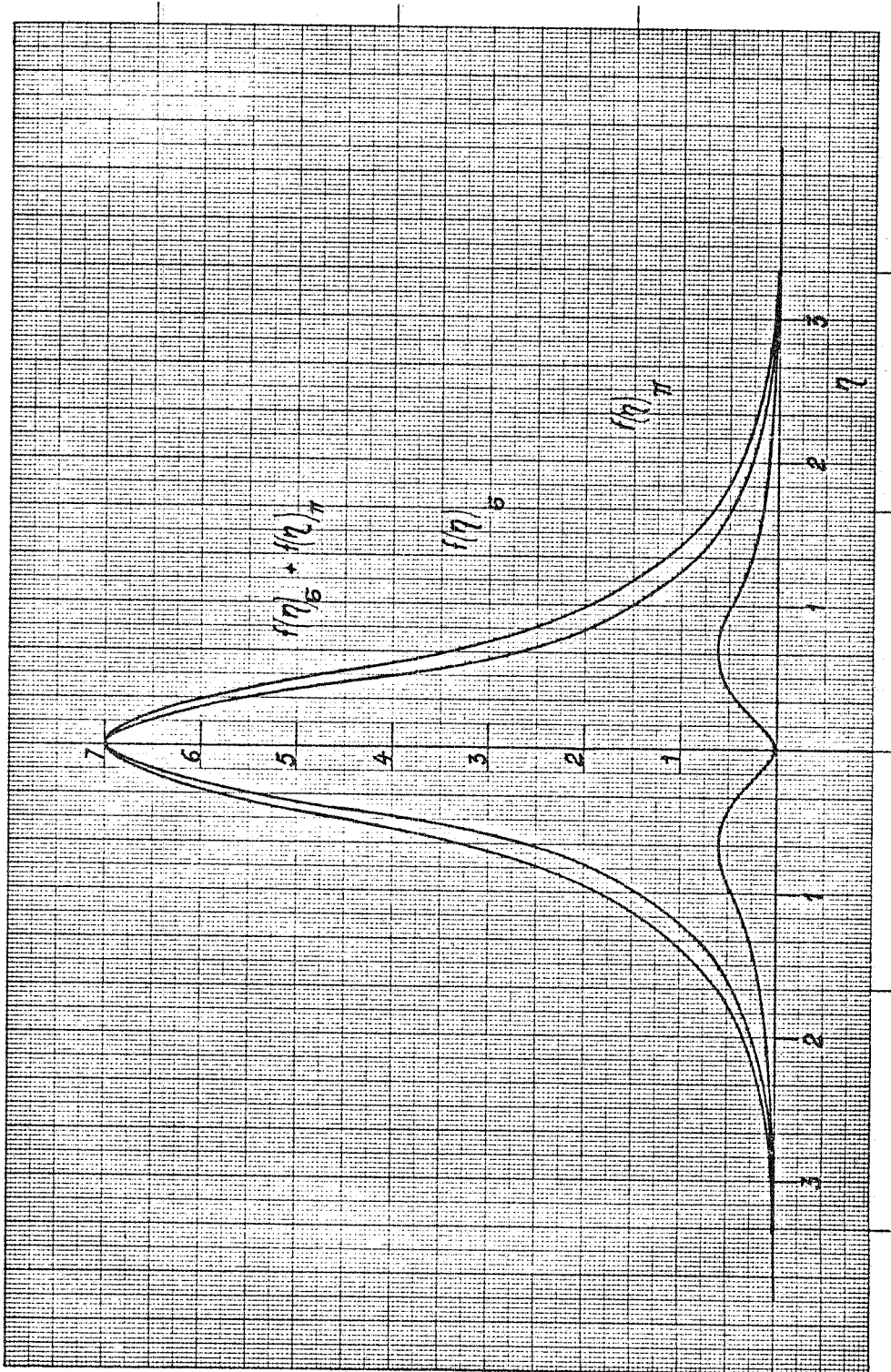


FIG. 17 - The adimensional functions of the total emitted energy.

$$(5.9) \quad \frac{d(\eta, \frac{\lambda_m}{\lambda})}{d\psi x d\lambda} = \frac{9}{32\pi} \frac{c e_o^2 \times 10^{-6}}{(5.12 \times 10^{-4})^8} \frac{E^8(\text{GeV})}{R^3(\text{m})} \times$$

$$\times \left[(1 + \eta^2)^2 (2.38 \frac{\lambda_m}{\lambda})^4 \left\{ I^2\left(\frac{2}{3}\right) + \frac{\eta^2}{1 + \eta^2} I^2\left(\frac{1}{3}\right) \right\} \right]$$

where

$$I^2\left(\frac{1}{k}\right) = \frac{3}{\pi^2} K_{1/k}^2(x)$$

the term in brackets is calculated as a function of the parameter η and at some values of the ratio λ_m/λ .

The constant value is:

$$\frac{9}{32\pi} \frac{c e_o^2 \times 10^{-6}}{(5.12 \times 10^{-4})^8} \frac{E^8(\text{GeV})}{R^3(\text{m})} = 1.31 \times 10^{-7} \frac{E^8(\text{GeV})}{R^3(\text{m})} \frac{\text{Watt.}}{\text{\AA mrad}}$$

Then, after fixing a radius and an orbital energy, one can determine λ_m (also λ_m/λ) by using the curves of Fig. 9. After choosing a curve of Fig. 18, the elevation angle, ψ , corresponds to a value of η given by (5.8) and tabulated in Fig. 19; then by multiplying the ordinate of that η by the value of the constant corresponding to the wanted $E(\text{GeV})$, $R(\text{m})$ the value of the radiated energy is obtained.

The polarization of the emitted radiation is defined by the relation:

$$p = \frac{W(\sigma) - W(\pi)}{W(\sigma) + W(\pi)}$$

or

$$p = \frac{K_{2/3}^2(x) - \frac{\eta^2}{1 + \eta^2} K_{1/3}^2(x)}{K_{2/3}^2(x) + \frac{\eta^2}{1 + \eta^2} K_{1/3}^2(x)}$$

which is calculated in Fig. 20 for some λ_m/λ ratios.

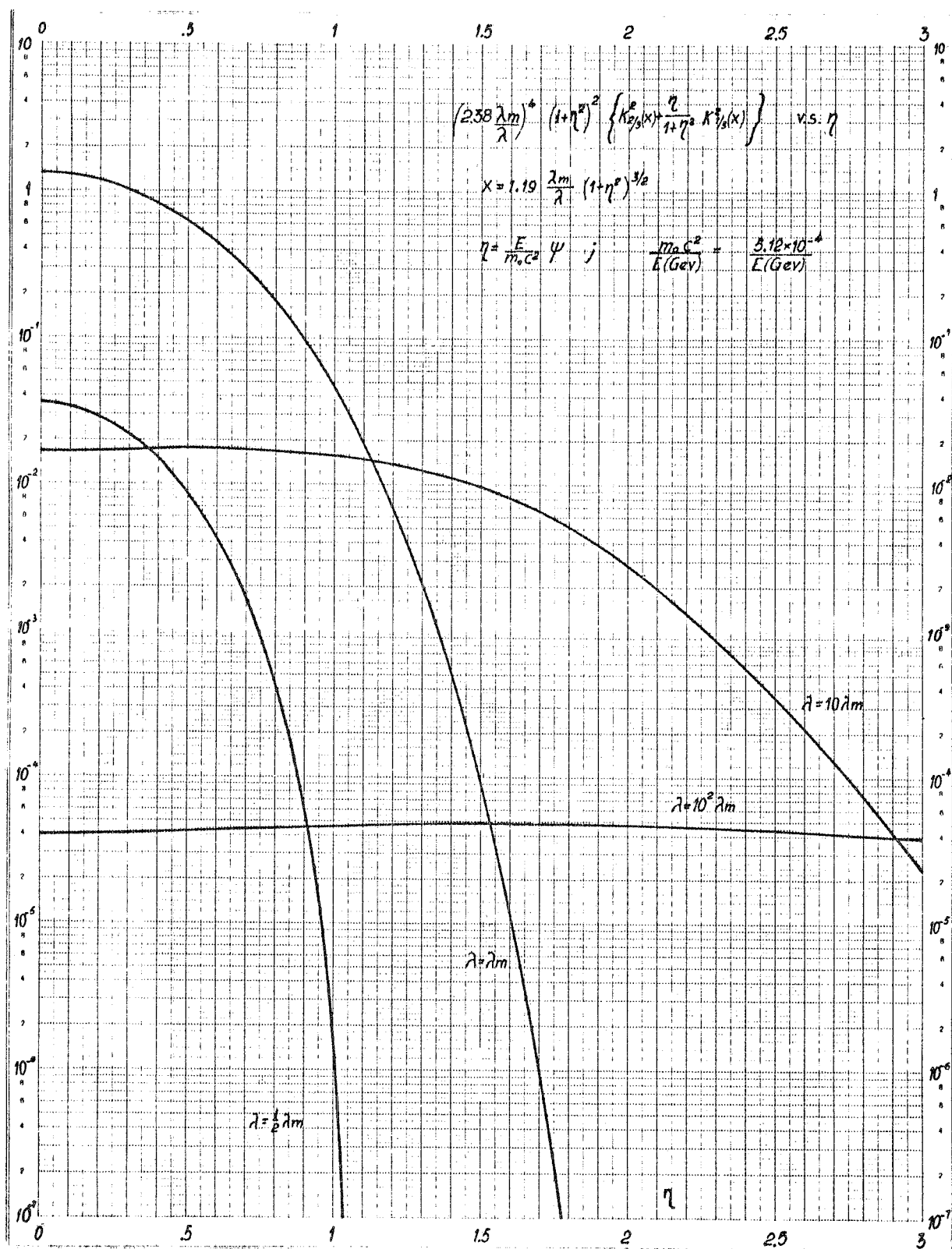


FIG. 18 - Graphical representation of the adimensional part of the function

$$\frac{d(\eta, \lambda_m/\lambda)}{d\psi \cdot d\lambda}$$

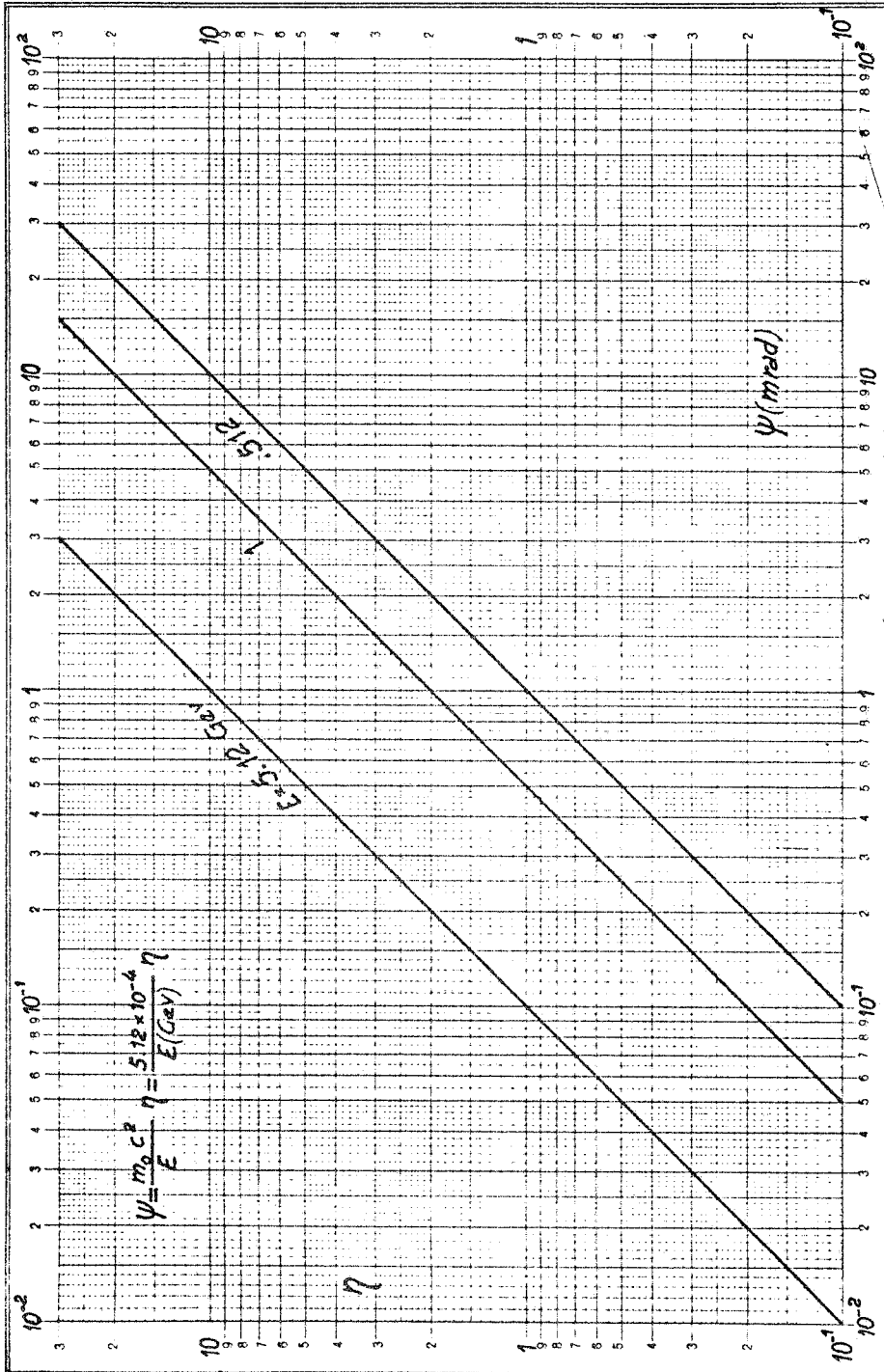


FIG. 19 - Graphical representation of the curve $\psi = (m_0 c^2 / E) \eta$

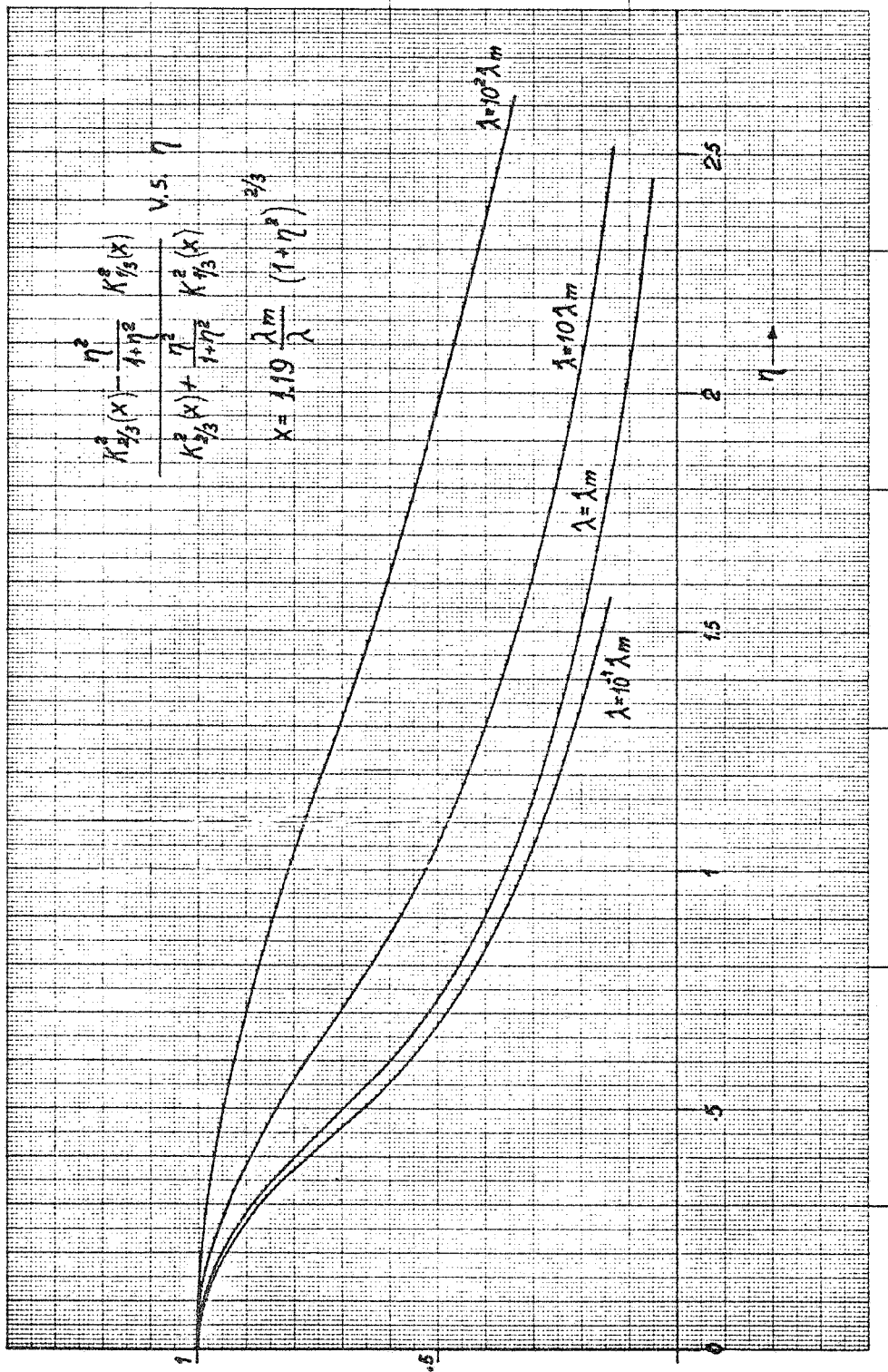


FIG. 20 - Graphical representation of the polarization for a few ratios of λ / λ_m .

APPENDIX. -

The most important functions are those of the radiated energy and the corresponding emitted photons.

We write here these functions with the numerical factors.

For the energy one has:

$$\begin{aligned} \frac{dW(\lambda)}{d\lambda} &= (4\pi)^{1/3} (0.42)^{7/3} c e_o^2 \frac{G(\lambda/\lambda_c)}{R^{2/3} \lambda_m^{7/3}} \\ \frac{dW(\lambda_m)}{d\lambda} &= (4\pi)^{1/3} (0.42)^{7/3} c e_o^2 \frac{G(\lambda = 0.42 \times \lambda_c)}{R^{2/3} \lambda_m^{7/3}} = \\ &= 5.4 \times 10^{-7} [R(m)]^{-2/3} [\lambda_m(\text{\AA})]^{-7/3} \frac{\text{Watt}}{\text{\AA}} \\ \frac{dW(\lambda, \lambda \gg \lambda_c)}{d\lambda} &= (4\pi)^{1/3} \times 2.12 \frac{c e_o^2}{R^{2/3} \lambda^{7/3}} \\ &= 7.3 \times 10^{-6} [R(m)]^{-2/3} [\lambda(\text{\AA})]^{-7/3} \frac{\text{Watt}}{\text{\AA}} \end{aligned}$$

where the universal function $G(\lambda/\lambda_c)$ is tabulated in Fig. 7, and :

$$\frac{dW(\lambda = \lambda_m; \lambda \gg \lambda_c)}{dW(\lambda_m)} = 13.1$$

For the photons:

$$\begin{aligned} \frac{dn(\lambda)}{d\lambda} &= \frac{3^{1/6}}{2^{1/3} \pi^{2/3}} (0.70)^{4/3} \frac{e_o}{\hbar} \frac{\phi(\lambda/\lambda_c)}{R^{2/3} \lambda_m^{4/3}} \\ \frac{dn(\lambda_m^F)}{d\lambda} &= \frac{3^{1/6}}{2^{1/3} \pi^{2/3}} (0.70)^{4/3} \frac{e_o}{\hbar} \frac{\phi(\lambda = 0.70 \lambda_c)}{R^{2/3} \lambda_m^{4/3}} = \\ &= 8.42 \times 10^8 [R(m)]^{-2/3} [\lambda_m^F(\text{\AA})]^{-4/3} \frac{\#}{\text{\AA sec}} \\ \frac{dn(\lambda, \lambda \gg \lambda_c)}{d\lambda} &= \frac{3^{1/6}}{2^{1/3} \pi^{2/3}} \times 2.12 \frac{e_o^2}{\hbar} \frac{1}{R^{2/3} \lambda^{4/3}} = 4.55 \times 10^9 [R(m)]^{-2/3} [\lambda(\text{\AA})]^{-4/3} \frac{\#}{\text{\AA sec}} \end{aligned}$$

where $\phi(\lambda/\lambda_c)$ is tabulated in Fig. 12a and

$$\frac{dW(\lambda = \lambda_m^F; \lambda \gg \lambda_c)}{dW(\lambda_m^F)} = 5.25$$

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These two pages are the "transparent" sheets for determining the point A.

They are not transparent, so it is necessary to recopy them on a transparent paper.

